NAME:___________________

MATH 5110
Midterm

Do all four problems (25 points each). Write readable answers on the test, but feel free to use or hand in additional paper if necessary. One page of notes, no calculators, no cheating.

1. One of the great challenges during the summer is balancing one's supply of tomatoes and hot peppers. Suppose that peppers or tomatoes are either eaten or saved for the following day, but never rot or get thrown away (very dry climate). In particular, $1/4$ of peppers are eaten each day, as are half of the tomatoes. Each day, you harvest half as many peppers as you had tomatoes the day before, and $1/4$ as many tomatoes as you had peppers the day before.

   a. Write these rules as a matrix equation.

   b. Find the eigenvalues.

   c. What will the ratio of tomatoes to peppers be in the long run?

   d. If you wanted the numbers of peppers and tomatoes to decrease as quickly as possible by changing one of the entries in the matrix by a small amount, which entry would you choose?

2. Suppose the number of peppers in the house follows the the discrete-time dynamical system

$$P_{t+1} = 4 + \frac{P_t}{1 + \frac{P_t^2}{64}}.$$

a. Give one simple interpretation of the two terms in this model.

b. Show that $P^* = 8$ is an equilibrium.

c. Graph P_{t+1} as a function of P_t.

d. Find the stability of $P^* = 8$.

e. Cobweb starting from $P_0 = 0$.

f. Consider the generalized model

$$P_{t+1} = 4 + \frac{(1+b)P_t}{2(1 + b\frac{P_t^2}{64})}.$$

where $b > -1$. Show that $P^* = 8$ is still an equilibrium. What bifurcations, if any, can this system have?

3. Suppose the probability that a pepper is not eaten depends on how hot it is, according to

$$\Pr(\text{Not Eaten}) = 1 - e^{-\alpha s}$$

where s represents the hotness in Scoville units. The probability density function of s is $f(s) = \beta e^{-\beta s}$. Both α and β are positive.

a. Find the fraction of peppers not eaten as a function of α and β.

b. Suppose 4 peppers are purchased each day. Write the discrete-time dynamical system.

c. Find the equilibria and their stability.

d. Explain their dependence on α and β. That is, is the equilibrium an increasing or decreasing function of each parameter, and why?

4. Suppose the numbers of peppers P_t and the number of tomatoes T_t obey

$$
\begin{aligned}
P_{t+1} &= 2 + \frac{P_t}{1 + \frac{P_t^2}{64}} + \frac{T_t}{8} \\
T_{t+1} &= 4 + \frac{3}{2} P_t + g(T_t - 2P_t).
\end{aligned}
$$

a. What conditions must the function g satisfy for the model to be internally consistent (that is, to make sure that T will not be negative)?

b. Suppose that $P^* = 8$ at an equilibrium. Find T^* and the value of $g(0)$.

c. Find the equilibria. What are the conditions on the parameters for a positive equilibrium to exist?

d. Find the Jacobian.

e. What are the conditions on the function g for the equilibrium to be stable? What happens when those conditions are violated?