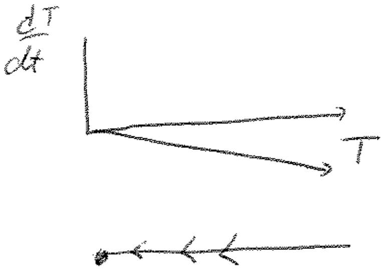
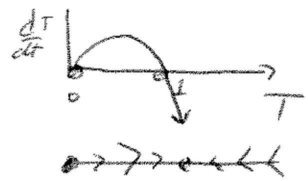
Santa test answers

2a. When $M=-1$, $\frac{dT}{dt} < 0$ for all $T > 0$



$\Rightarrow$ no toys this year!

b. With $M=3$, $\frac{dT}{dt} = T(3-2T-T^2) = T(1-T)(3+T)$

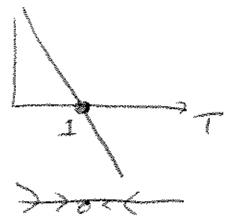


toys!

c. The linear system is the tangent line to $T(3-2T-T^2)$ at $T=1$

$\frac{d}{dT} T(3-2T-T^2) = 3-4T-3T^2 \Rightarrow \frac{d}{dT} T(3-2T-T^2)\Big|_{T=1} = -4$. Tangent line is then $-4(T-1)$

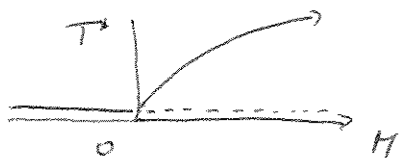
Linear system is $\frac{dT}{dt} = -4(T-1)$.



Phase-line is similar near $T=1$, but is quite different further away (only 1 equilibrium, for example)

d. We didn't do this

e.



Transcritical bifurcation at $M=0$.

3. a) Elf reproduction is reduced by large numbers of toys
σ is the fraction of toys left over, r = number of toys made per elf.

b. $E=0, T=0$ and $\left(E^* = \frac{(1-\sigma)(\lambda-1)}{r},\ T^* = \lambda - 1\right)$

c. $J = \begin{pmatrix} \frac{\lambda}{1+T^*} & \frac{-\lambda E^*}{(1+T^*)^2} \\ r & \sigma \end{pmatrix}$ At $(0,0)$: $\begin{pmatrix} \lambda & 0 \\ r & \sigma \end{pmatrix}$ $\Rightarrow$ unstable if $\lambda > 1$

At (E^*, T^*): $\begin{pmatrix} 1 & -E^*/\lambda \\ r & \sigma \end{pmatrix}$ $\det(J) = \sigma + \frac{rE^*}{\lambda} = \sigma + \frac{(1-\sigma)(\lambda-1)}{\lambda} < 1$

$Tr(J) = 1 + \sigma > 0$

Check if $\det(J) > -1 + Tr(J)$

$\sigma + \frac{(1-\sigma)(\lambda-1)}{\lambda} > -1 + 1 + \sigma = \sigma$

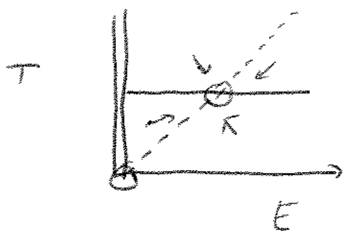
Yes! Stable.

d. Santa should increase λ to get more toys.
If a fraction $(1-\sigma)$ of toys are delivered, he should deliver them all (set $\sigma = 0$)

1. a. The first term says, as before, that per capita elf reproduction declines with the number of toys. Strangely, the $-aET$ seems to say that elves die more when there are more toys (perhaps due to accidents). r is the rate at which elves make toys, and μ is the rate at which toys disappear.

b. E-nullcline: $E = 0$ and $\frac{\lambda}{1+T} = aT$, a quadratic with positive solution T^*

T-nullcline: $T = \frac{rE}{\mu}$



c. $$J = \begin{pmatrix} \frac{\lambda}{1+T} - aT & \frac{-\lambda E}{(1+T)^2} - aE \\ r & -\mu \end{pmatrix}$$

At $(0,0)$: $\begin{pmatrix} \lambda & 0 \\ r & -\mu \end{pmatrix}$ Unstable if $\lambda > 0$

At (T^*, E^*) $\begin{pmatrix} 0 & \text{negative} \\ r & -\mu \end{pmatrix}$ $\begin{matrix} Tr < 0 \\ det > 0 \end{matrix} \Rightarrow$ stable

5. a) $$T + S_1 \xrightarrow{\alpha} S_2 \xrightarrow{K_2} S_3 + P \quad (S_3 \xrightarrow{K_3} S_1)$$

b) $$\frac{dT}{dt} = -\alpha T S_1$$

$$\frac{dS_1}{dt} = -\alpha T S_1 + K_3 S_3$$

$$\frac{dS_2}{dt} = \alpha T S_1 - K_2 S_2$$

$$\frac{dS_3}{dt} = K_2 S_2 - K_3 S_3$$

Conserved quantities: $S_1 + S_2 + S_3 = S_1(0)$. Could scale T by $T(0)$, S_1 and S_2 by $S_1(0)$

Let $\bar{T} = T/T(0)$, $\bar{S}_1 = \frac{S_1}{S_1(0)}$, $\bar{S}_2 = \frac{S_2}{S_2(0)}$

c) $$\frac{d\bar{T}}{dt} = -\alpha S_1(0) \bar{T} \bar{S}_1$$

$$\frac{d\bar{S}_1}{dt} = -\alpha T(0) \bar{T} \bar{S}_1 + K_3 (1 - \bar{S}_2 - \bar{S}_1)$$

$$\frac{d\bar{S}_2}{dt} = \alpha T(0) \bar{T} \bar{S}_1 - K_2 \bar{S}_2$$

THIS WILL NOT APPEAR ON THE TEST

d) Scale time by setting $\tau = \alpha S_1(0) \cdot t \Rightarrow$

$$\frac{d\bar{T}}{d\tau} = -\bar{T}\bar{S}_1$$

$$\frac{d\bar{S}_1}{d\tau} = -\frac{T(0)}{S_1(0)} \bar{T}\bar{S}_1 + \frac{K_3}{\alpha S_1(0)} (1 - \bar{S}_2 - \bar{S}_1)$$

$$\frac{d\bar{S}_2}{d\tau} = \frac{T(0)}{S_1(0)} \bar{T} \bar{S}_1 - \frac{K_2}{\alpha S_1(0)} \bar{S}_2$$