Modeling the Dynamics of Life: Calculus and Probability for Life Scientists

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2.11 Supplementary Problems for Chapter 2

♠ Find the limits of the following functions or say why you can't.

• EXERCISE **2.1**

$\lim_{x\to 0} 1+x^2$.

• EXERCISE **2.2**

$\lim_{x\to -2} 1+x^2$.

• EXERCISE **2.3**

$\lim_{x\to 1} \dfrac{1}{x^2-1}$.

• EXERCISE **2.4**

$\lim_{x\to 0} \dfrac{e^x}{1+e^{2x}}$.

• EXERCISE **2.5**

$\lim_{x\to 1^+} \dfrac{1}{x-1}$.

• EXERCISE **2.6**

$\lim_{x\to 1^-} 1/(x-1)$.

♠ Find the derivatives of the following functions. Note any points where the derivative does not exist.

• EXERCISE **2.7**

$F(y) = y^4 + 5y^2 - 1$.

• EXERCISE **2.8**

$a(x) = 4x^7 + 7x^4 - 28$.

• EXERCISE **2.9**

$H(c) = \dfrac{c^2}{1+2c}$.

• EXERCISE **2.10**

$h(z) = \dfrac{z}{1+\ln(z^2)}$ for $z \geq 0$.

• EXERCISE **2.11**

For $y \geq 0$, $b(y) = \frac{1}{y^{0.75}}$.

• EXERCISE **2.12**

For $z \geq 0$, $c(z) = \frac{z}{(1+z)(2+z)}$.

♠ Find the derivatives of the following functions

• EXERCISE **2.13**

$g(x) = (4+5x^2)^6$.

• EXERCISE **2.14**

$c(x) = (1+\frac{2}{x})^5$.

• EXERCISE **2.15**

$s(t) = \ln(2t^3)$.

• EXERCISE **2.16**

$p(t) = t^2 e^{2t}$.

• EXERCISE **2.17**

$s(x) = e^{-3x+1} + 5\ln(3x)$.

• EXERCISE **2.18**

$g(y) = e^{3y^3+2y^2+y}$.

♠ Find the derivatives and other requested quantities for the following functions.

• EXERCISE **2.19**

$f(t) = e^t \cos(t)$. Find one critical point.

• EXERCISE **2.20**

$g(x) = \ln(1+x^2)$. Find one point where $g(x)$ is decreasing.

• EXERCISE **2.21**
$h(y) = \frac{1-y}{(1+y)^3}$. Find all values where h is increasing.
• EXERCISE **2.22**
$c(z) = \frac{e^{2z}-1}{z}$. What is $c(0)$? (You need to take the limit).

♠ Find all critical points and points of inflection of the following. Sketch graphs.

• EXERCISE **2.23**
$f(x) = e^{-x^3}$.
• EXERCISE **2.24**
$g(x) = e^{-x^4}$.
• EXERCISE **2.25**
$h(y) = \cos(y) + \frac{y}{2}$.
• EXERCISE **2.26**
$F(c) = e^{-2c} + e^c$.

♠ Solve the following.

• EXERCISE **2.27**
A population has size

$$N(t) = 1000 + 10t^2.$$

where t is measured in years.

a. What units should follow the 1000 and the 10?
b. Graph the population between $t = 0$ and $t = 10$.
c. Find the population after 7 yr and 8 yr. Find the approximate growth rate between these two times. Where does this approximate growth rate appear on your graph?
d. Find and graph the derivative $N'(t)$.
e. Find the per capita rate of growth. Is it increasing?

• EXERCISE **2.28**
Suppose the volume of a cell is described by the function $V(t) = 2 - 2t + t^2$ where t is measured in minutes and V is measured in 1000's of μm^3.

a. Graph the secant line from time $t = 2$ to $t = 2.5$.
b. Find the equation of this line.
c. Find the value of the derivative at $t = 2$, and express it in both differential and prime notation. Don't forget the units.
d. What is happening to cell volume at time $t = 2$?
e. Graph the derivative of V as a function of time.

• EXERCISE **2.29**
Suppose a machine is invented to measure the amount of knowledge in a student's head in units called "factoids". One student is measured at $F(t) = t^3 - 6t^2 + 9t$ factoids at time t, where t is measured in weeks.

a. Find the rate at which the student is gaining (or losing) knowledge as a function of time (be sure to give the units).
b. During what time between $t = 0$ and 11 is the student losing knowledge?
c. Sketch a graph of the function $F(t)$.

• EXERCISE **2.30**
The following measurements are made of a plant's height in centimeters and its rate of growth.

Day	Height	Growth rate
1	8.0	5.2
2	15.0	9.4
3	28.0	17.8
4	53.0	34.6

a. Graph these data. Make sure to give units.
b. Write the equation of a secant line connecting two of these data points and use it to guess the height on day 5. Why did you pick the points you did?
c. Write the equation of a tangent line and use it to guess the height on day 5.
d. Which guess do you think is better?
e. How do you think the "growth rate" might have been measured?

• EXERCISE **2.31**
Suppose the fraction of mutants in a population follows the discrete-time dynamical system

$$p_{t+1} = \frac{2.0p_t}{2.0p_t + 1.5(1 - p_t)}.$$

Suppose you wish the fraction at time $t = 1$ to be close to $p_1 = 0.4$.

a. What would p_0 have to be to hit 0.4 exactly?
b. How close would p_0 have to be to this value to produce p_1 within 0.1 of the target?
c. What happens to the input tolerance as the output tolerance becomes smaller (as p_1 is required to be closer and closer to 0.4)?

• EXERCISE **2.32**
Suppose the position of an object attached to a spring is

$$x(t) = 2.0 \cos\left(\frac{2\pi t}{3.2}\right).$$

a. Find the derivative of $x(t)$. What does it mean physically?
b. Find the second derivative of $x(t)$. What does this mean physically?
c. What is the position and velocity at $t = 0$?
d. What is the position and velocity at $t = 1.6$?
e. What is the position and velocity at $t = 3.2$?
f. What differential equation does this object follow?

• EXERCISE **2.33**
Suppose a system exhibits hysteresis. While increasing the temperature T, the voltage response follows

$$V_i(T) = \begin{cases} T & \text{if } T \leq 50 \\ 100 & \text{if } 50 < T \leq 100 \end{cases}$$

While decreasing the temperature, however, the voltage response follows

$$V_d(T) = \begin{cases} T & \text{if } T \leq 20 \\ 100 & \text{if } 20 < T \leq 100 \end{cases}$$

a. Graph these functions.
b. Find the left and right hand limits of V_i and V_d as T approaches 50.
c. Find the left and right hand limits of V_i and V_d as T approaches 20.

• EXERCISE **2.34**
Suppose the total product generated by a chemical reaction is

$$P(t) = \frac{t}{1 + 2t}$$

where t is measured in hours.

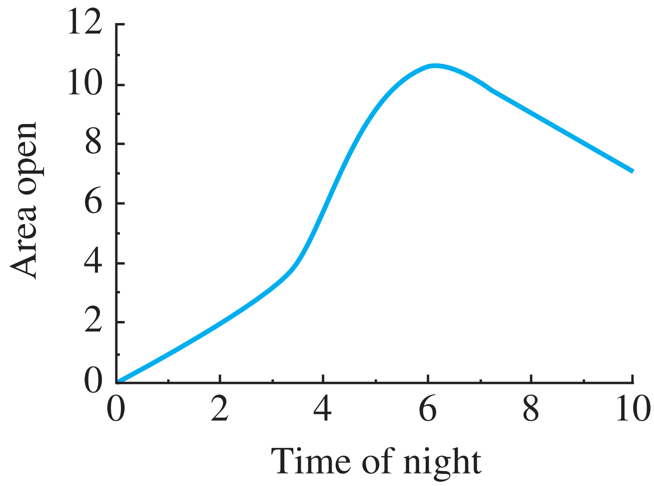
a. Find the average rate of change between $t = 1$ and $t = 2$.
b. Find the equation of the secant line between these times.

c. Graph the secant line and the function $P(t)$.
d. Sketch a graph of the tangent line at $t = 2$. Based on your graph, is the slope of the secant larger, smaller, or the same as the slope of the tangent?
e. Write the limit you would take to find the instantaneous rate of change at $t = 2$.

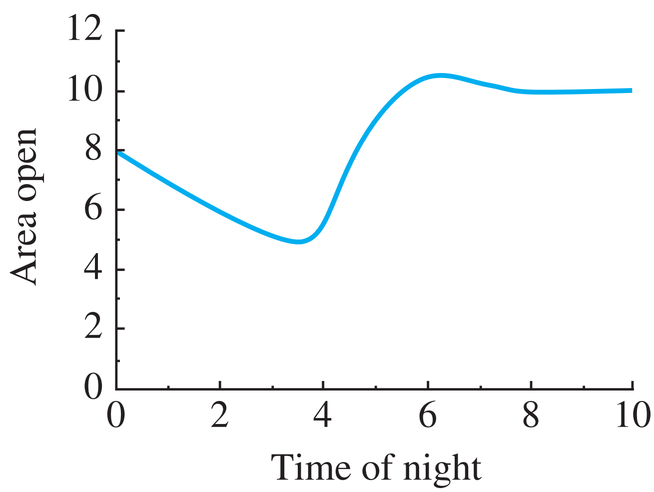
♠ In each of the following cases, suppose a window is described by the graph. Time is measured in time after going to sleep at 10:00 p.m., and the area open is in square feet. In each case,

a. Label a point where the first derivative is positive and the second derivative is negative,
b. Label a point of inflection,
c. Sketch a graph of the rate of change,
d. Sketch a graph of the second derivative.

• EXERCISE **2.35**



• EXERCISE **2.36**



Chapter 9

Answers

2.1. We can plug in because the function is continuous. The limit is 1.
2.2. We can plug in and the limit is 5.
2.3. This increases to infinity.
2.4. The limit is 0.5.
2.5. This increases to infinity.
2.6. This decreases to negative infinity.
2.7. This is a polynomial, so we can use the sum and power rules to find $F'(y) = 4y^3 + 10y$.
2.8. This is a polynomial, so we can use the sum and power rules to find $a'(x) = 28x^6 + 28x^3$.
2.9. This is a quotient of $u(c) = c^2$ and $v(c) = 1 + 2c$ with derivatives $u'(c) = 2c$ and $v'(c) = 2$. By the quotient rule $H'(c) = \frac{(1+2c)2c - 2c^2}{(1+2c)^2} = \frac{2c(1+c)}{(1+2c)^2}$. This is only defined when $c \neq -1/2$.
2.10. This is a quotient of $u(z) = z$ and $v(z) = 1 + \ln(z^2) = 1 + 2\ln(z)$ (from the laws of logs) with derivatives $u'(z) = 1$ and $v'(z) = 2/z$. By the quotient rule

$$h'(z) = \frac{1 + 2\ln(z) - z \cdot 2/z}{(1 + \ln(z^2))^2} = \frac{2\ln(z) - 1}{(1 + \ln(z^2))^2}.$$

2.11. This is a power function with power $p = -0.75$, so $b'(y) = -0.75y^{-1.75}$ for $y \geq 0$.
2.12. This is a quotient of $u(z) = z$ and $v(z) = (1 + z)(2 + z)$. The derivatives are $u'(z) = 1$ and $v'(z) = 3 + 2z$ (using the product rule or multiplying out). Then

$$c'(z) = \frac{(1+z)(2+z) - z(3+2z)}{(1+z)^2(2+z)^2} = \frac{2 - z^2}{(1+z)^2(2+z)^2}$$

2.13. This is a composition $g(x) = f(h(x))$ where $h(x) = 4 + 5x^2$ and $f(h) = h^6$ with derivatives $h'(x) = 10x$ and $f'(h) = 6h^5$. Therefore, $g'(x) = 10x \cdot 6h(x)^5 = 60x(4 + 5x^2)^5$.
2.14. This is a composition $c(x) = f(g(x))$ where $g(x) = 1 + 2/x$ and $f(g) = g^5$ with derivatives $g'(4) = -2/x^2$ and $f'(g) = 5g^4$. Therefore, $c'(x) = -10(1 + \frac{2}{x})^4/x^2$.
2.15. Expanding with the laws of logs gives $s(t) = \ln(2) + 3\ln(t)$ with derivative $s'(t) = 3/t$.
2.16. This is a product of $f(t) = t^2$ and $g(t) = e^{2t}$. Because $f'(t) = 2t$ and $g'(t) = 2e^{2t}$, we have that $p'(t) = f'(t)g(t) + g'(t)f(t) = (2t^2 + 2t)e^{2t}$.

2.17. This can be broken down with the sum rule. The first term is the composition $f(g(x))$ with $g(x) = -3x + 1$ and $f(g) = e^g$ with derivatives $g'(x) = -3$ and $f'(g) = e^g$. Therefore

$$\frac{d}{dx}e^{-3x+1} = -3e^g = -3e^{-3x+1}.$$

The second term can be rewritten as $5\ln(3x) = 5\ln(3) + 5\ln(x)$ with derivative $5/x$. Therefore $s'(x) = -3e^{-3x+1} + 5/x$.

2.18. This is the composition $F(G(y))$ with $G(y) = 3y^3 + 2y^2 + y$ and $F(G) = e^G$ with derivatives $G'(y) = 9y^2 + 4y + 1$ and $F'(G) = e^G$. Therefore $g'(y) = F'(G)G'(y) = (9y^2 + 4y + 1)e^{3y^3+2y^2+y}$.

2.19. This function is the production of $u(t) = e^t$ and $v(t) = \cos(t)$. Then $u'(t) = e^t$ and $v'(t) = -sin(t)$, so $f'(t) = u'(t)v(t) + v'(t)u(t) = (\cos(t) - \sin(t))e^t$. To find a critical point we must solve $f'(t) = 0$, which occurs when $\cos(t) - \sin(t) = 0$ or $\cos(t) = \sin(t)$. Because $\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$, the point $t = \pi/4$ is a critical point.

2.20. This is the composition $f(h(x))$ with $f(h) = \ln(h)$ and $h(x) = 1 + x^2$. Because $f'(h) = 1/h$ and $h'(x) = 2x$, we have that $g'(x) = f'(h)h'(x) = \frac{2x}{1+x^2}$. $g(x)$ is decreasing when this derivative is negative. Because the denominator $1 + x^2$ is always positive, this occurs when $x < 0$.

2.21. This is the quotient of $u(y) = 1 - y$ and $v(y) = (1 + y)^3$ with derivatives $u'(y) = -1$ and $v'(y) = 3(1 + y)^2$. By the quotient rule,

$$h'(y) = \frac{-(1+y)^3 - 3(1-y)(1+y)^3}{(1+y)^6} = \frac{2(y-2)}{(1+y)^4}.$$

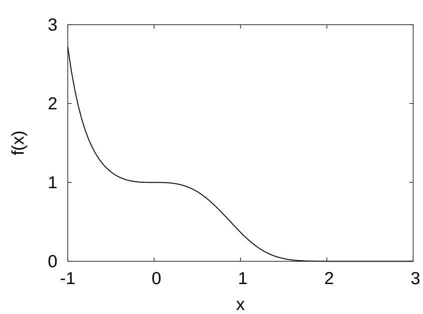
$h(y)$ is increasing this is positive. Because the denominator is always positive, this occurs when the numerator is positive, or $y > 2$.

2.22. This is a quotient of $u(z) = e^{2x} - 1$ and $v(z) = z$ with derivatives $u'(z) = 2e^{2x}$ and $v'(z) = 1$. Therefore,

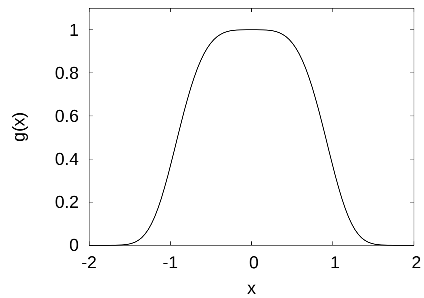
$$c'(z) = \frac{e^{2z} - 1 - 2ze^{2z}}{z^2}.$$

$c(z)$ is the slope of the secant line to the function $f(z) = e^{2z}$ connecting the point $(0, 1)$ to the point $(z, f(z)$. The limit is $f'(0) = 2e^{2\cdot 0} = 2$. This could also be found with L'Hopital's rule.

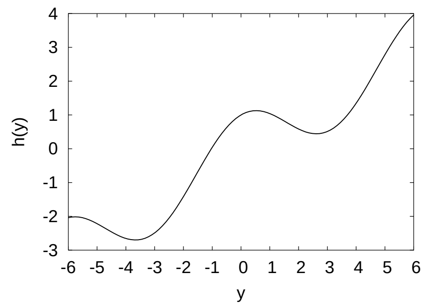
2.23. $f'(x) = -3x^2e^{-x^3}$ which is 0 at $x = 0$. $f''(x) = (9x^4 - 6x)e^{-x^3}$ which is 0 at $x = 0$ and $x = 0.873$.



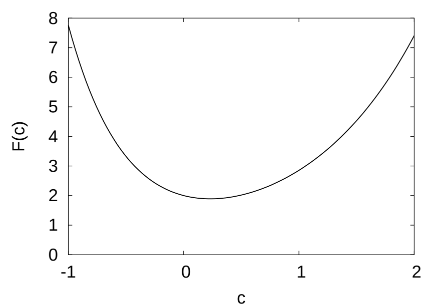
2.24. $g'(x) = -4x^3e^{-x^4}$ which is 0 at $x = 0$. $g''(x) = (16x^6 - 12x^2)e^{-x^4}$ which is 0 at $x = 0$ and $x = \pm 0.931$.



2.25. $h'(y) = -\sin(y) + \frac{1}{2}$ which is 0 at $y = \frac{\pi}{6} + 2n\pi$ and $y = \frac{t\pi}{6} + 2n\pi$ for any integer n. $h''(y) = -\cos(y)$, which is 0 at $y = \frac{\pi}{2} + n\pi$ for any integer n.



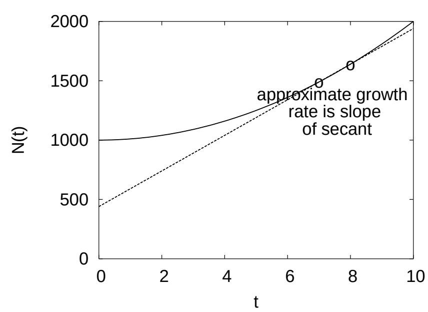
2.26. $F'(c) = -2e^{-2c} + e^c$ which is 0 where $c = \ln(2)/3$. $F''(c) = 4e^{-2c} + e^c$ which is always positive. There are no points of inflection.



2.27.

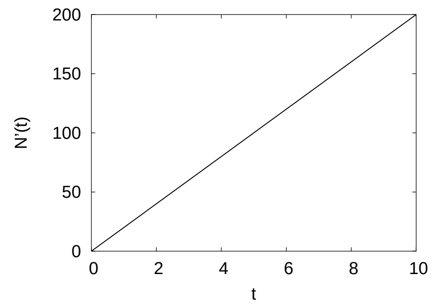
a. The 1000 has units of individuals, and the 10 has units of individuals/yr^2.

b.



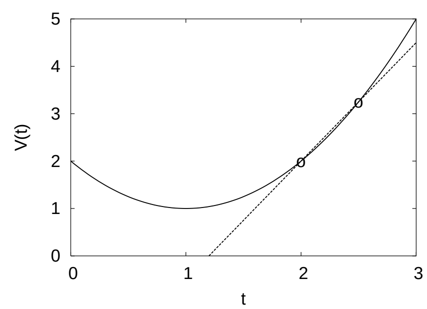
c. The population is 1490 after 7 yr and 1640 after 8 yr. The approximate growth rate is 150 during this year.

d. The derivative is $20t$.



e. The per capita rate of growth is $\frac{20t}{(1000 + 10t^2)}$. It starts at 0 and increases for a while. Eventually, however, it will decrease.

2.28.

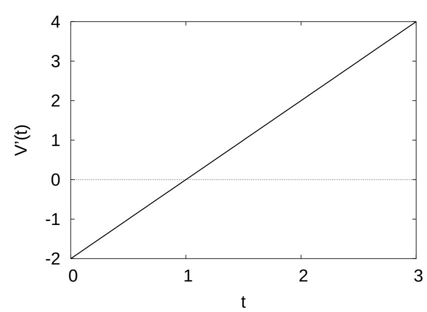


a.

b. The secant line has slope $(V(2.5) - V(2))/0.5 = 2.5$. Because $V(2) = 2$, the equation is $V_s(t) = 2.5(t - 2) + 2$.

c. $V'(t) = 2t - 2$, so $V'(2) = 2$ in units of $\frac{\mu\text{m}}{\text{min}}$.

d. At $t = 2$, cell volume is increasing by $2\frac{\mu\text{m}}{\text{min}}$.

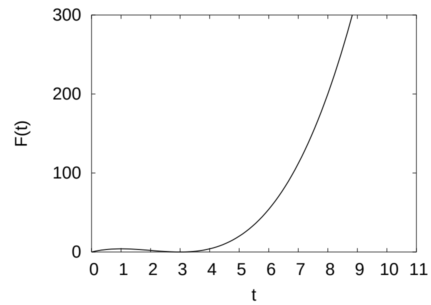


e.

2.29.

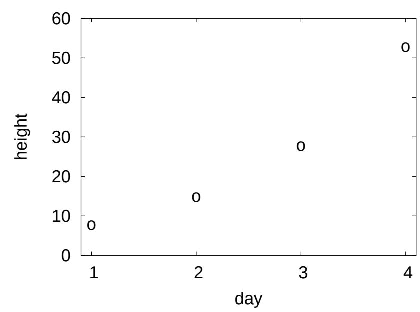
a. $F'(t) = 3t^2 - 12t + 9$ is the derivative, with units of factoids per week.

b. With the quadratic formula, we find that $F'(t) = 0$ when $t = 1$ or $t = 3$. Because $F'(t)$ is a parabola with goes up, it must be negative between times 1 and 3. This is the time when knowledge is being lost.



c.

2.30.



a.

- **b.** I picked days 3 and 4 because they are closest to day 5. The second line has equation $\hat{h}(t) = 28.0 + 25.0(t - 3)$, and $\hat{h}(5) = 78.0$.
- **c.** On day 4, the tangent line is $\hat{h}(t) = 53.0 + 34.6(t - 4)$, and $\hat{h}(5) = 77.6$.
- **d.** Probably the tangent line because it uses the latest information.
- **e.** Maybe they went out and measured the change in plant height over one hour.

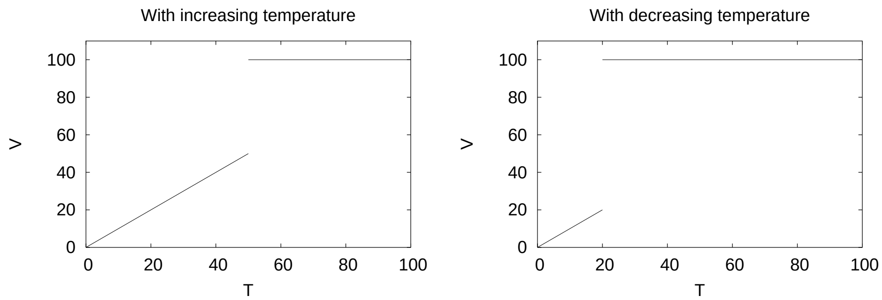
2.31.

- **a.** $p_0 = 0.333$.
- **b.** p_0 would have to be between 0.243 and 0.428, or within about 0.1.
- **c.** It would get smaller and smaller also.

2.32.

- **a.** $x'(t) = -1.25\pi \sin(\frac{2\pi t}{3.2})$. This is the velocity $v(t)$.
- **b.** $x''(t) = -0.78\pi^2 \cos(\frac{2\pi t}{3.2})$. This is the acceleration $a(t)$.
- **c.** At $t = 0$, the position is $x = 2$ and the velocity is 0.
- **d.** At $t = 1$, the position is $x(1) = -0.765$ and the velocity is $v(1) = -3.628$.
- **e.** At $t = 2$, the position is $x(2) = -1.414$ and the velocity is $v(2) = 2.777$.
- **f.** $\dfrac{d^2x}{dt^2} = -3.855x(t)$.

2.33.



a.

b. $\lim_{T\to 50^+} V_i(T) = 100$. $\lim_{T\to 50^-} V_i(T) = 50$. $\lim_{T\to 50^+} V_d(T) = \lim_{T\to 50^-} V_d(T) = 100$.

c. $\lim_{T\to 20^+} V_i(T) = \lim_{T\to 20^-} V_i(T) = 20$. $\lim_{T\to 20^+} V_d(T) = 100$. $\lim_{T\to 20^-} V_d(T) = 20$.

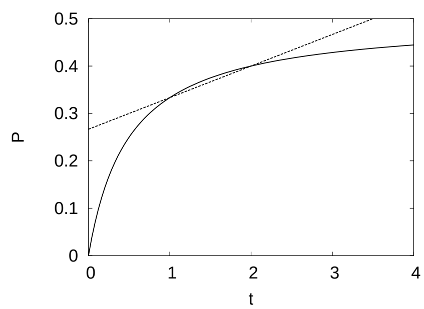
2.34.

a. The average rate of change is

$$\frac{\Delta P}{\Delta t} = \frac{P(2) - P(1)}{2-1} = \frac{2/5 - 1/3}{1} = 1/15.$$

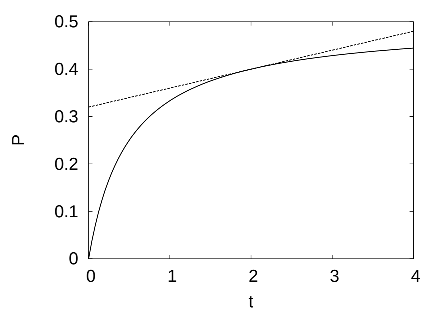
b. The secant line passes through the point $(1, 1/3)$ with slope $1/15$, and has equation

$$P_s(t) = \frac{1}{3} + \frac{1}{15}(t-1).$$



c.

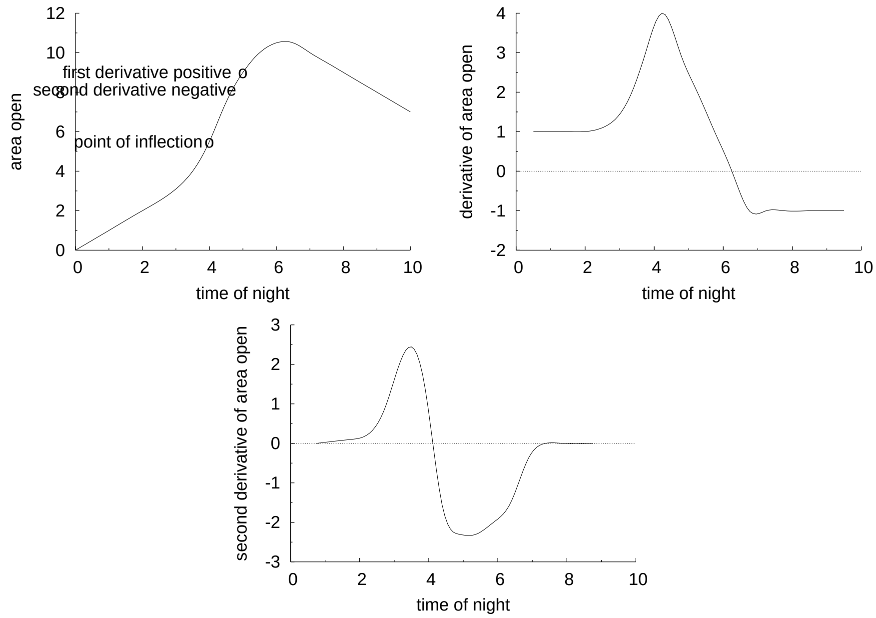
d. I'd say it is less steep because the curve gets less and less steep for larger values of t.



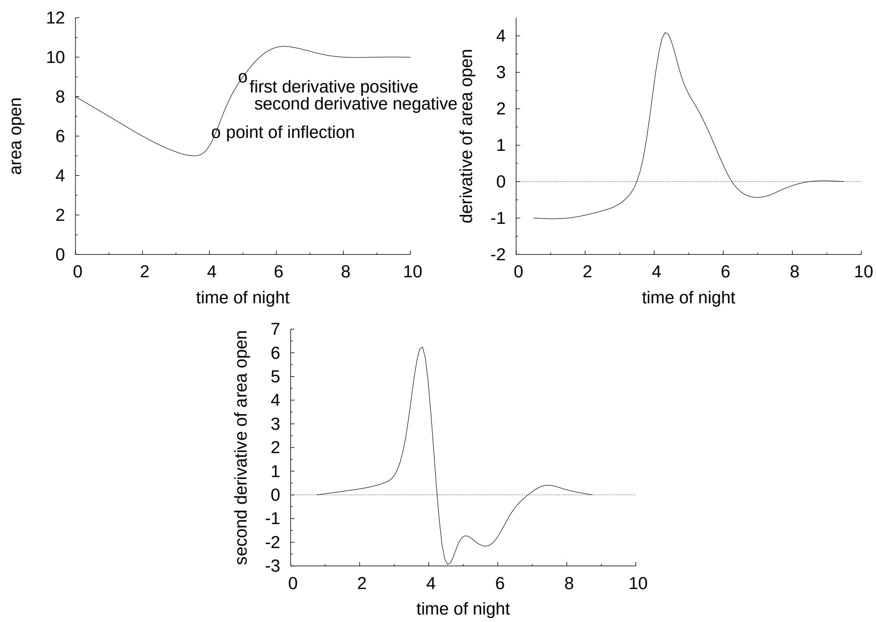
e.

$$\lim_{\Delta t \to 0} \frac{P(2+\Delta t) - P(2)}{\Delta t}.$$

2.35.


first derivative positive
second derivative negative
point of inflection
area open
time of night
derivative of area open
second derivative of area open

2.36.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