Math 5110: Homework Assignment 2
Due September 13, 2005

1. Consider a discrete-time model of the concentration of medication in the bloodstream, M_t,

$$M_{t+1} = M_t - f(M_t)M_t + S,$$

where $f(M_t)$ is the fraction absorbed and S is the daily dosage. Suppose that

$$f(M) = \frac{M^n}{K^n + M^n}.$$

Set $K = 2$ and $S = 1$. Show that the equilibrium level is $M^* = 2$ for any value of n. For which values of n will solutions oscillate toward the equilibrium? Are there any values of n for which the equilibrium is unstable? Sketch a cobweb diagram of an illuminating case.

2. Newton's method for numerically solving equations can be formalized as a discrete-time dynamical system. For example, to find a numerical value of $\sqrt{3}$, we must solve $x^2 = 3$. Think of this as solving the equation $f(x) = 0$ where $f(x) = x^2 - 3$. Newton's method consists of picking a guess (like $x_0 = 2$), and replacing the equation $f(x) = 0$ with the linear equation $\hat{f}(x) = 0$ where $\hat{f}(x)$ is the tangent line to $f(x)$ at the first guess x_0.

   a. Graph $f(x)$ and $\hat{f}(x)$ at the guess $x_0 = 2$.

   b. Solve $\hat{f}(x)$ to find x_1.

   c. Find the tangent line at x_1, and use it to find the next guess x_2.

   d. Write down the discrete-time dynamical system that gets from one guess to the next. Graph the updating function.

   e. Find the equilibrium and its stability.

   f. Newton's method converges to the correct solution with astonishing speed. Do you have any idea why?

3. The following model describes a pest population of size N_t that is being controlled by the annual release of S sterile males,

$$N_{t+1} = \left(\frac{R}{1 + \frac{(R-1)N_t}{K}}\right)\left(\frac{N_t}{N_t + S}\right) N_t.$$

The first term describes the per capita egg production of a female as a function of crowding. The second term gives the probability that the male she mates with is fertile (females mate only once). If the female mates with a sterile male her eggs do not hatch. Show that there is a critical value of S above which the population goes extinct. Sketch a bifurcation diagram that describes what happens to this population as S is increased. Would pests disappear gradually or catastrophically as S passes the critical value?