Math 5110: Homework Assignment 8
Due November 8, 2005

The first three problems concern a population following the equation

$$\frac{dN}{dt} = rN - \mu N. \tag{1}$$

Individuals reproduce at per capita rate r and die at per capita rate μ.

1. This problem shows that a particular form of distributed delay shares solutions with an ordinary differential equation.

   a. If an individual is born at some time t_0, find the probability $F(\tau)$ it is still alive at time $t_0 + \tau$.

   b. The probability $f(\tau)$ it dies at age τ is

   $$f(\tau) = -\frac{dF}{d\tau}.$$

   Explain why this is the case. Find $f(\tau)$.

   c. Explain why the equation with distributed delay

   $$\frac{dN}{dt} = rN - r\int_{\tau=0}^{\infty} f(\tau)N(t-\tau)d\tau \tag{2}$$

   should have the same solution as equation 1.

   d. Find the solution of equation 1 with $N(0) = N_0$ and show that it is also a solution of equation 2.

2. Suppose that individuals reproduce at per capita rate r but die at age T.

   a. Assume that the population grows according to $N(t) = e^{\lambda t}$. Write the equation that λ must satisfy.

   b. Find a necessary and sufficient condition on r and T for there to be a positive solution for λ. Explain this condition (think about what is required for a population to replace itself).

3. In equation 1, individuals die at rate μ. Suppose that $\mu = 1$.

   a. Find and graph the growth rate of the population as a function of r.

   b. Find the average time that an individual remains alive (if you can't figure out how to do this, try guessing).

   c. We can now compare a population where individuals remain alive for time T **on average** (equation 1) with one where individuals remain alive for time T **exactly** (exercise **2**). Use the average you found in part **b** as T, and numerically solve for the growth rate λ (found in exercise **2**) for five interesting values of r. Make sure to use at least one value of r which produces a small growth rate.

   d. Plot these values on your graph from part **a**. Do they match? How do they differ? Can you explain the difference?

4. Suppose chemical A turns into B at rate α. The body likes to maintain the amount of B at a low level. It would like to set the reversion rate equal to μB, but due to delays in the system the reversion rate is a function of B a time T ago. Suppose the initial conditions are $A(0) = A_0$ and $B(0) = 0$.

   a. Write differential equations for the amounts of A and B.

   b. Write a single differential equation for B.

   c. Find the equilibrium. How does it depend on the parameters α, μ and T?

   d. Suppose $\alpha = \mu = 1$ and $A_0 = 2$. Can the equilibrium be destabilized by the delay?

5. THIS PROBLEM IS NOT REQUIRED, AND IS NOT EXTRA CREDIT. IT'S JUST FOR THOSE WHO WANT AN EXTRA PROBLEM TO DO.
   Consider the spruce budworm model, but suppose that the predator functional response depends on the prey population a time T earlier, or

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \rho(N(t-T))P.$$

   a. Without plugging in a specific form for ρ, linearize this system around its equilibria.

   b. In the case with a type 2 functional response and 3 equilibria, there are two stable equilibria. Can either of them be destabilized by the delay?

   c. In the case with a type 3 functional response and 4 equilibria, there are two stable equilibria. Which, if either, is destabilized by the smallest delay?