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MATH 1170: The Final

Do all five problems, points as indicated. Write readable answers on the test, but feel free to use or hand in additional paper if necessary. If you do, **make sure I can find both the answer and how you got it**. You can use five sides of notes, but no calculators, books or other aids. Extra credit indicated with stars. Don't spend too much time on any one problem!

1. (40 points)

a. Find the indefinite integral of $I(x) = 23e^{3-2x}$.

b. Find the derivative of $s(t) = \ln(1 + \cos(2\pi t)) + 53$.

c. Graph $c(x) = 2x^3 - 3x^2 + 5$ for $-1 \geq x \geq 2$, identifying critical points, points of inflection and the global maximum.

d. Use the first two terms of the Taylor series for e^x to estimate $e^{0.7}$ (the exact answer is 2.014). Why is it so close to 2?

e. *If you are assigned homework in a math class, you should _ _ _ _.

a. $\int 23e^{3-2x}dx$

Let $u = 3-2x$

$\frac{du}{dx} = -2$

$\Rightarrow dx = -\frac{1}{2}du$

$= \int -\frac{23}{2}e^u du = -\frac{23}{2}e^u + C$

$= -\frac{23}{2}e^{3-2x} + C$

b. $s(t) = f(g(h(t))) + 53$ where $h(t) = 2\pi t$ $\quad h'(t) = 2\pi$

$g(h) = 1+\cos(h) \quad g'(h) = -\sin(h)$

Derivative of 53 is zero, so

$f(g) = \ln(g) \quad f'(g) = \frac{1}{g}$

$s'(t) = f'(g)\cdot g'(h)\cdot h'(t) = -\frac{1}{g}\cdot\sin(h)\cdot 2\pi = \frac{-1}{1+\cos(2\pi t)}\cdot\sin(2\pi t)\cdot 2\pi = -2\pi\frac{\sin(2\pi t)}{1+\cos(2\pi t)}$

c. $c'(x) = 6x^2 - 6x$ $\quad$ Critical points where $c'(x) = 0$ or $6x(x-1) = 0$, or $x = 0$, $x = 1$

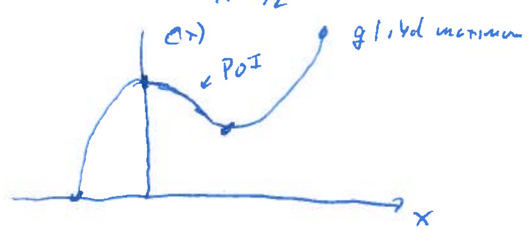
$c''(x) = 12x - 6$ $\quad$ Point of inflection when $12x - 6 = 0$ or $x = \frac{1}{2}$

At four values: $c(-1) = -2 - 3 + 5 = 0$

$c(0) = 5$

$c(1) = 4$

$c(2) = 2\cdot 2^3 - 3\cdot 2^2 + 5 = 9$



d. Let $f(x) = e^x$, then $f'(x) = f''(x) = e^x$. Taylor polynomial at $x = 0$ is $f_2(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2}x^2 = 1 + x + \frac{x^2}{2}$

Then $f(0) = f'(0) = f''(0) = 1$

$f_2(0.7) = 1 + 0.7 + \frac{0.7^2}{2} = 1 + 0.7 + \frac{0.49}{2} = 1.7 + 0.245 = 1.945$

Close to 2 because 0.7 is close to $\ln(2)$

e. do it

2. (40 points) The velocity of traffic on the highway to Mesquite, Nevada after the holidays obeys

$$v(t) = \frac{75}{1 + 0.5t}$$

where t is measured in hours and $v(t)$ in miles per hour. The distance to Mesquite is, strangely, $150\ln(10) \approx 345$ miles.

a. Graph velocity as a function of time.

b. Find distance traveled as a function of time since starting.

c. Would a car ever get to Mesquite at this rate? If so, how long would it take?

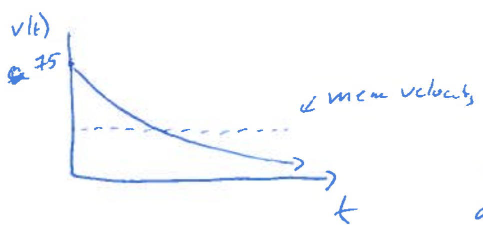
d. Graph position as a function of time. Indicate on your graph the value guaranteed by the Mean Value Theorem (you don't have to solve for this value).

don't worry about this

e. Indicate the value guaranteed by the Mean Value Theorem on your graph of the velocity. What does it correspond to?

f. *Who was the greatest mathematician of the ancient world?

a.



$u = 1+0.5t$

$du = 0.5\,dt$

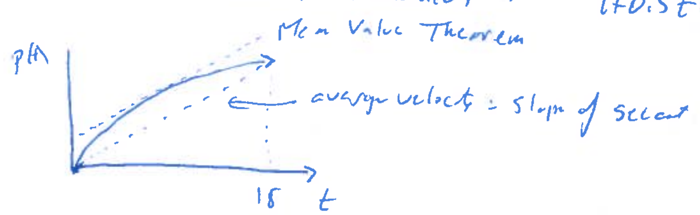
$dt = 2\,du$

b. $\frac{dp}{dt} = v(t) \Rightarrow p(t) = \int v(t)\,dt = \int \frac{75}{1+0.5t}\,dt = \int \frac{150}{u}\,du = 150\ln(u) = 150\ln(1+0.5t)$

Distance travelled is $p(t) - p(0) = 150\ln(1+0.5t)$

c. Solve $150\ln(1+0.5t) = 150\ln(10) \Rightarrow$ $1+0.5t = 10 \Rightarrow 0.5t = 9 \Rightarrow t = 18$ hours

d.



e. This is the mean velocity $= \frac{345\text{m}}{18\text{h}} \approx 18$mph.

3. (40 points) When faced by slow traffic, many people turn to bifwaze, a smartphone app that guides people onto less traveled routes, in particular, to Bif's cutoff (which goes past Bif's truck stop). Driving this route takes one hour under ideal circumstances. If the number of cars driving on Bif's cutoff in hour t is B_t, then

$$B_{t+1} = \begin{cases} 1000 - \frac{B_t^2}{500} & \text{if } 1000 - \frac{B_t^2}{500} > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Show (without using any algebra) that this discrete-time dynamical system has exactly one positive equilibrium.

b. Show that this equilibrium is $B^* = 500$.

c. Find the stability of the equilibrium using the stability theorem.

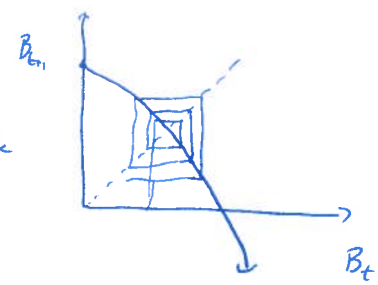
d. Sketch a cobweb diagram starting from $B_0 = 450$.

e. Graph the solution, and explain in words what it means.

f. *Name one animal that was **not** used as an example in class.

a. Let $f(B) = 1000 - \frac{B^2}{500}$ $f(0) = 1000$, and $f'(B) = \frac{-2B}{500} < 0$

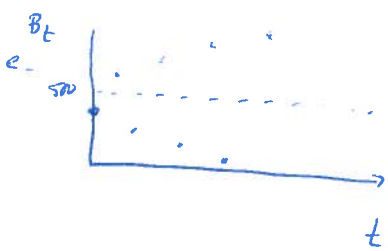
This decreasing function must cross the diagonal exactly once.



b. $f(500) = 1000 - \frac{500^2}{500} = 1000 - 500 = 500$ ✓

c. $f'(500) = \frac{-2 \cdot 500}{500} = -2 < -1$ Unstable

e.



The number of cars alternates between low values and high values

4. (40 points) You decide to serve 100 kg of jello to your friends in Mesquite at an enormous New Year's Eve bash. Jello costs \$1/kg in Salt Lake City and \$3/kg in Mesquite. The cost of driving is $0.5e^{0.007J}$ dollars per mile, where J is the weight of jello in your vehicle. Your goal is to minimize the combined cost of driving and of jello. Suppose the drive is 400 miles.

a. What would jello and driving cost if you bought all of the jello in Mesquite?

b. What would jello and driving cost if you bought all of the jello in Salt Lake City? Use problem 1d to help find a good approximate answer (feel free to use a simple nearby integer).

c. Find the cost if you buy J kg of the jello in Salt Lake City. Call this function $C(J)$.

d. Find the derivative of $C(J)$ at $J = 0$ and estimate it at $J = 100$ (use the same number you used in part **b**). What does this tell you about the minimum?

e. Find the value of J that minimizes the cost (again, don't find an exact number).

f. Sketch a graph of cost as a function of J.

g. *What do troglodytes have to teach us about calculus?

a. let $D(J)$ = Cost of driving as a fcn of jello in car, where J = weight of jello bought in Salt Lake

let $P(J)$ = \$ cost of jello $= \underbrace{1 \cdot J}_{\text{Salt Lake}} + \underbrace{3(100 - J)}_{\text{Mesquite}} = 300 - 2J$

If $J = 0$, then $C(J) = D(J) + P(J) = D(0) + P(0) = 400 \cdot 0.5e^{0.007 \cdot 0} + (300 - 2 \cdot 0) =$

$= 200 + 300 = \$500$

b. $C(100) = D(100) + P(100) = 400 \cdot 0.5e^{0.007 \cdot 100} + (300 - 2 \cdot 100) = 200e^{0.7} + 100 \approx \$500$

c. $C(J) = 200e^{0.007 \cdot J} + (300 - 2J)$

d. $C'(J) = 1.40e^{0.007 \cdot J} - 2$ $\quad C'(0) = 1.4e^{0} - 2 = -0.6 < 0$ decreasing at $J = 0$

$C'(100) = 1.4e^{0.7} - 2 \approx 2.8 - 2 = 0.8 > 0$ increasing at $J = 100$

Must be a minimum in between

e. Solve $C'(J) = 0$

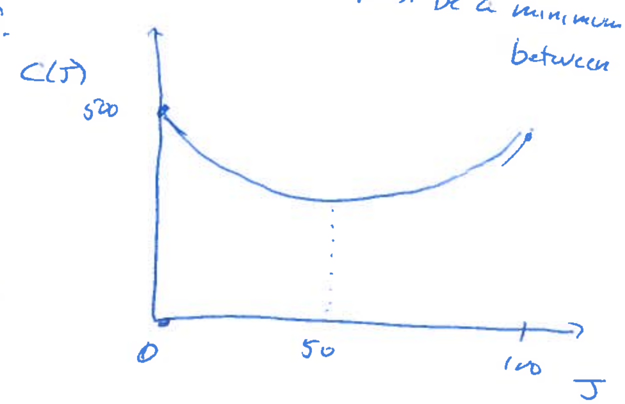
$1.4e^{0.007J} = 2$

$e^{0.007J} = 2/1.4 \approx 1.4 \approx \sqrt{2}$

$0.007J = \ln(\sqrt{2}) = \frac{1}{2}\ln(2) \approx 0.35$

$J = \frac{0.35}{0.007} = \frac{350}{7} = 50$

f.



Out of curiosity: $C(50) = 200e^{0.007 \cdot 50} + (300 - 2 \cdot 50) \approx 280 + 200 = \$480$

5. (40 points) Traffic density, in cars per mile, is

$$\rho(x) = 20 + 6\sqrt{x}$$

where x is measured in miles from Salt Lake City, $0 \leq x \leq 400$. Parts **d** and **e** are a bit harder, so save them for last if you have time.

a. Find the total number of cars on the road. Remember that $\sqrt{400} = 20$.

b. Find the average traffic density.

c. Write (using numbers) the expression you would evaluate to estimate the total number of cars using the left hand estimate with $n = 5$. Where are the break points?

d. Suppose traffic speed, in miles per hour, decreases according to $80 - 0.5\rho$ as a function of density. At what rate will cars pass a given point as a function of the density?

e. Where along the road will cars pass at the maximum rate?

f. *What do 3-year-olds have to teach us about applied mathematics?

g. *Simplify "Merry $\sqrt{2}$mas" where $X^2 - 2 = 0$.

a. $T = \int_0^{400} 20 + 6\sqrt{x}\, dx = 20x + 4x^{3/2}\Big|_0^{400} = 20\cdot 400 + 4\cdot 400^{3/2} = 20\cdot 400 + 4\cdot 8000 = 40{,}000$

b. $\text{Average} = \frac{T}{400} = \frac{40000}{400} = 100$

c. Using $n=5$, $x_0 = 0,\ x_1 = 80,\ x_2 = 160,\ x_3 = 240,\ x_4 = 320,\ x_5 = 400$

$\Delta x = \frac{400}{5} = 80$

$LHE = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$= (20 + 6\sqrt{0})\cdot 80 + (20 + 6\sqrt{80})\cdot 80 + (20 + 6\sqrt{160})80 + (20 + 6\sqrt{240})80 + (20 + 6\sqrt{320})80$

d. $v(\rho) = 80 - 0.5\rho$

Rate at which cars pass is velocity $\times$ density $= (80 - 0.5\rho)\cdot\rho = R(\rho)$

$= (80 - 0.5(20 + 6\sqrt{x}))\cdot(20 + 6\sqrt{x})$

e. Maximize $R(\rho)$: $\frac{dR}{d\rho} = 80 - \rho \Rightarrow \rho = 80$ $\qquad \rho = 80 = 20 + 6\sqrt{x}$

$\Rightarrow 60 = 6\sqrt{x}$

$\Rightarrow 10 = \sqrt{x}$

$\boxed{x = 100}$