Modeling the Dynamics of Life: Calculus and Probability for Life Scientists

Frederick R. Adler [1]

©Frederick R. Adler, 2011

[1]Department of Mathematics and Department of Biology, University of Utah, Salt Lake City, Utah 84112

4.8 Supplementary Problems for Chapter 4

• EXERCISE **4.1**

The voltage v of a neuron follows the differential equation

$$\frac{dv}{dt} = 1.0 + \frac{1}{1+0.02t} - e^{0.01t}$$

over the course of 100 ms, where t is measured in ms and v in millivolts. We start at $v(0) = -70$.

a. Sketch a graph of the rate of change. Indicate on your graph the times when the voltage reaches minima and maxima (you don't need to solve for the numerical values).
b. Sketch a graph of the voltage as a function of time.
c. What is the voltage after 100 ms?

• EXERCISE **4.2**

Consider again the differential equation in the previous problem,

$$\frac{dv}{dt} = 1.0 + \frac{1}{1+0.02t} - e^{0.01t}$$

with $v(0) = -70$.

a. Use Euler's method to estimate the voltage after 1 millisecond, and again 1 millisecond after that.
b. Estimate the voltage after 2 ms using left-hand and right-hand Riemann sums.
c. Which of your estimates matches Euler's method and why?

• EXERCISE **4.3**

A neuron in your brain sends a charge down an 80 centimeter long axon (a long skinny thing) toward your hand at a speed of 10 meters per second. At the time when the charge reaches your elbow, the voltage in the axon is -70 millivolts except on the 6 centimeter long piece between 47 and 53 centimeters from your brain. On this piece, the voltage is

$$v(x) = -70.0 + 10.0(9.0 - (x - 50.0)^2)$$

where $v(x)$ is the voltage at a distance of x centimeters from the brain.

a. How long will it take the information to get to your hand? How long did it take to reach your elbow?
b. Sketch a graph of the voltage along the whole axon.
c. Find the average voltage of the 6 centimeter piece.
d. Find the average voltage of the whole axon.

• EXERCISE **4.4**

The charge in a dead neuron decays according to

$$\frac{dv}{dt} = \frac{1}{\sqrt{1+4t}} - \frac{2}{(1+4t)^{\frac{3}{2}}}$$

starting again from $v(0) = -70$ at $t = 0$.

a. Is the voltage approaching 0 as $t \to \infty$? How do you know that it will eventually reach 0?
b. Write an equation (but don't solve it) for the time when the voltage reaches 0.
c. What is wrong with this model?

• EXERCISE **4.5**

Consider the differential equations

$$\frac{db}{dt} = 2b$$

and

$$\frac{dB}{dt} = 1 + 2t.$$

a. Which of these is a pure-time differential equation? Describe circumstances when you might find each of these equations.
b. Suppose $b(0) = B(0) = 1$. Use Euler's method to find estimates for $b(0.1)$ and $B(0.1)$.
c. Use Euler's method again to find estimates for $b(0.2)$ and $B(0.2)$.

• EXERCISE **4.6**
Consider the differential equation

$$\frac{dp}{dt} = e^{-4t}$$

where $p(t)$ is product in moles at time t and t is measured in seconds.

a. Explain in words what is going on.
b. Suppose $p(0) = 1$. Find $p(1)$.

• EXERCISE **4.7**
Consider the differential equation

$$\frac{dV}{dt} = 4 - t^2$$

where $V(t)$ is volume in liters at time t and t is measured in minutes.

a. Explain in words what is going on.
b. At what time is the volume a maximum?
c. Break the interval from $t = 0$ to $t = 3$ into three parts and find the left-hand and right-hand estimates of the volume at $t = 3$ (assume $V(0) = 0$).
d. Write down the definite integral expressing volume at $t = 3$ and evaluate.

• EXERCISE **4.8**
Find the area under the curve $f(x) = 3 + (1 + x/3)^2$ between $x = 0$ and $x = 3$.

• EXERCISE **4.9**
The population density of trout in a stream is

$$\rho(x) = |-x^2 + 5x + 50|$$

where ρ is measured in trout per mile and x is measured in miles. x runs from 0 to 20.

a. Graph $\rho(x)$ and find the minimum and maximum.
b. Find the total number of trout in the stream.
c. Find the average density of trout in the stream.
d. Indicate on your graph how you would find where the actual density is equal to the average density.

• EXERCISE **4.10**
The amount of product is described by the differential equation

$$\frac{dp}{dt} = \frac{1}{\sqrt{1 + 3t}}$$

starting at time $t = 0$. Suppose p is measured in moles, t in hours, and that $p(0) = 0$.

a. Find the limiting amount of product.
b. Find the average rate at which product is produced as a function of time, and compute the limit as $t \to \infty$.
c. Find the limit as $t \to 0$ of the average rate at which product is produced.

• EXERCISE **4.11**
A student is hooked up to an EEG during a test, and her α brain wave power follows

$$A(t) = \frac{50}{2.0 + 0.3t} + 10e^{0.0125t}$$

where t runs from 0 to 120 minutes.

a. Convince yourself that brain wave activity has a minimum value some time during the test. Sketch a graph of the function. Find the maximum.
b. Find the total brain wave power during the test.
c. Find the average brain wave power during the test. Sketch the corresponding line on your graph.
d. Estimate the minimum value from your average value.
e. Draw a graph showing how you would estimate the total using the right hand approximation with $n = 6$. Write down the associated sum. Do you think your estimate is high or low?

• EXERCISE **4.12**
The density of sugar in a hummingbird's 20mm long tongue is

$$s(x) = \frac{1.2}{1.0 + 0.2x}$$

where x is measured in mm from the end of the tongue and s is measured in moles per meter.

a. Find the total amount of sugar in the hummingbird's tongue.
b. Find the average density of sugar in the tongue.
c. Compare the average with the minimum and maximum density. Does your answer make sense?

• EXERCISE **4.13**
Consider the function $G(h)$ giving the density of nutrients in a plant stem as a function of the height h

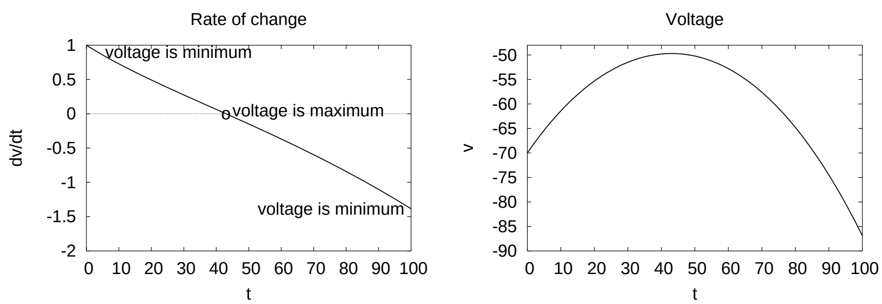
$$G(h) = 5 + 3e^{-2h}$$

where G is measured in mol/m and h is measured in m.

a. Find the total amount of nutrient if the stem is 2.0 m tall.
b. Find the average density in the stem.
c. Find the exact and approximate amount between 1.0 and 1.01 m.

Answers

4.1.

a. 

c. Solving with the indefinite integral, we get

$$v(t) = t + 50\ln(1 + 0.02t) - 100e^{0.01t} + c.$$

Substituting the initial condition, we have that $c = 30$. Therefore, $v(100) = 100 + 50\ln(3.0) - 100e^1 + 30 = -86.9$.

4.2.

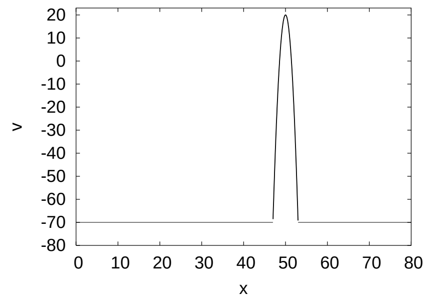
a. $\hat{v}(1) = v(0) + v'(0) \cdot (1 - 0) = -70 + 1.0 = -69.0$. After another ms, we find $\hat{v}(2) = \hat{v}(1) + v'(1) \cdot (2 - 1) = -69.0 + 0.97 = -68.03$.

b. The voltage after 2 ms is $-70 + \int_0^2 t + 50\ln(1 + 0.02t) - 100e^{0.01t}\,dt$. Denoting the integrand by $f(t)$, the left-hand Riemann sum is $-70 + f(0) + f(1) = -70 + 1.0 + 0.97 = -68.03$. The right-hand Riemann sum is $-70 + f(1) + f(2) = -70 + 0.97 + 0.94 = -68.09$.

c. The left-hand estimate matches because it uses the same information as Euler's method.

4.3.

a. At a speed of $10\frac{\text{m}}{\text{s}} = 1000\frac{\text{cm}}{\text{s}}$, it takes $t = 80\text{cm}/(1000\text{cm}/\sec) = 0.08\,\text{sec}$ to reach your hand. Similarly, your elbow seems to be 50 cm from the brain, so it takes the signal 0.05 s to get there.



b.

c. average= $\frac{1}{6}\int_{47}^{53} -70.0 + 10.0(9.0 - (x - 50.0)^2)dx$. Substituting $y = x - 50$, we have that $dy = dx$ and limits of integration from $y = -3$ to $y = 3$, or

$$\begin{aligned}
\text{average} &= \frac{1}{6}\int_{-3}^{3} -70.0 + 10.0(9.0 - y^2)dy \\
&= \frac{1}{6}\int_{-3}^{3} 20.0 - 10.0y^2 dy \\
&= \frac{1}{6}(20.0y - \frac{10.0}{3}y^3)|_{-3}^{3} \\
&= \frac{1}{6}\left((60.0 - \frac{10.0}{3}3^3) - (-60.0 - \frac{10.0}{3}(-3)^3)\right) = -10.
\end{aligned}$$

d. The total voltage along the 6 cm piece is -60. Along the rest (74 cm) the total is $74 \cdot (-70) = -5180$. The total along the whole thing is $-5180 - 60 = -5240$, so the average is $-5240/80 = -65.6$.

4.4.

a. As $t \to \infty$, the rate of change acts like the leading behavior of the right hand side, or $\frac{1}{\sqrt{1+4t}}$, which in turn acts more or less like $t^{-1/2}$. The integral of such a function approaches infinity. By the Intermediate Value Theorem, the voltage must cross 0 on its way from -70 to infinity.

b. If T represents the time when the voltage reaches 0, we require an increase in voltage of exactly 70. The equation is then

$$\int_0^T \frac{1}{\sqrt{1+4t}} - \frac{2}{(1+4t)^{\frac{3}{2}}}dt = 70.$$

c. It seems rather unlikely that the voltage of a dead neuron will approach infinity.

4.5.

a. The second is a pure-time differential equation. The first might describe a population of bacteria that are autonomously reproducing, and the second might describe a population being supplemented from outside at an ever increasing rate.

b. $\hat{b}(0.1) = b(0) + b'(0) \cdot 0.1 = 1 + 2 \cdot 0.1 = 1.2$. Similarly, $\hat{B}(0.1) = B(0) + B'(0) \cdot 0.1 = 1 + 2 \cdot 0.1 = 1.2$.

c. $\hat{b}(0.2) = \hat{b}(0.1) + 2\hat{b}(0.1) \cdot 0.1 = 1.2 + 2 \cdot 1.2 \cdot 0.1 = 1.44$. Similarly, $\hat{B}(0.2) = \hat{B}(0.1) + B'(0.1) \cdot 0.1 = 1.2 + 1.2 \cdot 0.1 = 1.32$.

4.6.

a. The rate of production is decreasing exponentially, even though the total amount of product continues to increase.

b. $p(1) = 1 + \int_0^1 e^{-4t}dt = 1 - 0.25e^{-4t}|_0^1 = 1.24.$

4.7.

a. The volume is increasing until time $t = 2$, and decreases thereafter.

b. The volume is a maximum when the rate of change is 0, or at $t = 2$.

c. LHE$=V(0) \cdot 1 + V(1) \cdot 1 + V(2) \cdot 1 = 7$ and RHE$=V(1) \cdot 1 + V(2) \cdot 1 + V(3) \cdot 1 = -2$.

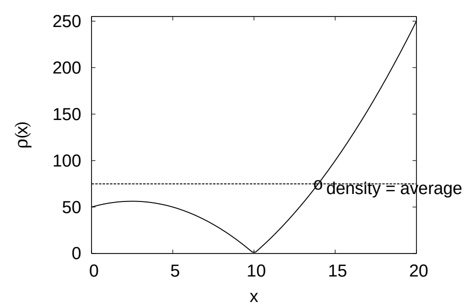
d. $V(3) = \int_0^3 4 - t^2 dt = 4t - \frac{t^3}{3}|_0^3 = 3.$

4.8. Area $= \int_0^3 3 + (1 + x/3)^2 dx$. We substitute $y = 1 + x/3$, finding $dy = dx/3$, and limits of integration from $y = 1$ to $y = 2$. So

$$\begin{aligned} \text{area} &= \int_0^3 3 + (1 + x/3)^2 dx = \int_1^2 (3 + y^2)3dy \\ &= 9y + y^3|_1^2 = (9 \cdot 2 + 2^3) - (9 \cdot 1 + 1^3) = 16. \end{aligned}$$

4.9.

a. The function hits 0 when $x = 10$. The derivative is 0 at $x = 2.5$, where the density is 56.25. At the endpoints, we have densities of 50 (at $x = 0$) and 250 (at $x = 20$). The maximum is thus at $x = 20$, with the minimum at $x = 10$.



b. Taking into account the absolute values, the total number is

$$\begin{aligned} \text{total} &= \int_0^{20} |-x^2 + 5x + 50|dx \\ &= \int_0^{10} -x^2 + 5x + 50dx + \int_{10}^{20} x^2 - 5x - 50dx \\ &= -x^3/3 + 5x^2/2 + 50x|_0^{10} + x^3/3 - 5x^2/2 - 50x|_{10}^{20} = 1500. \end{aligned}$$

c. The average density is 75.0.

d. Shown in figure for **a**.

4.10.

a. The amount of product increases to infinity.

b. We have that

$$p(t) = \int_0^t \frac{1}{\sqrt{1+3s}} ds.$$

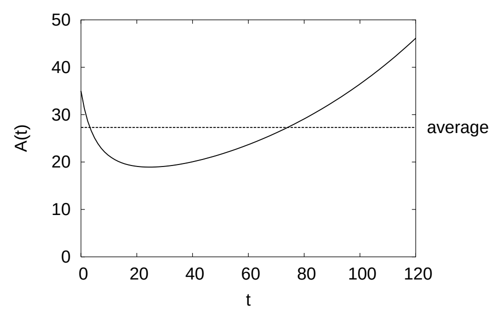
After substituting $y = 1 + 3s$, we find that $p(t) = \frac{2}{3}(\sqrt{1+3t} - 1)$. The average rate is $p(t)/t$. Because $p(t)$ acts like $\sqrt{t}$, which increases more slowly than t this average approaches 0 as $t \to \infty$.

c. Because both $p(t)$ and t approach 0 as $t \to 0$, we can use L'Hopital's rule as

$$\lim_{t\to 0} \frac{p(t)}{t} = \lim_{t\to 0} \frac{p'(t)}{1} = \lim_{t\to 0} \frac{1}{\sqrt{1+3t}} = 1.$$

4.11.

a. $A'(t) = -15/(2 + .3t)^2 + 0.125e^{0.125t}$, so $A'(0) = -3.625$. Also, $A(0) = 35.0$ and $A(120) = 46.13$. The graph thus begins decreasing and then increases. The maximum is at $t = 120$.



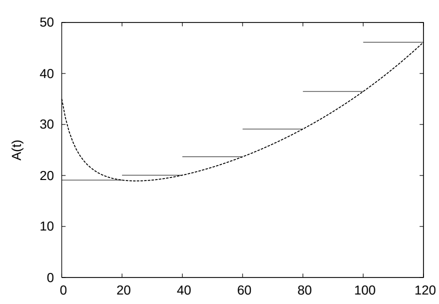
b.

$$\int_0^{120} A(t)dt = \frac{50}{0.3} \ln(2 + 0.3t) + 800e^{0.0125t}|_0^{120} = 3276.$$

c. The average is $3276/120 = 27.3$.

d. The minimum value must be less than 27.3.

e. RHE$=\sum_{i=1}^{6} A(20i)20$. I bet the estimate is high.



4.12.

a. Because distances are given in millimeters, we must convert densities into moles per millimeter, so $s(x) = 0.0012/(1.0 + 0.2x)$. Integrating to find the total, we get

$$\begin{aligned} \text{total} &= \int_0^{20} \frac{0.0012}{1.0 + 0.2x} dx \\ &= 0.006 \ln(1.0 + 0.2x)|_0^{20} = 0.006(\ln(5) - \ln(1)) = 0.00966. \end{aligned}$$

b. The average density is $0.00966/20 = 0.00048$ mol/mm.

c. The density decreases along the tongue from a maximum of 0.0012 at $x = 0$ to a minimum of 0.00024 at $x = 20$. The average lies between these values, as it must.

4.13.

a. Total=$\int_0^2 G(h)dh = 5h - 1.5e^{-2h}|_0^2 = 11.4$ mol.

b. Average=11.4 mol/2 m=5.7 mol/m.

c. Exact amount between 1.0 and 1.01 is $\int_{1.0}^{1.01} G(h)dh5h - 1.5e^{-2h}|_{1.0}^{1.01} = 0.0540$. The approximate amount is $G(1.0) \cdot 0.01(5 + 3e^{-2.0}) \cdot 0.01 = 0.0541$.