Modeling the Dynamics of Life: Calculus and Probability for Life Scientists

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5.9 Supplementary Problems for Chapter 5

• EXERCISE **5.1**
Consider the differential equation

$$\frac{dC}{dt} = 3(\Gamma - C) + 1$$

where C is the concentration of some chemical in a cell, measured in moles per liter and Γ is a constant.

a. What kind of differential equation is this? Explain the terms in the equation.
b. Draw the phase-line diagram.
c. Verify the stability of the equilibrium by using the derivative.
d. Sketch solutions as functions of time starting from two initial conditions: $C(0) = 0$ and $C(0) = \Gamma + 1$.

• EXERCISE **5.2**
Consider again the differential equation

$$\frac{dC}{dt} = 3(\Gamma - C) + 1$$

where Γ is a constant.

a. Solve the equation when $C(0) = 0$.
b. Check your answer.
c. Find $C(0.4)$.
d. After what time will the solution be within 5% of its limit?

• EXERCISE **5.3**
Consider the system of equations

$$\begin{aligned}\frac{d\Gamma}{dt} &= (C - \Gamma) - \frac{\Gamma^2}{3}\\ \frac{dC}{dt} &= 3(\Gamma - C) + 1.\end{aligned}$$

C is the internal concentration of a chemical and Γ the external concentration.

a. Describe in words the two processes affecting concentration in the external environment. How big is the external environment relative to the cell?
b. Draw a phase plane replete with nullclines, equilibria, and direction arrows. HINT. It is easier to put C on the vertical axis.

• EXERCISE **5.4**
Describe conditions when you might observe population growth described by the following.

a. One dimensional autonomous differential equation.
b. One dimensional nonautonomous differential equation.
c. One dimensional pure-time differential equation.
d. Two dimensional autonomous differential equation.

• EXERCISE **5.5**
Consider the differential equation

$$\frac{dV}{dt} = 12 - t^2$$

where $V(t)$ is volume in liters at time t and t is measured in seconds.

a. What kind of differential equation is this?
b. Graph the rate of change function and use it to sketch a graph of the solution.
c. At what time does V take on its maximum?

d. Suppose $V(0) = 0$. Use Euler's method to estimate $V(0.1)$.
e. Suppose $V(0) = 0$. Find the time T when $V(t)$ is 0 again.
f. What is the average volume between 0 and T.

• EXERCISE **5.6**
Consider the differential equation

$$\frac{dx}{dt} = 3x(x-1)^2$$

a. Draw the phase-line diagram of this equation.
b. Find the stability of the equilibria using the derivative.
c. Sketch trajectories of x as a function of time for initial conditions $x(0) = -0.5, x(0) = 0.5, x(0) = 1.5$.

• EXERCISE **5.7**
Suppose the per capita production rate of a bacterial population is given by

$$\text{per capita production} = \frac{1}{\sqrt{b}}$$

where $b(t)$ is the population size at time t and t is measured in hours.

a. Find the differential equation describing this population.
b. Solve the equation and check your answer.
c. What is the population after 2 hours if the population starts at $b(0) = 10000$?
d. Does this population grow faster or slower than one growing exponentially? Why?

• EXERCISE **5.8**
Differential equations to describe an epidemic are sometimes given as

$$\begin{aligned} \frac{dS}{dt} &= \beta(S+I) - cSI \\ \frac{dI}{dt} &= cSI - \delta I. \end{aligned}$$

Here S measures the number of susceptible people and I the number of infected people.

a. Compare with the predator-prey equations. What is different in these equations? What biological process does each term on the right hand side describe?
b. Sketch the nullclines and find the equilibria if $\beta = 1$ and $c = \delta = 2$. (Draw only the parts where S and I are positive).
c. Sketch direction arrows on your phase-plane diagram.

• EXERCISE **5.9**
Consider the following differential equation describing the concentration of sodium ions in a cell following consumption of a bag of Doritos at time $t = 0$ seconds.

$$\frac{dN}{dt} = 2 - 10t.$$

Suppose $N(0) = 50$ mm per cm^3.

a. What kind of differential equation is this?
b. Sketch a graph of the rate of change as a function of time.
c. Sketch a graph of the concentration as a function of time.
d. Use Euler's method to estimate $N(0.1)$.
e. Find $N(1)$ exactly.
f. At what times is $N(t) = 50$?

• EXERCISE **5.10**
Consider the following differential equation describing the concentration of sodium ions in a cell.

$$\frac{dN}{dt} = 2(N-50) - (N-50)^2.$$

a. What kind of differential equation is this? What does each term mean?
b. Sketch a graph of the rate of change as a function of concentration.
c. Draw the phase-line diagram.
d. Sketch solutions starting from $N(0) = 48$, $N(0) = 51$ and $N(0) = 54$.
e. Suppose $N(0) = 51$. Estimate $N(0.1)$.
f. What method would you use to solve this equation?

• EXERCISE **5.11**
Two types of bacteria with populations a and b are living in a culture. Suppose

$$\begin{aligned}\frac{da}{dt} &= 2a\left(1 - \frac{a}{500} - \frac{b}{200}\right)\\ \frac{db}{dt} &= 3b\left(1 + \frac{a}{1000} - \frac{b}{100}\right).\end{aligned}$$

a. What kind of differential equation is this? Explain the terms.
b. Draw the phase-plane, including nullclines, equilibria and direction arrows.

• EXERCISE **5.12**
Suppose the scaled potential v and level of potassium channel opening w in a neuron were described by

$$\begin{aligned}\frac{dv}{dt} &= v(1-v) - w\\ \frac{dw}{dt} &= v - 2w\end{aligned}$$

a. Draw a phase line for v assuming that w is fixed at 0.
b. Draw the phase plane for the full system including equilibria, nullclines, and direction arrows.

• EXERCISE **5.13**
The length L of a microtubule is found to follow the differential equation

$$\frac{dL}{dt} = -L(2-L)(1-L)$$

where L is measured in microns.

a. Draw the phase-line diagram.
b. Check the stability of the equilibria using the derivative.
c. Sketch trajectories starting from $L(0) = 0.5$, $L(0) = 1.5$ and $L(0) = 2.5$.

• EXERCISE **5.14**
A different lab finds that

$$\begin{aligned}\frac{dL}{dt} &= -2.0L + 5.4LE\\ \frac{dE}{dt} &= 3.5 - LE\end{aligned}$$

where E is the level of some component of the microtubules and L is the length of the microtubule.

a. Explain the terms in these equations.
b. Draw the phase plane diagram, including nullclines, equilibria, and direction arrows.

• EXERCISE **5.15**

Consider the differential equation describing a population of mathematically sophisticated bacteria,

$$\frac{db}{dt} = b\ln(\frac{2b+1}{2+b}).$$

a. What kind of differential equation is this?
b. Draw the derivative as a function of the state variable and draw the phase-line diagram.
c. Sketch trajectories starting from $b(0) = 0.8$ and $b(0) = 1.2$.
d. Check the stability of the equilibrium at $b = 0$ by taking the derivative of the rate of change function.
e. Use Euler's method to estimate $b(0.01)$ if $b(0) = 0.5$.

• EXERCISE **5.16**

Suppose a population of size N has per capita production equal to $1 - 2e^{-0.01N}$.

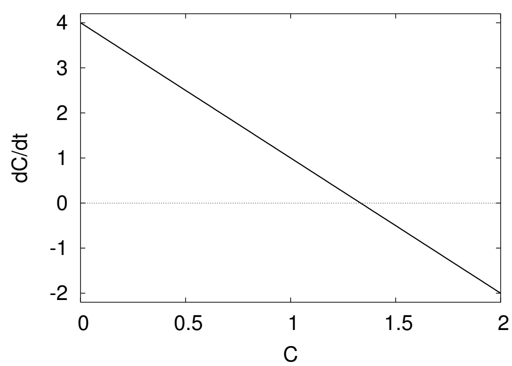
a. Write the differential equation for population size.
b. Find the equilibrium or equilibria.
c. Use the stability theorem to determine stability.
d. Based on these calculations, draw a phase-line diagram.

Answers

5.1.

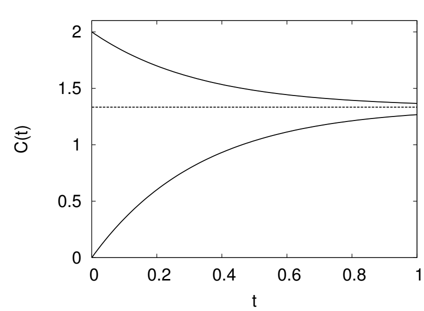
a. If Γ is constant, this is an autonomous differential equation. The first term represents diffusion of chemical, and the second term describes a constant increase from an outside source.

b.



c. The equilibrium is the solution of $3(\Gamma - C) + 1 = 0$, or $c = \Gamma + 1/3$. If $f(C) = 3(\Gamma - C) + 1$, then $f'(C) = -3$, which is clearly negative. Therefore, the equilibrium is stable.

d. With $\Gamma = 1.0$,



5.2.

a. Separating variables, we have

$$\frac{dC}{3(\Gamma - C) + 1} = dt$$

$$
\begin{aligned}
\frac{-1}{3}\ln(3(\Gamma - C) + 1) &= t + k \\
\ln(3(\Gamma - C) + 1) &= -3t - 3k \\
3(\Gamma - C) + 1 &= e^{-3t-3k} \\
3(\Gamma - C) &= e^{-3t-3k} - 1 \\
C &= \Gamma - \frac{1}{3}(e^{-3t-3k} - 1).
\end{aligned}
$$

To find k, plug in the initial condition $C(0) = 0$, finding $e^{-3k} = 3\Gamma + 1$, so $C(t) = \Gamma - (\Gamma + 1/3)(e^{-3t}) + 1/3$ or $C(t) = (\Gamma + 1/3)(1 - e^{-3t})$.

b. Differentiating, we find that $C'(t) = (3\Gamma + 1)e^{-3t}$. Also,

$$
\begin{aligned}
3(\Gamma - C) + 1 &= 3\Gamma - 3(\Gamma + 1/3)(1 - e^{-3t}) + 1 \\
&= 3\Gamma - 3\Gamma + 3\Gamma e^{-3t} - 1 + e^{-3t} + 1 \\
&= 3\Gamma e^{-3t} + e^{-3t} = (3\Gamma + 1)e^{-3t}.
\end{aligned}
$$

They match!

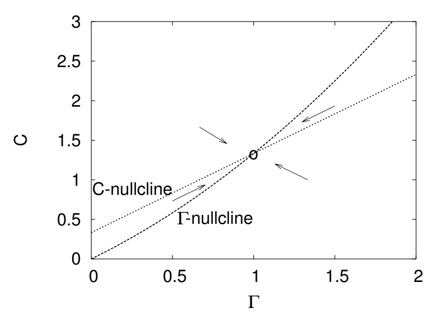
c. Substitute $t = 0.4$, finding $C(0.4) = (\Gamma + 1/3)(1 - e^{-1.2}) = 0.7(\Gamma + 1/3)$.

d. Need to find when $C(t) > 0.95(\Gamma + 1/3)$, which occurs after the time when $1 - e^{-3t} = 0.95$ or when $e^{-3t} = 0.05$, which has solution $t = -\ln(0.05)/3 = 1.0$.

5.3.

a. The first terms look like diffusion. The factor of 3 means that C changes more quickly than Γ, so it must have a smaller volume. The $\Gamma^2/3$ term means that Γ is being depleted the larger its concentration. The $+1$ is an outside supplement to C.

b.



5.4.

a. When one quantity determines its own rate of change.

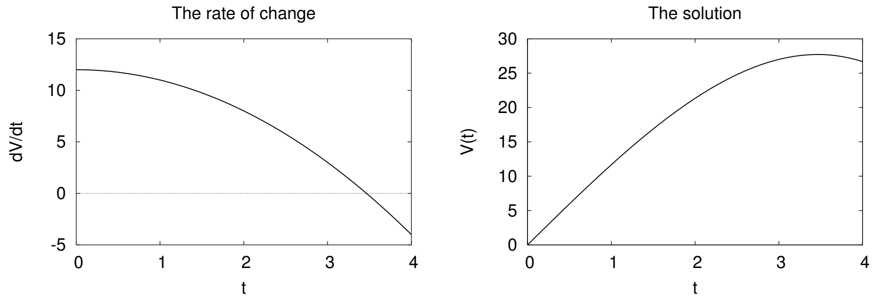
b. When the rate of change of one quantity is determined in part by external factors.

c. When the rate of change of a quantity is determined by external factors.

d. When two quantities mutually determine their rates of change.

5.5.

a. This is a pure-time differential equation because the rate of change depends only on the time.



b.

c. V takes on a maximum when $dV/dt = 0$, or at $t = \sqrt{12}$.

d. $\hat{V}(0.1) = V(0) + V'(0) \cdot (0.1) = 0 + 12 \cdot 0.1 = 1.2$.

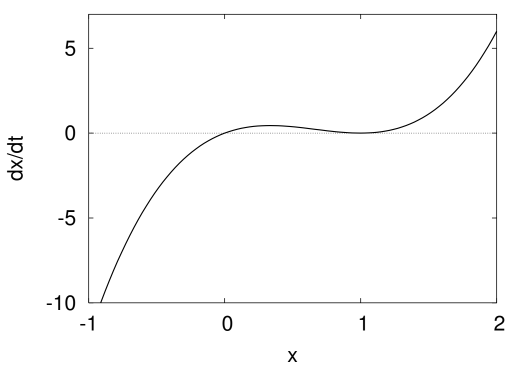
e. Integrating, we find $V(t) = 12t - t^3/3 + c$, and the constant c must be 0. Solving $V(t) = 0$ gives $t = 6$.

f. The average volume is

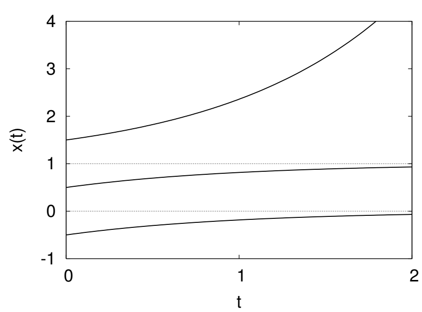
$$\frac{1}{6}\int_0^6 12t - \frac{t^3}{3}dt = \frac{1}{6}\left(6t^2 - \frac{t^4}{12}|_0^6\right) = 18.$$

5.6.

a. We begin by sketching the graph of $3x(x-1)^2$, which is negative for $x < 0$, and positive for $x > 0$ except at $x = 1$.



b. Equilibria are $x = 0$ and $x = 1$. Setting $f(x) = 3x(x-1)^2$, we have $f'(x) = 3(x-1)^2 + 6x(x-1)$. $f'(0) = 3$, so 0 is unstable, and $f'(1) = 0$, so we can't tell whether 1 is stable.



c.

5.7.

a. $\frac{db}{dt} = \frac{1}{\sqrt{b}} \cdot b = \sqrt{b}$.

b. Using separation of variables, we get

$$\begin{aligned} \frac{db}{\sqrt{b}} &= dt \\ 2\sqrt{b} &= t + c \\ b &= \left(\frac{t+c}{2}\right)^2. \end{aligned}$$

Checking, $b'(t) = (t + c)/2$, which is indeed $\sqrt{b}$.

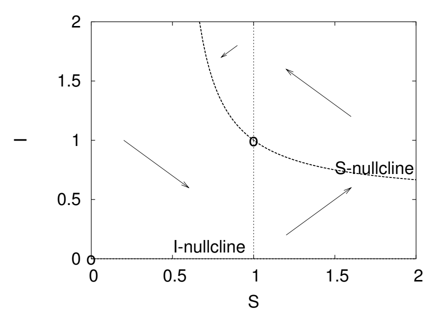
c. If $b(0) = 10000$, $(c/2)^2 = 10000$, so $c = 200$. Substituting $t = 2$, we find $b = 101^2 = 10201$.

d. A quadratic function increases more slowly than an exponential function, so this population grows more slowly. This occurs because per capita reproduction decreases with larger populations.

5.8.

a. The difference is that I's can reproduce and turn into S's. The $\beta(S + I)$ term represents reproduction of both S and I individuals into the S class, meaning that new offspring are not infected. The cSI term represents transfer from the S into the I class through the infection process. The $-\delta I$ term is death of the infected individuals.

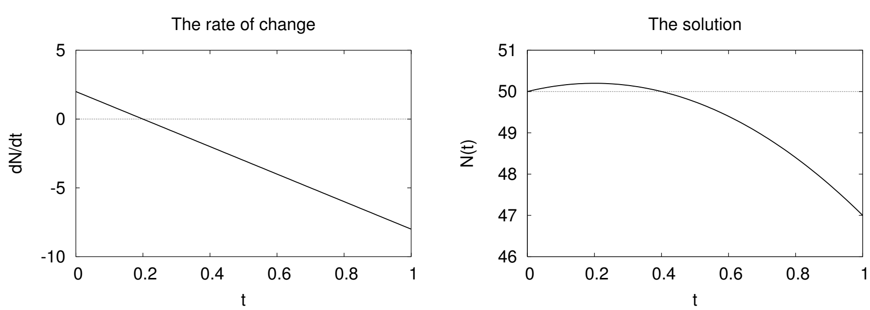
b. The S-nullcline is $I = \beta S/(cS - \beta)$, and the I-nullcline is $I = 0$ and $S = \delta/c$. The equilibria are $(0, 0)$ and $(1, 1)$.



5.9.

a. This is a pure-time differential equation because the rate of change depends only on time, not on N.

b.



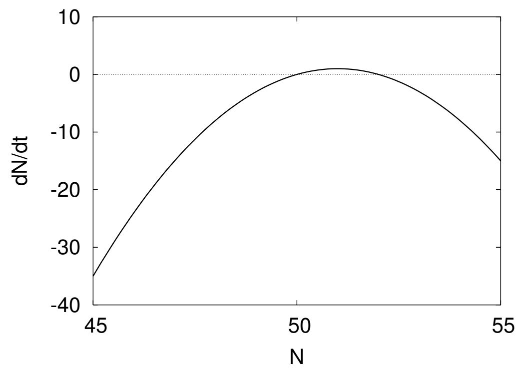
d. $\hat{N}(0.1) = N(0) + N'(0) \cdot 0.1 = 50 + 2 \cdot 0.1 = 50.2$.

e. Integrating, we get $N(t) = 2t - 5t^2 + c$. The constant is 50 from the initial condition. Substituting, we find that $N(1) = 47$.

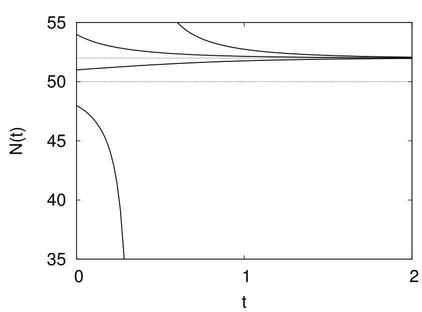
f. $N(t) = 50$ at $t = 0$ and at $t = 0.4$.

5.10.

a. This is an autonomous differential equation because the rate of change depends only on N, not on time. The first term looks kind of like diffusion, but the sign is wrong. The cell concentrates sodium when $N > 50$ and removes it when $N < 50$. The second term is a removal term – more sodium is removed the farther the concentration gets from 50.



b.



d.

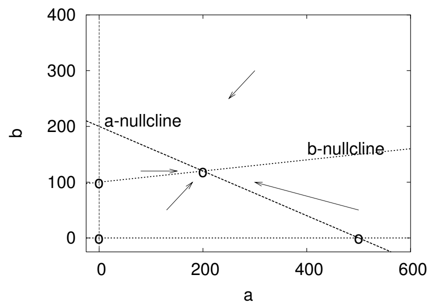
e. $\hat{N}(0.1) = N(0) + N'(0) \cdot 0.1 = 51 + 1 \cdot 0.1 = 51.1$.

f. I'd use **separation of variables** to solve it. Or ask my friendly local computer.

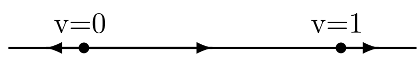
5.11.

a. A system of coupled autonomous differential equations. This looks like a system in which type a has a lower per capita production, and the per capita production of type a is reduced by its own presence and the presence of the other, and rather more by b, which probably eats a. The per capita production of type b is increased by type a and decreased by itself. Perhaps b's eat a's but fight with each other.

b. The a-nullcline is $a = 0$ and $b = 200 - 0.4a$. The b-nullcline is $b = 0$ and $b = 100 + 0.1a$. The equilibria are $(0, 0)$, $(0, 100)$, $(500, 0)$, $(200, 120)$.

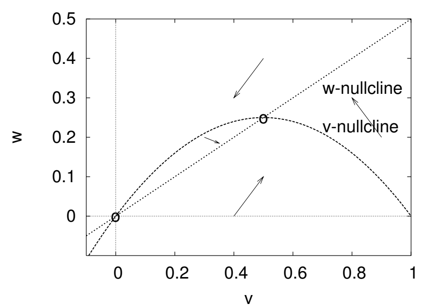


5.12.



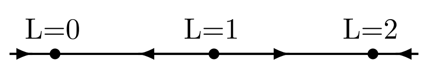
a.

b. Placing w on the vertical axis, the v-nullcline is $w = v(1-v)$ and the w-nullcline is $w = v/2$. $dv/dt > 0$ if w is below the nullcline, and $dw/dt > 0$ also occurs when w is below the nullcline,



5.13.

a. The equilibria are $L = 0$, $L = 1$ and $L = 2$. At $L = -0.5$, the derivative is positive, at $L = 0.5$, the derivative is negative, at $L = 1.5$ it is positive, and at $L = 2.5$ it is again negative.

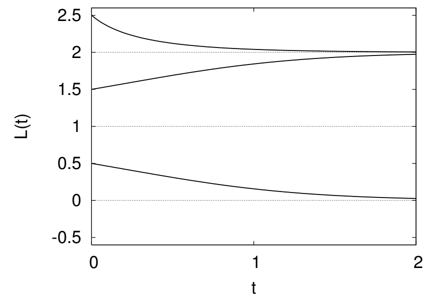


b. Taking the derivative of the rate of change function with respect to L,

$$\frac{d}{dt}\left(-L(2-L)(1-L)\right) = -(2-L)(1-L) + L(1-L) + L(2-L).$$

At $L = 0$, this is -2, at $L = 1$ it is 1 and at $L = 2$ it is -2. Therefore the equilibrium at $L = 1$ is unstable and the other two are stable.

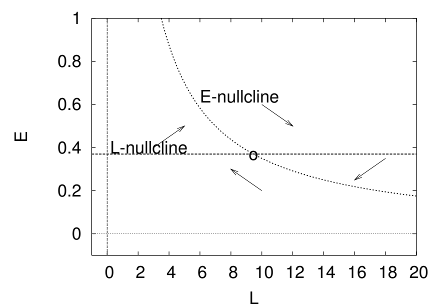
c.



5.14.

a. The first term in the L equation indicates that microtubules break down faster when they are larger, but build up faster in the presence of the component E. The component E is constantly supplemented from outside and seems to be converted into microtubule.

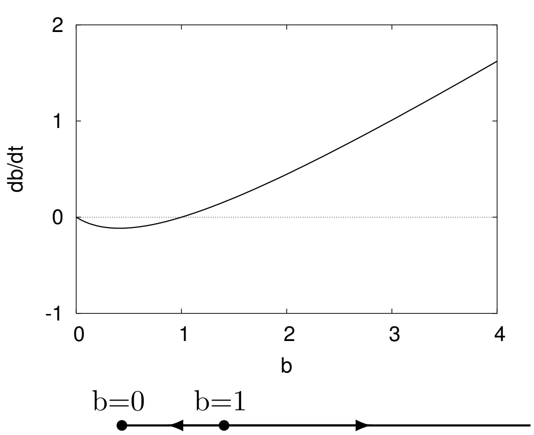
b. With E on the vertical axis, the L-nullcline is at $L = 0$ or $E = 2.0/5.4 = 0.37$. The E-nullcline is $E = 3.5/L$. L increases when E is above the nullcline, and E increases when E is below the nullcline.

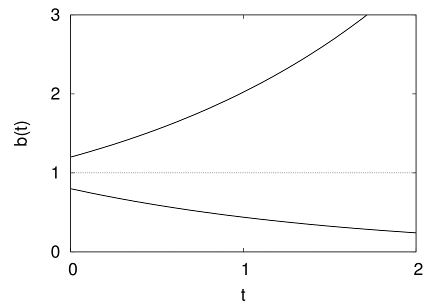


5.15.

a. This is an autonomous differential equation.

b.





c.

d. The derivative is $-\ln(2) < 0$, and the equilibrium is stable.

e. The rate of change if $b = 0.5$ is -0.111. Therefore,

$$\hat{b}(0.01) = 0.5 - 0.111 \cdot 0.01 = 0.49889.$$

5.16.

a.

$$\frac{dN}{dt} = (1 - 2e^{-0.01N})N$$

b. Equilibria occur where $N = 0$ and where $1 - 2e^{-0.01N} = 0$ or $N^* = 69.3$.

c. The derivative of $f(N) = (1 - 2e^{-0.01N})N$ is $f'(N) = (1 - 2e^{-0.01N}) + 2e^{-0.01N}N$. Then $f'(0) = -1$ and $f'(N^*) = 69.3$. Therefore, the equilibrium at 0 is stable and the one at N^* is unstable.