Modeling the Dynamics of Life: Calculus and Probability for Life Scientists

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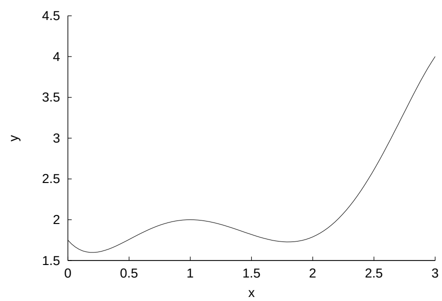
3.10 Supplementary Problems for Part 3

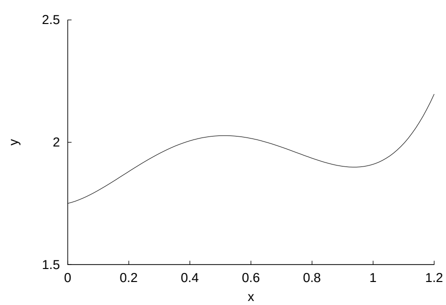
♠ For the functions shown:

a. Sketch the derivative.
b. Label local and global maxima.
c. Label local and global minima.
d. Subsets of the domain with positive second derivative.

• EXERCISE **3.1**



• EXERCISE **3.2**



♠ Use the tangent line and the quadratic Taylor polynomial to find approximate values of the following. Make sure to write down the function or functions you use and the equation of the tangent line. Check with your calculator.

• EXERCISE **3.3**
$1/(3 + 1.01^2)$.

• EXERCISE **3.4**
$e^{3(1.02)^2+2(1.02)}$.

♠ Write down the tangent line approximation for the following functions and estimate the requested values.

• EXERCISE **3.5**
$f(x) = \dfrac{1+x}{1+e^{3x}}$. Estimate $f(-0.03)$.

• EXERCISE **3.6**
$g(y) = (1 + 2y)^4 \ln(y)$. Estimate $g(1.02)$.

♠ Sketch graphs of the following functions. Find all critical points, and state whether they are minima or maxima. Find the limit of the function as $x \to \infty$.

• EXERCISE **3.7**
$(x^2 + 2x)e^{-x}$ for positive x.

• EXERCISE **3.8**
$\ln(x)/(1 + x)$ for positive x. Do not solve for the maximum, just show that there must be one.

♠ Find the Taylor polynomial of degree 2 approximating each of the following.

• EXERCISE **3.9**
$f(x) = \frac{1 + x}{1 + x^2}$ for x near 0.

• EXERCISE **3.10**
$f(x) = \frac{1 + x}{1 + x^2}$ for x near 1.

• EXERCISE **3.11**
$g(x) = \frac{1 + x}{1 + e^x}$ for x near 0.

• EXERCISE **3.12**
$h(x) = \frac{x}{2 - e^x}$ for x near 0.

♠ Combine the Taylor polynomials from the previous set of problems with the leading behavior of the functions for large x to sketch graphs

• EXERCISE **3.13**
$g(x)$.

• EXERCISE **3.14**
$h(x)$.

• EXERCISE **3.15**
Between days 0 and 150 (measured from November 1), the snow at a certain ski resort is given by

$$S(t) = -\frac{1}{4}t^4 + 60t^3 - 4000t^2 + 96000t$$

where S is measured in microns (one micron is 10^{-4} centimeters).

a. A yetti tells you that this function has critical points at $t = 20$, $t = 40$ and $t = 120$. Confirm this assertion.
b. Find the global maximum and global minimum amounts of snow in feet. Remember that 1 inch = 2.54 centimeters.
c. Use the second derivative test to identify the critical points as local maxima and minima.
d. Sketch the function.

• EXERCISE **3.16**
Let $r(x)$ be the function giving the per capita production as

$$\text{per capita production } = \frac{4x}{1 + 3x^2}$$

as a function of population size x.

a. Find the population size that produces the highest per capita production.
b. Find the highest per capita production.
c. Check with the second derivative test.

• EXERCISE **3.17**
An organism is replacing 25% of the air in its lung each breath and the external concentration of a chemical is $\gamma = 5.0 \times 10^{-4}$ moles/liter. Suppose the body uses a fraction α of the chemical just after breathing. That is, the chemical follows

$$c_t \to (1 - \alpha)c_t \to \text{mix of 75\% used air and 25\% ambient air}$$

a. Write the discrete-time dynamical system for this process.
b. Find the equilibrium level in the lung as a function of α.
c. Find the amount absorbed by the body each breath at equilibrium as a function of α.
d. Find the value of α that maximizes the amount of chemical absorbed at equilibrium.
e. Explain your result in words.

• EXERCISE **3.18**
An organism is replacing a fraction q of the air in its lung each breath and the external concentration of a chemical is $\gamma = 5.0 \times 10^{-4}$ moles/liter. Suppose the body uses a fraction $\alpha = 1 - q$ of the chemical just after breathing. The chemical follows

$$c_t \overset{\text{absorption}}{\longrightarrow} (1-\alpha)c_t \overset{\text{breathing}}{\longrightarrow} \text{mix of used and ambient air}$$

a. Write the discrete-time dynamical system for this process.
b. Find the equilibrium level in the lung as a function of q. Is it stable?
c. Find the amount absorbed by the body each breath at equilibrium as a function of q.
d. Find the value of q that maximizes the amount of chemical absorbed at equilibrium.
e. Explain your result in words.

• EXERCISE **3.19**
Suppose the volume of a plant cell follows $V(t) = 1000(1 - e^{-t})\mu\text{m}^3$ for t measured in days. Suppose the fraction of cell in a vacuole (a water-filled portion of the cell) is $H(t) = e^t/(1 + e^t)$.

a. Sketch a graph of the total size of the cell as a function of time.
b. Find the volume of cell outside the vacuole.
c. Find and interpret the derivative of this function. Don't forget the units.
d. Find when the volume of the cell outside the vacuole reaches a maximum.

• EXERCISE **3.20**
Consider the function

$$F(t) = \frac{\ln(1+t)}{t+t^2}.$$

a. What is $\lim_{t\to 0} F(t)$?
b. What is $\lim_{t\to 0} F'(t)$?
c. What is $\lim_{t\to 0} F''(t)$?
d. Sketch a graph of this function.

• EXERCISE **3.21**
During Thanksgiving dinner, the table is replenished with food every 5 minutes. Let F_t represent the fraction of the table laden with food.

$$F_{t+1} = F_t - \text{ amount eaten } + \text{ amount replenished } .$$

Suppose that

$$\begin{aligned} \text{amount eaten} &= \frac{bF_t}{1+F_t} \\ \text{amount replenished} &= a(1-F_t) \end{aligned}$$

and that $a = 1.0$ and $b = 1.5$.

a. Explain the terms describing amount eaten and amount replenished.
b. If the table starts out empty, how much food is there after 5 minutes? How much is there after 10 minutes?
c. Use the quadratic formula to find the equilibria.
d. How much food will there be five minutes after the table is 60% full? Sketch the solution.

• EXERCISE **3.22**
Let N_t represent the difference between the sodium concentration inside and outside a cell at some time. After one second, the value of N_{t+1} is

$$\begin{cases} N_{t+1} = 0.5N_t & \text{if } N_t < 2 \\ N_{t+1} = 4.0N_t - 7 & \text{if } 2 < N_t < 4 \\ N_{t+1} = -0.25N_t + 10 & \text{if } 4 < N_t. \end{cases}$$

a. Graph the updating function and show that it is continuous.
b. Find the equilibria and their stability.
c. Find all initial conditions which end up at $N = 0$.

• EXERCISE **3.23**
Consider looking for a positive solution of the equation

$$e^x = 2x + 1.$$

a. Draw a graph and pick a reasonable starting value.
b. Write down the Newton's method iteration for this equation.
c. Find your next guess.
d. Show explicitly that the slope of the updating function for this iteration is zero at the solution.

• EXERCISE **3.24**
Consider trying to solve the equation

$$\ln(x) = \frac{x}{3}.$$

a. Convince yourself there is indeed a solution and find a reasonable guess.
b. Use Newton's method to update your guess twice.
c. What would be a bad choice of an initial guess?

• EXERCISE **3.25**
Suppose a bee gains an amount of energy

$$F(t) = \frac{3t}{1+t}$$

after it has been on a flower for time t, but that it uses $2t$ energy units in that time (it has to struggle with the flower).

a. Find the net energy gain as a function of t.
b. Find when the net energy gain per flower is maximum.
c. Suppose the travel time between flowers is $\tau = 1$. Find the time spent on the flower that maximizes the rate of energy gain.
d. Draw a diagram illustrating the results of parts **b** and **c**. Why is the answer to **c** smaller?

• EXERCISE **3.26**
Consider the function for net energy gain from the previous problem.

a. Use the Extreme Value Theorem to show that there must be a maximum.
b. Use the Intermediate Value Theorem to show that there must be a residence time t that maximizes the rate of energy gain.

• EXERCISE **3.27**
A peculiar variety of bacteria enhances its own per capita production. In particular, the number of offspring per bacteria increases according to the function

$$\text{per capita production } = r(1 + \frac{b_t}{K}).$$

Suppose that $r = 0.5$ and that $K = 10^6$.

a. Graph per capita production as a function of population size.
b. Find the discrete-time dynamical system for this population and graph the updating function.
c. Find the equilibria.
d. Find their stability.

• EXERCISE **3.28**
A type of butterfly has two morphs, a and b. Each type reproduces annually after predation. 20% of type a are eaten, and 10% of type b are eaten. Each type doubles its population when it reproduces. However, the types do not breed true. Only 90% of the offspring of type a are of type a, the rest being of type b. Only 80% of the offspring of type b are of type b, the rest being of type a.

a. Suppose there are 10,000 of each type before predation and reproduction. Find the number of each type after predation and reproduction.
b. Find the discrete-time dynamical systems for types a and b.
c. Find the discrete-time dynamical system for the fraction p of type a.
d. Find the equilibria.
e. Find their stability.

• EXERCISE **3.29**
A population of size x_t follows the rule

$$\text{per capita production } = \frac{4x_t}{1+3x_t^2}.$$

a. Find the updating function for this population.
b. Find the equilibrium or equilibria.
c. What is the stability of each equilibrium?
d. Find the equation of the tangent line at each equilibrium.
e. What is the behavior of the approximate dynamical system defined by the tangent line at the middle equilibrium?

• EXERCISE **3.30**
Consider a population following the discrete-time dynamical system

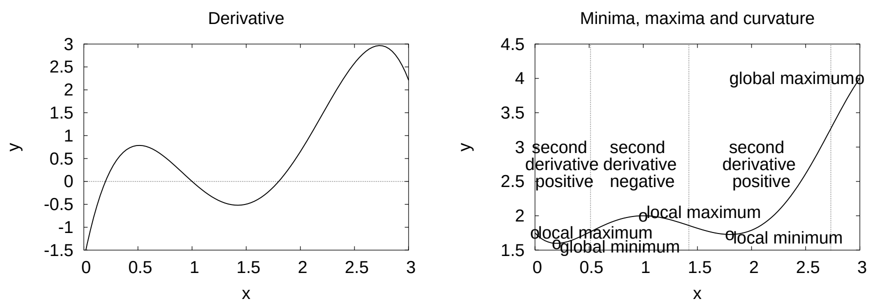
$$N_{t+1} = \frac{rN_t}{1+N_t^2}.$$

a. What is the per capita production?
b. Find the equilibrium as a function of r.
c. Find the stability of the equilibrium as a function of r.
d. Does this positive equilibrium become unstable as r becomes large?

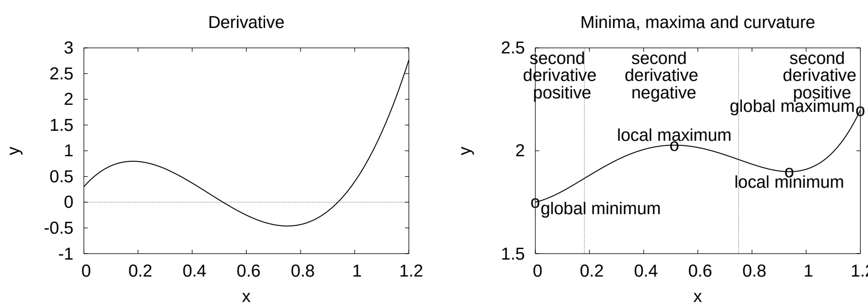
Chapter 9

Answers

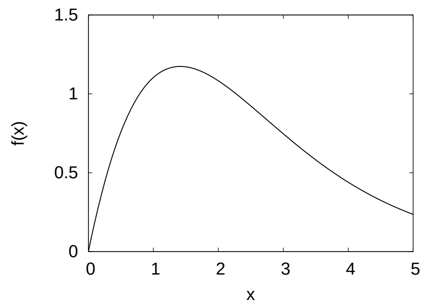
3.1.



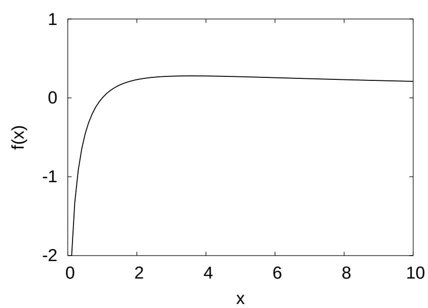
3.2.



3.3. Let $f(x) = 1/(3 + x^2)$. Then $f'(x) = -2x/(3 + x^2)^2$. Substituting $x = 1$, we find $f(1) = 1/4$ and $f'(1) = -1/8$, so $\hat{f}(x) = 1/4 - 1/8(x - 1)$, and $\hat{f}(1.01) = 1/4 - 1/8(1.01 - 1) = 0.24875$.
3.4. Let $f(x) = e^{3x^2+2x}$. Then $f'(x) = (6x + 2)e^{3x^2+2x}$. Substituting $x = 1$, we find $f(1) = e^5$ and $f'(1) = 8e^5$, so $\hat{f}(x) = e^5 + 8e^5(x - 1)$ and $\hat{f}(1.02) = e^5 + 8e^5(1.02 - 1) = 1.16e^5$.
3.5. $\hat{f}(x) = 0.5 - 0.25x$. $\hat{f}(-0.03) = 0.5075$. In this case, $f(-0.03) \approx 0.5068$.
3.6. $\hat{g}(y) = 81(y - 1)$. $\hat{g}(1.02) = 1.62$.
3.7. $f'(x) = (2 - x^2)e^{-x}$ which is 0 at $x = \sqrt{2}$. Because $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 0$ (exponential declines faster than quadratic increases), this must be a maximum.



3.8. This function is negative for $x < 1$, zero at $x = 1$ and positive for $x > 1$. Also, its limit at infinity is zero. It must indeed have a positive maximum between 1 and infinity.



3.9. We have that

$$f'(x) = \frac{1 - 2x - x^2}{(1 + x^2)^2}, \quad f''(x) = \frac{2(-1 - 3x + 3x^2 + x^3)}{(1 + x^2)^3}.$$

Therefore, $f'(0) = 1$ and $f''(0) = -2$, and $\hat{f}(x) = 1 + x - x^2$.

3.10. Using the derivative in the previous problem, $f'(1) = -1/2$ and $f''(1) = 0$, so $\hat{f}(x) = 1 - \frac{1}{2}(x - 1)$.

3.11.

$$g'(x) = \frac{1 - xe^x}{(1 + e^x)^2}, \quad g''(x) = \frac{e^x(-3 - x + xe^x - e^x)}{(1 + e^x)^3}.$$

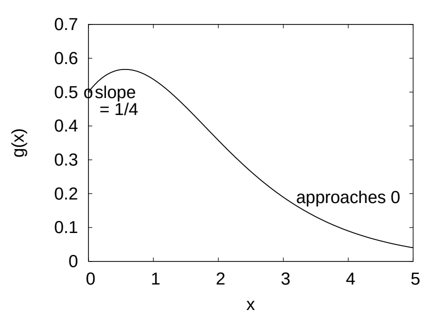
Then $g(0) = 1/2$, $g'(0) = 1/4$ and $g''(0) = -1/2$ and $\hat{g}(x) = \frac{1}{2} + \frac{1}{4}x - \frac{1}{4}x^2$.

3.12.

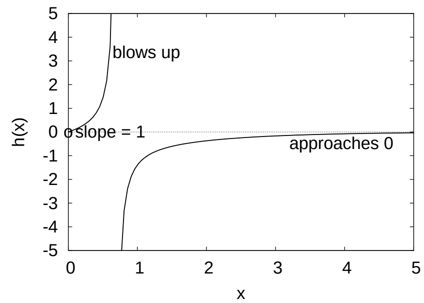
$$h'(x) = \frac{2 - e^x + xe^x}{(-2 + e^x)^2}, \quad h''(x) = \frac{e^x(-4 - 2x - xe^x + 2e^x)}{(-2 + e^x)^3}.$$

Then $h(0) = 0$, $h'(0) = 1$ and $h''(0) = 2$ and $\hat{h}(x) = x + x^2$.

3.13.



3.14.

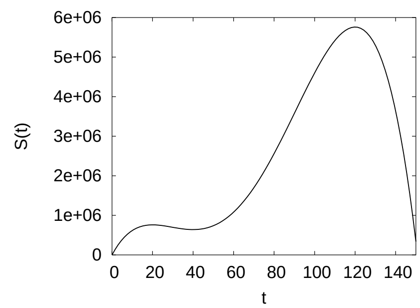


3.15.

a. $S'(t) = -t^3 + 180t^2 - 8000t + 96000$, which is 0 at $t = 20$, $t = 40$ and $t = 120$.

b. Substituting the endpoints ($t = 0$ and $t = 150$) and the critical points into the function $S(t)$, we find a maximum of 576,000 at $t = 120$.

c. $S''(t) = -3t^3 + 360t - 8000$. Then $S''(20) = -2000$, $S''(40) = 1600$, and $S''(120) = -8000$. The first and last are maxima and the middle one is a minimum.



d.

3.16.

a. If $r(x) = \frac{4x}{1+3x^2}$, then

$$r'(x) = \frac{4(3x^2-1)}{(1+3x^2)^2}.$$

The critical points are then at $x = \pm\sqrt{1/3}$. Only positive values make sense, so we need only consider $x \approx 0.577$. Because $r(0) = 0$ and $\lim_{x\to 0} r(x) = 0$, this must be a maximum.

b. The value is $2/\sqrt{3} \approx 1.155$.

c. The second derivative is

$$r''(x) = \frac{72x(x^2-1)}{(1+3x^2)^2}$$

which is negative at $x = \sqrt{1/3}$.

3.17.

a. $c_{t+1} = 0.75(1-\alpha)c_t + 0.25\gamma$.

b. $c^* = 0.25\gamma/(0.25 + 0.75\alpha)$.

c. $\alpha c^* = 0.25\alpha\gamma/(0.25 + 0.75\alpha)$.

d. This has its maximum at $\alpha = 1$.
e. Sure, absorb as much as you can as fast as you can.

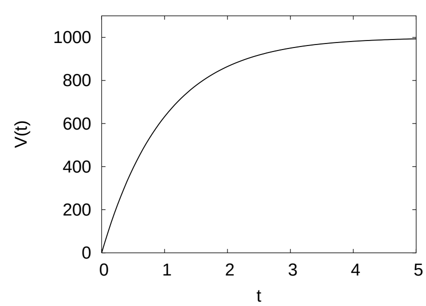
3.18.

a. $c_{t+1} = (1-q)(1-\alpha)c_t + q\gamma$.
b. $c^* = q\gamma/(q+(1-q)\alpha$. Definitely stable because the slope of the updating function is $(1-q)(1-\alpha) < 1$.
c. $\alpha c^* = q\alpha\gamma/(q+(1-q)\alpha$.
d. Taking the derivative with respect to q, we find the derivative is

$$\frac{\alpha^2\gamma}{(\alpha+q-\alpha q)^2}$$

which is always positive. The maximum is at $q = 1$.
e. Sure, absorb as much as you can by breathing as deeply as you can.

3.19.



a.
b. The fraction outside is $1 - H(t) = 1/(1+e^t)$, so the total volume outside is

$$\frac{1000(1-e^{-t})}{1+e^t}.$$

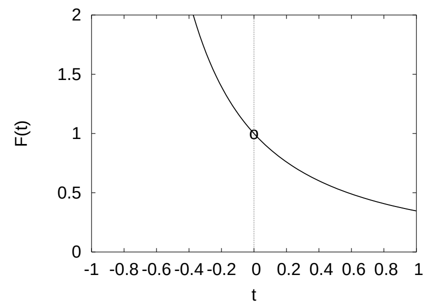
Call this function $V_o(t)$.
c.

$$V_o'(t) = \frac{1000(2+e^{-t}-e^t)}{(1+e^t)^2}.$$

d. This is a bit tricky. The maximum is the critical point where $V_o'(t) = 0$, or where $2+e^{-t}-e^t = 0$. Letting $x = e^t$, this is $2+1/x-x = 0$. Multiplying both sides by x, we get the quadratic $2x+1-x^2 = 0$, which can be solved with the quadratic formula to give $x = 1+\sqrt{2}$, so that $t = \ln(1+\sqrt{2}) \approx 0.88$ days.

3.20.

a. $\lim_{t\to 0} F(t) = 1$.
b. $\lim_{t\to 0} F'(t) = \frac{-3}{2}$.
c. $\lim_{t\to 0} F''(t) = \frac{11}{3}$.



3.21.

a. The more food there is, the more is eaten, up to a limit of b. The replenishment is enough to refill the table.

b. The system is

$$F_{t+1} = F_t - \frac{1.5F_t}{1+F_t} + 1 - F_t = 1 - \frac{1.5F_t}{1+F_t}.$$

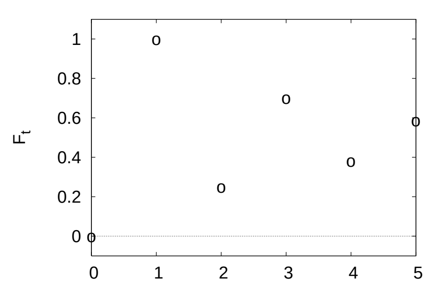
If $F_0 = 0$, then $F_1 = 1$ and $F_2 = 1 - \frac{1.5}{2} = 0.25$.

c. Solving for the equilibrium F^* with

$$\begin{aligned} F^* &= 1 - \frac{1.5F^*}{1+F^*} \\ F^*(1+F^*) &= 1 + F^* - 1.5F^* \\ F^* + (F^*)^2 &= 1 - 0.5F^* \\ 1.5F^* + (F^*)^2 - 1 &= 0 \\ (F^* - 0.5)(F^* + 2) &= 0. \end{aligned}$$

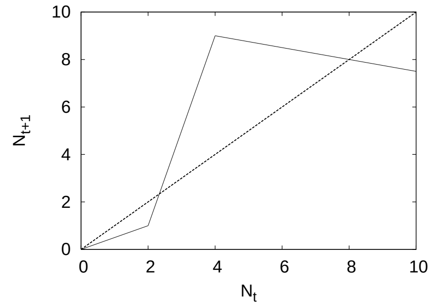
The only positive equilibrium is at $F = 0.5$.

d. $G(0.6) = 0.4375$.



3.22.

a. Each piece is continuous, being linear. And at the jump points, they match. That is, at $N_t = 2$, $0.5N_t = 4N_t - 7 = 1$, and at $N_t = 4$, $4N_t - 7 = -0.25N_t + 10 = 9$.

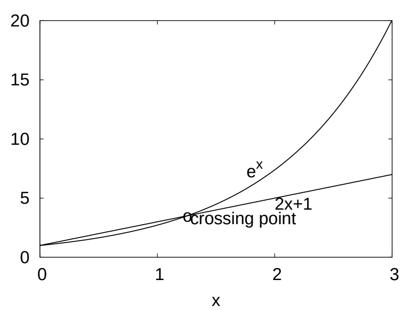


b. Along the first piece, there is an equilibrium at $N = 0$, which is stable because the slope is 0.5. The diagonal intersects the second piece at $N = 2.333$, another equilibrium. This one is unstable. The third piece hits the diagonal at $N = 8$, a stable equilibrium with negative slope of -0.25.

c. Anything below 2.333 definitely does, because it just decreases. And anything that drops below 2.333 on the second step, or if $-0.25N_t + 10 < 2.333$ or $N_t > 30.67$. All the values between 2.333 and 30.67 end up at the 8.

3.23.

a. It looks like $x_0 = 1$ is a good guess.



b. We need to solve $g(x) = e^x - 2x - 1 = 0$, so

$$x_{t+1} = x_t - \frac{g(x_t)}{g'(x_t)} = x_t - \frac{e^x - 2}{e^x - 2x - 1}.$$

c. $x_1 = 1.3922$.

d. The derivative of the updating function is

$$\frac{e^x(e^x - 2x - 1)}{(e^x - 2)^2},$$

which is 0 when $e^x - 2x - 1 = 0$.

3.24.

a. At $x = 1$, the left hand side is smaller. At $x = e$, the left hand side is larger. And at $x = e^2$, the left hand side is smaller again. There must be at least two solutions, one between 1 and e and another between e and e^2. To find the lower one, I bet that 2 is a good guess.

b. Our function is $f(x) = \ln(x) - x/3$. The Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} = x_t - \frac{\ln(x_t) - x_t/3}{1/x_t - 1/3}.$$

Substituting $x_0 = 2$, we find $x_1 = 1.8411$, and then that $x_2 = 1.8570$. To five decimal places, the exact answer is 1.8571.

c. $f'(3) = 0$. This would be a rather bad guess.

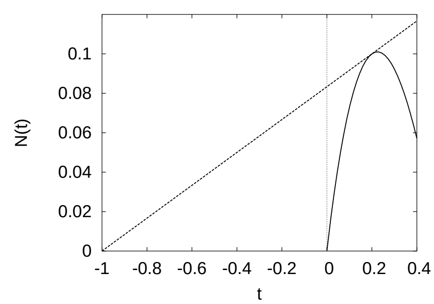
3.25.

a. Let $N(t)$ be net energy gain. Then $N(t) = F(t) - 2t$.

b. When $N'(t) = 0$ or $F'(t) - 2 = \dfrac{3}{(1+t)^2} - 2 = 0$. This has solution $t = \sqrt{3/2} - 1 \approx 0.225$.

c. We need to maximize $\dfrac{N(t)}{1+t}$, which occurs when $t = 0.2$.

d. The answer to **c** is smaller because the bee has other options besides sucking as much nectar as possible out of the flower.



3.26.

a. $N(0) = 0$ as does $N(0.5)$. Because the values are positive in between, there must be a maximum.

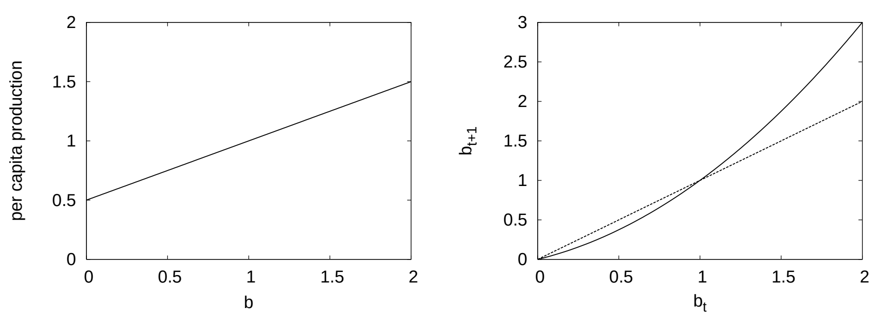
b. The derivative of $\dfrac{N(t)}{1+t}$ is

$$\frac{1-5t}{(1+t)^2}$$

which is positive for $t = 0$ and negative for $t = 1$. There must be critical point in between, which must be a maximum because the function switches from increasing to decreasing.

3.27.

a.



b. The discrete-time dynamical system is $b_{t+1} = rb_t(1 + \frac{b_t}{K})$.
c. Equilibria at $b^* = 0$ and $b^* = 10^6$.
d. Equilibrium at $b^* = 0$ is stable and equilibrium at $b^* = 10^6$ is unstable.

3.28.

a. For type a: there are 8,000 after predation, they reproduce to make 16,000, 90% or 14,400 of which are a's and 10% or 1600 of which are b's. For type b: there are 9,000 after predation, they reproduce to make 18,000, 80% or 14,400 of which are b's and 20% or 3600 of which are a's.
b. $a_{t+1} = 0.9 \cdot 2 \cdot 0.8a_t + 0.2 \cdot 2 \cdot 0.9b_t = 1.44a_t + 0.36b_t$. $b_{t+1} = 0.1 \cdot 2 \cdot 0.8a_t + 0.8 \cdot 2 \cdot 0.9b_t = 0.16a_t + 1.44b_t$.
c.

$$p_{t+1} = \frac{1.44p_t + 0.36(1 - p_t)}{1.6p_t + 1.8(1 - p_t)}.$$

d. $p^* = 0.6$.
e. $f'(p^*) = 0.714$ and the equilibrium is stable.

3.29.

a. The updating function f is $f(x) = 4x^2/(1 + 3x^2)$, multiplying the per capita production by x, the number of individuals.
b. First, find that 0 is a solution. The rest is a quadratic, which has solutions at $x = 1/3$ and $x = 1$.
c. We find that $f'(x) = 8x/(1 + 3x^2)^2$. Then $f'(0) = 0$, $f'(1/3) = 3/2$ and $f'(1) = 1/2$. Therefore, the equilibria at 0 and 1 are stable and the one at 1/3 is unstable.
d. The tangent line at 0 is $\hat{f}(x) = 0$. The tangent line at $x = 1/3$ is $\hat{f}(x) = 1/3 + 3/2(x - 1/3)$. The tangent line at $x = 1$ is $\hat{f}(x) = 1 + 1/2(x - 1)$.
e. This dynamical system shoots off to infinity for starting points greater than 1/3, and off to negative infinity for starting points less than 1/3. This is different from the behavior of the original system, in which such solutions approach $x = 1$ and $x = 0$ respectively.

3.30.

a. It is $\frac{r}{1 + N_t^2}$.
b. $N^* = \sqrt{r - 1}$.
c. The derivative at the equilibrium is $\frac{2 - r}{r}$.
d. No, this is always greater than -1.