

NAME: _____

MATH 5110
The Final

Do all five problems. Write readable answers on the test, but feel free to use or hand in additional paper if necessary. **Make sure I can find both the answer and how you got it.** Open book and notes, no calculators. 40 points each.

1. Suppose two proteins inhibit each other's production, but function on a daily time scale according to the discrete-time dynamical system

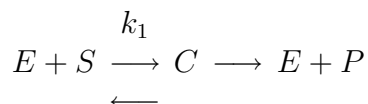
$$\begin{aligned}x_{t+1} &= f(y_t) + \sigma_1 x_t \\ y_{t+1} &= g(x_t) + \sigma_2 y_t\end{aligned}$$

where x and y represent the concentrations of the two proteins, f and g are positive decreasing functions, and σ_1 and σ_2 are the fraction of x and y respectively that persist until the next day. Find the condition on $f'(y^*)$ and $g'(x^*)$ that makes an equilibrium (x^*, y^*) unstable, and explain what it means graphically. Does the existence of an unstable equilibrium guarantee that this system works as a switch?

2. Suppose two organisms, with population sizes N_1 and N_2 , are competing for the resource R according to the system

$$\begin{aligned}\frac{dN_1}{dt} &= \frac{r_1 N_1 R}{1 + \alpha_1 R} - \delta_1 N_1 \\ \frac{dN_2}{dt} &= \frac{r_2 N_2 R}{1 + \alpha_2 R} - \delta_2 N_2 \\ \frac{dR}{dt} &= S - r_1 N_1 R - r_2 N_2 R - bR.\end{aligned}$$

- a. Describe (in 15 words or fewer) how the per capita growth of the two species depends on R .
 - b. Scale R by its equilibrium value in the absence of the two organisms (if $N_1 = N_2 = 0$).
 - c. Scale time in all three equations using the rate constant in the scaled R equation.
 - d. Suppose that r_1, r_2, δ_1 , and δ_2 are all of similar magnitude, but that b can differ. For what values of b are resources in quasi-steady state? Write and label the equations associated with the inner and outer solutions in this case, and find the inner solution.
3. Consider the basic reaction



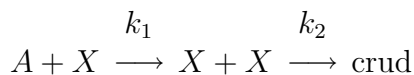
Suppose, however, that the complex C does not break down at a constant rate, but has a probability $G(\tau)$ of still persisting a time τ after being created. Do not assume that $G(\tau)$ is an exponential function. When it breaks down, a fraction q becomes product P and enzyme E , and a fraction $1 - q$ returns to being substrate S and enzyme E .

- a. Write equations describing this process.
 - b. Suppose also that the k_1 at time t is a decreasing function of the concentration of product P at time $t - T$. Write a model that includes this additional factor.
 - c. Which mechanism might destabilize the model? Describe the possible resulting behavior.
4. Consider a model of prey with population size N and predators with population size P

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - aNf(P)P \\ \frac{dP}{dt} &= caNf(P)P - \delta P\end{aligned}$$

where $f(P)$ is an increasing, bounded function with $f(0) = 0$.

- a. What does the inclusion of $f(P)$ mean (in 10 words or fewer)?
 - b. What bifurcations can this system have?
 - c. Draw the phase-plane, complete with direction arrows, in the case with the most equilibria.
5. Consider an autocatalytic enzyme that obeys



After the reaction, molecules of X are monomers (singletons) and decay independently at rate k_2 . Assume that A is held constant (a constant supply of substrate is always available). Suppose that k_1 is a function of X with the form

$$k_1 = \frac{k_0}{h(x)}$$

- a. Suppose first that $h(x)$ is an increasing function. Describe all possible behaviors of the differential equation for X and the bifurcations that occur as a function of A .
- b. What other form (other than increasing) of the function $h(X)$ is needed for this system to be able to function as a switch? What additional bifurcations occur in this case?