NAME:____________________

MATH 5110
The Final

Do all six problems. Write readable answers on the test, but feel free to use or hand in additional paper if necessary. **Make sure I can find both the answer and how you got it**. Open book and notes, no calculators. 33 points each, plus 2 for writing your name.

1. An organism seeks to maintain an ideal body temperature of P by setting an internal thermostat to a target temperature T (which may not be equal to P). Let H represent the current body temperature. Suppose that H approaches T at rate α (as in Newton's Law of Cooling), and that T follows the differential equation

$$\frac{dT}{dt} = \beta(P - H).$$

- **a.** Write the differential equation followed by H.
- **b.** Find the equilibria and their stability for the system of equations followed by H and T.
- **c.** Draw the phase plane replete with direction arrows.
- **d.** Sketch a solution with a large value of α starting from initial conditions with $H < P < T$. Find the inner and outer solutions in this case, and describe the result in words.

2. As in problem **1**, suppose an organism seeks to maintain an ideal body temperature $H(t) = P$ by setting an internal thermostat to a target temperature T. Suppose that H approaches T at rate α, but now that $T(t) = 2P - H(t - \tau)$.

- **a.** Write a differential equation for H.
- **b.** Find the equilibria and their stability. Is there any value of τ for which the system becomes unstable?
- **c.** Why might a system like this be better than just setting $T = P$ in the first place? (Compare the rate of approach to equilibrium when $\tau = 0$).

3. As in problem **2**, an organism seeks to maintain an ideal body temperature of $H_t = P$ by setting a thermostat T_t, but it follows the discrete-time system

$$\begin{aligned} H_{t+1} &= \frac{H_t + T_t}{2} \\ T_{t+1} &= 2P - H_t \end{aligned}$$

where t is measured in hours.

- **a.** Explain the equations. Do they model the same process as in problem **2**?
- **b.** Find the equilibria and their stability.
- **c.** Write a model describing a thermostat that is reset based on the body temperature two hours earlier. Does this change the stability?

4. Consider the Fitzhugh-Nagumo equations

$$\frac{dv}{dt} = f(v) - w$$
$$\frac{dw}{dt} = \epsilon(v - \gamma w).$$

$f(v)$ is a function with $f(0) = f(a) = f(1) = 0$ for $0 < a < 1$ and $f(v) > 0$ if and only if $v < 0$ or $a < v < 1$.

a. Sketch the phase plane in a case where there are three equilibria.

b. Find the stability of the equilibria.

c. What good might a system of this sort be? How does it differ from the one-dimensional model $\frac{dv}{dt} = f(v)$?

d. Find an equation for the critical value of γ where the number of equilibria changes. What sort of bifurcation is this?

5. Little Anglers pond is stocked with fish for young enthusiasts to practice the art of fly-fishing. Fish are added at probabilistic rate λ, independent of the number of fish present. They are captured and removed at a per capita probabilistic rate of μ.

a. Draw a little diagram illustrating this process.

b. Write differential equations for $p_i(t)$, the probability that the pond has exactly i fish at time t.

c. Find the equilibrium for the p_i and the expected number of fish at equilibrium.

d. Write a differential equation for the expected number of fish as a function of time. Does the equilibrium match what you found in **c**?

6. After extensive training, the fish in Little Anglers pond learn to reproduce by themselves. Suppose the number of fish $N(t)$ obeys the differential equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - hN$$

where r is the maximum per capita reproduction, K is the carrying capacity, and h is the harvest effort.

a. Find the equilibria and their stability. What values of h produce a positive equilibrium?

b. What value of h maximizes the total number of fish harvested at the positive equilibrium?

c. Suppose one child is already fishing with harvest effort h_1. What value of h_2 for a second competing child is the best reply, in the sense of maximizing her harvest rate at equilibrium?

d. Find values h_1 and h_2 that are best replies to each other. Who might be happy about this situation?