Math and Medicine: Homework Assignment 4
Answers

1. Suppose people enter a simple transplant waiting list (all people are ranked equally) at rate $\lambda = 10.0$/year, organs become available at rate $\sigma = 20.0$/year, and people die at rate $\mu = 0.05$/year.

   a.

$$
\begin{aligned}
q_1 &= \frac{\lambda}{\sigma+\mu} q_0 \\
q_2 &= \frac{\lambda}{\sigma+2\mu} q_1 = \left(\frac{\lambda}{\sigma+\mu}\right)\left(\frac{\lambda}{\sigma+2\mu}\right) q_0 \\
q_3 &= \frac{\lambda}{\sigma+3\mu} q_2 = \left(\frac{\lambda}{\sigma+\mu}\right)\left(\frac{\lambda}{\sigma+2\mu}\right)\left(\frac{\lambda}{\sigma+3\mu}\right) q_0.
\end{aligned}
$$

   And so forth.

   b. With the given parameter values, I got

$$
\begin{aligned}
q_1 &= 0.4987 q_0,\ q_2 = 0.2481 q_0,\ q_3 = 0.1231 q_0,\ q_4 = 0.0609 q_0,\ q_5 = 0.0301 q_0 \\
q_6 &= 0.0148 q_0,\ q_7 = 0.0073 q_0,\ q_8 = 0.0035 q_0,\ q_9 = 0.0017 q_0,\ q_{10} = 0.0008 q_0.
\end{aligned}
$$

   This seemed small enough to me. Adding these up, gives

$$1 = q_0 + 0.9894 q_0$$

   so $q_0 = 0.503$. The list will be empty about half the time.

   c. If there is just one person on the list, the probability of dying is $\delta_1 = \mu/(\sigma+\mu)$. With two people, the probability is $\delta_2 = \mu/(\sigma/2+\mu)$ because you have only half a chance of getting the organ. Continuing in this way, we find $\delta_i = \mu/(\sigma/i+\mu)$. If we multiply these probabilities by the q_i's (but divided by $1-q_0$ because we are assuming somebody is on the list), we get

$$\frac{\sum \delta_i q_i}{1-q_0} = 0.0049.$$

   Less than 0.5% of people should die on the list, thanks to the low death rate and the high donation rate.

   d. An organ is wasted if it appears while the list is empty, which is roughly half the time. The price for the low death rate is many wasted organs.

2. My probabilities came out really close, with the simulated $q_0 = 0.502$. Also, out of 519 people, not a single one died the first time, and only one died out of 492 the next time. This seems a bit lower than the calculated value. But when I ran it with over 5000 patients (by setting `tmax` to be 500, the probability of death was 0.0051.

3. I found only one cycle of length 2 (P6 and P9), and only one cycle of length 3 (P6, P7 and P9). The longest chain seems to be length 8: P7, P9, P6, P10, P4, P5, P2, P3.