Derivation of Equation (6)

We want to solve the equations

$$\begin{aligned}\frac{dT^*}{dt} &= kV_I T_0 - \delta T^* \\ \frac{dV_I}{dt} &= -cV_I \\ \frac{dV_{NI}}{dt} &= \delta N T^* - cV_{NI}.\end{aligned}$$

where $T_0 = c/(kN)$, and with initial conditions $V_I(0) = V_0$, $V_{NI}(0) = 0$, and $T^*(0) = cV_0/(\delta N)$.

We used separation of variables in class to solve for V_I as

$$V_I = V_0 e^{-ct}.$$

Substituting into the T^* equation gives

$$\frac{dT^*}{dt} = kV_0 e^{-ct} T_0 - \delta T^* = \frac{cV_0}{N} e^{-ct} - \delta T^*.$$

Using the integrating factor trick, we have that

$$\frac{d}{dt} e^{\delta t} T^* = \frac{cV_0}{N} e^{(\delta - c)t}.$$

Integrating both sides gives

$$e^{\delta t} T^* = \frac{cV_0}{N(\delta - c)} e^{(\delta - c)t} + a.$$

Substituting in $t = 0$ and the initial conditions gives

$$T^*(0) = \frac{cV_0}{N(\delta - c)} + a$$

so that

$$a = T^*(0) - \frac{cV_0}{N(\delta - c)} = \frac{cV_0}{\delta N} - \frac{cV_0}{N(\delta - c)}$$

after a little algebra. Substituting back in and multiplying by $e^{-\delta t}$, we get

$$T^* = \frac{cV_0}{N(c - \delta)}(e^{-\delta t} - e^{-ct}) + \frac{cV_0}{\delta N} e^{-\delta t}.$$

With this in hand, we can solve for V_{NI} the same way.

$$\frac{dV_{NI}}{dt} = \frac{c\delta V_0}{c - \delta}(e^{-\delta t} - e^{-ct}) + cV_0 e^{-\delta t} - cV_{NI}$$

Using the integrating factor e^{ct}, we get

$$\frac{d}{dt}e^{ct}V_{NI} = \frac{c\delta V_0}{c-\delta}(e^{(c-\delta)t} - 1) + cV_0 e^{(c-\delta)t}$$

Integrating,

$$e^{ct}V_{NI} = \frac{c\delta V_0}{(c-\delta)^2}(e^{(c-\delta)t} - 1) - \frac{c\delta V_0 t}{c-\delta} + \frac{cV_0}{c-\delta}e^{(c-\delta)t} + a$$

Using the initial condition that $V_{NI}(0) = 0$, we can write this as

$$e^{ct}V_{NI} = \frac{c\delta V_0}{(c-\delta)^2}(e^{(c-\delta)t} - 1) - \frac{c\delta V_0 t}{c-\delta} + \frac{cV_0}{c-\delta}(e^{(c-\delta)t} - 1).$$

Multiplying by e^{-ct} and simplifying gives

$$\begin{aligned} V_{NI} &= \frac{cV_0}{c-\delta}\left(\frac{\delta}{c-\delta}(e^{-\delta t} - e^{-ct}) + (e^{-\delta t} - e^{-ct}) - \delta t e^{-ct}\right) \\ &= \frac{cV_0}{c-\delta}\left(\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta t e^{-ct}\right). \end{aligned}$$

Adding the expressions for $V_I(t)$ and $V_{NI}(t)$ gives equation (6) in the paper.