

Modeling the Dynamics of Life: Calculus and Probability for Life Scientists

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4.8 Supplementary Problems for Chapter 4

• EXERCISE 4.1

The voltage v of a neuron follows the differential equation

$$\frac{dv}{dt} = 1.0 + \frac{1}{1 + 0.02t} - e^{0.01t}$$

over the course of 100 ms, where t is measured in ms and v in millivolts. We start at $v(0) = -70$.

- Sketch a graph of the rate of change. Indicate on your graph the times when the voltage reaches minima and maxima (you don't need to solve for the numerical values).
- Sketch a graph of the voltage as a function of time.
- What is the voltage after 100 ms?

• EXERCISE 4.2

Consider again the differential equation in the previous problem,

$$\frac{dv}{dt} = 1.0 + \frac{1}{1 + 0.02t} - e^{0.01t}$$

with $v(0) = -70$.

- Use Euler's method to estimate the voltage after 1 millisecond, and again 1 millisecond after that.
- Estimate the voltage after 2 ms using left hand and right hand Riemann sums.
- Which of your estimates matches Euler's method and why?

• EXERCISE 4.3

A neuron in your brain sends a charge down an 80 centimeter long axon (a long skinny thing) toward your hand at a speed of 10 meters per second. At the time when the charge reaches your elbow, the voltage in the axon is -70 millivolts except on the 6 centimeter long piece between 47 and 53 centimeters from your brain. On this piece, the voltage is

$$v(x) = -70.0 + 10.0(9.0 - (x - 50.0)^2)$$

where $v(x)$ is the voltage at a distance of x centimeters from the brain.

- How long will it take the information to get to your hand? How long did it take to reach your elbow?
- Sketch a graph of the voltage along the whole axon.
- Find the average voltage of the 6 centimeter piece.
- Find the average voltage of the whole axon.

• EXERCISE 4.4

The charge in a dead neuron decays according to

$$\frac{dv}{dt} = \frac{1}{\sqrt{1 + 4t}} - \frac{2}{(1 + 4t)^{\frac{3}{2}}}$$

starting again from $v(0) = -70$ at $t = 0$.

- Is the voltage approaching 0 as $t \rightarrow \infty$? How do you know that it will eventually reach 0?
- Write an equation (but don't solve it) for the time when the voltage reaches 0.
- What is wrong with this model?

• EXERCISE 4.5

Consider the differential equations

$$\frac{db}{dt} = 2b$$

and

$$\frac{dB}{dt} = 1 + 2t.$$

- Which of these is a pure-time differential equation? Describe circumstances when you might find each of these equations.
- Suppose $b(0) = B(0) = 1$. Use Euler's method to find estimates for $b(0.1)$ and $B(0.1)$.
- Use Euler's method again to find estimates for $b(0.2)$ and $B(0.2)$.

• EXERCISE 4.6

Consider the differential equation

$$\frac{dp}{dt} = e^{-4t}$$

where $p(t)$ is product in moles at time t and t is measured in seconds.

- Explain in words what is going on.
- Suppose $p(0) = 1$. Find $p(1)$.

• EXERCISE 4.7

Consider the differential equation

$$\frac{dV}{dt} = 4 - t^2$$

where $V(t)$ is volume in liters at time t and t is measured in minutes.

- Explain in words what is going on.
- At what time is the volume a maximum?
- Break the interval from $t = 0$ to $t = 3$ into three parts and find the left hand and right hand estimates of the volume at $t = 3$ (assume $V(0) = 0$).
- Write down the definite integral expressing volume at $t = 3$ and evaluate.

• EXERCISE 4.8

Find the area under the curve $f(x) = 3 + (1 + x/3)^2$ between $x = 0$ and $x = 3$.

• EXERCISE 4.9

The population density of trout in a stream is

$$\rho(x) = |-x^2 + 5x + 50|$$

where ρ is measured in trout per mile and x is measured in miles. x runs from 0 to 20.

- Graph $\rho(x)$ and find the minimum and maximum.
- Find the total number of trout in the stream.
- Find the average density of trout in the stream.
- Indicate on your graph how you would find where the actual density is equal to the average density.

• EXERCISE 4.10

The amount of product is described by the differential equation

$$\frac{dp}{dt} = \frac{1}{\sqrt{1+3t}}$$

starting at time $t = 0$. Suppose p is measured in moles, t in hours, and that $p(0) = 0$.

- Find the limiting amount of product.
- Find the average rate at which product is produced as a function of time, and compute the limit as $t \rightarrow \infty$.
- Find the limit as $t \rightarrow 0$ of the average rate at which product is produced.

• EXERCISE 4.11

A student is hooked up to an EEG during a test, and her α brain wave power follows

$$A(t) = \frac{50}{2.0 + 0.3t} + 10e^{0.0125t}$$

where t runs from 0 to 120 minutes.

- a. Convince yourself that brain wave activity has a minimum value some time during the test. Sketch a graph of the function. Find the maximum.
- b. Find the total brain wave power during the test.
- c. Find the average brain wave power during the test. Sketch the corresponding line on your graph.
- d. Estimate the minimum value from your average value.
- e. Draw a graph showing how you would estimate the total using the right hand approximation with $n = 6$. Write down the associated sum. Do you think your estimate is high or low?

• EXERCISE 4.12

Consider the function $G(h)$ giving the density of nutrients in a plant stem as a function of the height h

$$G(h) = 5 + 3e^{-2h}$$

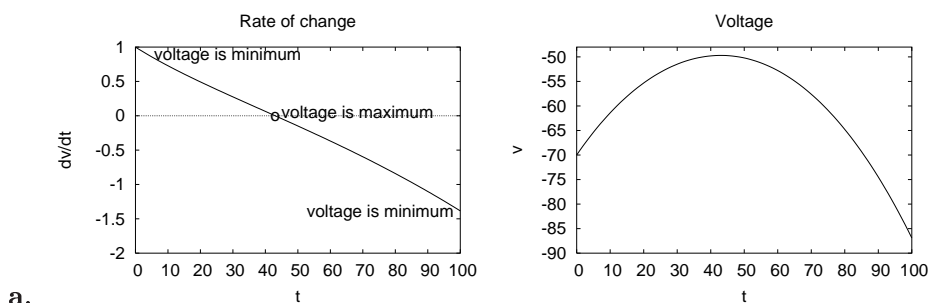
where G is measured in mol/m and h is measured in m.

- a. Find the total amount of nutrient if the stem is 2.0 m tall.
- b. Find the average density in the stem.
- c. Find the exact and approximate amount between 1.0 and 1.01 m.

Chapter 9

Answers

4.1.



a.

c. Solving with the indefinite integral, we get

$$v(t) = t + 50 \ln(1 + 0.02t) - 100e^{0.01t} + c.$$

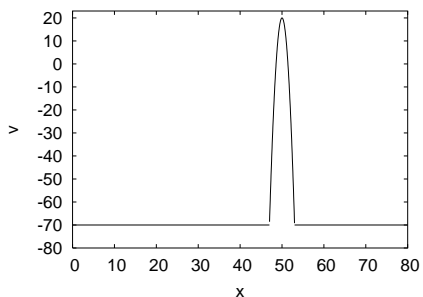
Substituting the initial condition, we have that $c = 30$. Therefore, $v(100) = 100 + 50 \ln(3.0) - 100e^1 + 30 \approx -86.9$.

4.2.

- $\hat{v}(1) = v(0) + v'(0) \cdot (1 - 0) = -70 + 1.0 = -69.0$. After another ms, we find $\hat{v}(2) = \hat{v}(1) + v'(1) \cdot (2 - 1) = -69.0 + 0.97 = -68.03$.
- The voltage after 2 ms is $-70 + \int_0^2 t + 50 \ln(1 + 0.02t) - 100e^{0.01t} dt$. Denoting the integrand by $f(t)$, the left hand Riemann sum is $-70 + f(0) + f(1) = -70 + 1.0 + 0.97 = -68.03$. The right hand Riemann sum is $-70 + f(1) + f(2) = -70 + 0.97 + 0.94 = -68.09$.
- The left hand estimate matches because it uses the same information as Euler's method.

4.3.

- At a speed of $10 \frac{\text{m}}{\text{s}} = 1000 \frac{\text{cm}}{\text{s}}$, it takes $t = 80\text{cm}/(1000\text{cm}/\text{sec}) = 0.08 \text{ sec}$ to reach your hand. Similarly, your elbow seems to be 50 cm from the brain, so it takes the signal 0.05 s to get there.



b.

c. average = $\frac{1}{6} \int_{47}^{53} -70.0 + 10.0(9.0 - (x - 50.0)^2) dx$. Substituting $y = x - 50$, we have that $dy = dx$ and limits of integration from $y = -3$ to $y = 3$, or

$$\begin{aligned} \text{average} &= \frac{1}{6} \int_{-3}^3 -70.0 + 10.0(9.0 - y^2) dy \\ &= \frac{1}{6} \int_{-3}^3 20.0 - 10.0y^2 dy \\ &= \frac{1}{6} \left(20.0y - \frac{10.0}{3}y^3 \right) \Big|_{-3}^3 \\ &= \frac{1}{6} \left(\left(60.0 - \frac{10.0}{3}3^3 \right) - \left(-60.0 - \frac{10.0}{3}(-3)^3 \right) \right) = -10. \end{aligned}$$

d. The total voltage along the 6 cm piece is -60. Along the rest (74 cm) the total is $74 \cdot (-70) = -5180$. The total along the whole thing is $-5180 - 60 = -5240$, so the average is $-5240/80 = -65.6$.

4.4.

- a. As $t \rightarrow \infty$, the rate of change acts like the leading behavior of the right hand side, or $\frac{1}{\sqrt{1+4t}}$, which in turn acts more or less like $t^{-1/2}$. The integral of such a function approaches infinity. By the Intermediate Value Theorem, the voltage must cross 0 on its way from -70 to infinity.
- b. If T represents the time when the voltage reaches 0, we require an increase in voltage of exactly 70. The equation is then

$$\int_0^T \frac{1}{\sqrt{1+4t}} - \frac{2}{(1+4t)^{\frac{3}{2}}} dt = 70.$$

c. It seems rather unlikely that the voltage of a dead neuron will approach infinity.

4.5.

- a. The second is a pure-time differential equation. The first might describe a population of bacteria that are autonomously reproducing, and the second might describe a population being supplemented from outside at an ever increasing rate.
- b. $\hat{b}(0.1) = b(0) + b'(0) \cdot 0.1 = 1 + 2 \cdot 0.1 = 1.2$. Similarly, $\hat{B}(0.1) = B(0) + B'(0) \cdot 0.1 = 1 + 2 \cdot 0.1 = 1.2$.
- c. $\hat{b}(0.2) = \hat{b}(0.1) + 2\hat{b}(0.1) \cdot 0.1 = 1.2 + 2 \cdot 1.2 \cdot 0.1 = 1.44$. Similarly, $\hat{B}(0.2) = \hat{B}(0.1) + B'(0.1) \cdot 0.1 = 1.2 + 1.2 \cdot 0.1 = 1.32$.

4.6.

- a. The rate of production is decreasing exponentially, even though the total amount of product continues to increase.
- b. $p(1) = 1 + \int_0^1 e^{-4t} dt = 1 - 0.25e^{-4t} \Big|_0^1 \approx 1.24$.

4.7.

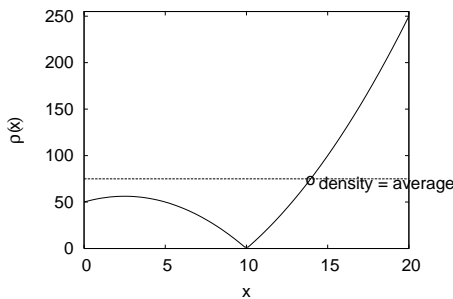
- a. The volume is increasing until time $t = 2$, and decreases thereafter.
- b. The volume is a maximum when the rate of change is 0, or at $t = 2$.
- c. LHE= $V(0) \cdot 1 + V(1) \cdot 1 + V(2) \cdot 1 = 7$ and RHE= $V(1) \cdot 1 + V(2) \cdot 1 + V(3) \cdot 1 = -2$.
- d. $V(3) = \int_0^3 4 - t^2 dt = 4t - \frac{t^3}{3} \Big|_0^3 = 3$.

4.8. Area = $\int_0^3 3 + (1 + x/3)^2 dx$. We substitute $y = 1 + x/3$, finding $dy = dx/3$, and limits of integration from $y = 1$ to $y = 2$. So

$$\begin{aligned} \text{area} &= \int_0^3 3 + (1 + x/3)^2 dx = \int_1^2 (3 + y^2) 3dy \\ &= 9y + y^3 \Big|_1^2 = (9 \cdot 2 + 2^3) - (9 \cdot 1 + 1^3) = 16. \end{aligned}$$

4.9.

- a. The function hits 0 when $x = 10$. The derivative is 0 at $x = 2.5$, where the density is 56.25. At the endpoints, we have densities of 50 (at $x = 0$) and 250 (at $x = 20$). The maximum is thus at $x = 20$, with the minimum at $x = 10$.



- b. Taking into account the absolute values, the total number is

$$\begin{aligned} \text{total} &= \int_0^{20} |-x^2 + 5x + 50| dx \\ &= \int_0^{10} -x^2 + 5x + 50 dx + \int_{10}^{20} x^2 - 5x - 50 dx \\ &= -x^3/3 + 5x^2/2 + 50x \Big|_0^{10} + x^3/3 - 5x^2/2 - 50x \Big|_{10}^{20} = 1500. \end{aligned}$$

- c. The average density is 75.0.
- d. Shown in figure for a.

4.10.

- a. The amount of product increases to infinity.
 b. We have that

$$p(t) = \int_0^t \frac{1}{\sqrt{1+3s}} ds.$$

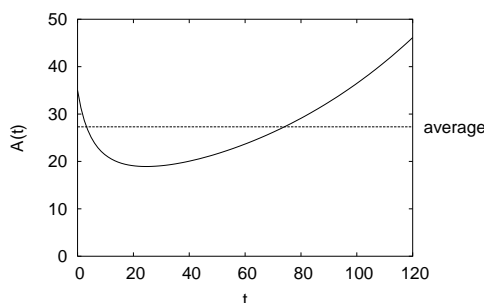
After substituting $y = 1 + 3s$, we find that $p(t) = \frac{2}{3}(\sqrt{1+3t} - 1)$. The average rate is $p(t)/t$. Because $p(t)$ acts like \sqrt{t} , which increases more slowly than t this average approaches 0 as $t \rightarrow \infty$.

- c. Because both $p(t)$ and t approach 0 as $t \rightarrow 0$, we can use L'Hopital's rule as

$$\lim_{t \rightarrow 0} \frac{p(t)}{t} = \lim_{t \rightarrow 0} \frac{p'(t)}{1} = \lim_{t \rightarrow 0} \frac{1}{\sqrt{1+3t}} = 1.$$

4.11.

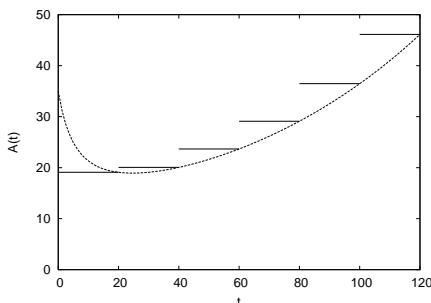
- a. $A'(t) = -15/(2 + .3t)^2 + 0.125e^{0.125t}$, so $A'(0) = -3.625$. Also, $A(0) = 35.0$ and $A(120) \approx 46.13$. The graph thus begins decreasing and then increases. The maximum is at $t = 120$.



- b.

$$\int_0^{120} A(t) dt = \frac{50}{0.3} \ln(2 + 0.3t) + 800e^{0.0125t} \Big|_0^{120} \approx 3276.$$

- c. The average is $3276/120 \approx 27.3$.
 d. The minimum value must be less than 27.3.
 e. RHE = $\sum_{i=1}^6 A(20i)20$. I bet the estimate is high.



4.12.

- a. Total = $\int_0^2 G(h) dh = 5h - 1.5e^{-2h} \Big|_0^2 \approx 11.4$ mol.
 b. Average = $11.4 \text{ mol} / 2 \text{ m} = 5.7 \text{ mol/m}$.
 c. Exact amount between 1.0 and 1.01 is $\int_{1.0}^{1.01} G(h) dh = 5h - 1.5e^{-2h} \Big|_{1.0}^{1.01} \approx 0.0540$. The approximate amount is $G(1.0) \cdot 0.01 = (5 + 3e^{-2.0}) \cdot 0.01 \approx 0.0541$.