Outer space and the work of Karen Vogtmann

$Out(F_n)$

$F_n = < a_1, a_2, \cdots, a_n >$ is the free group of rank n.

$$Out(F_n) = Aut(F_n)/Inn(F_n)$$

- contains $MCG(S)$ for punctured surfaces S
- maps to $GL_n(\mathbb{Z})$

The study of mapping class groups and arithmetic groups is an inspiration in the study of $Out(F_n)$.

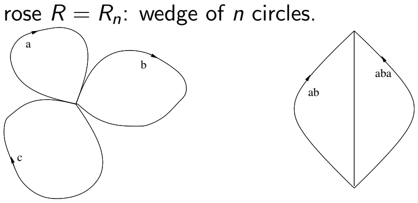
Theorem (Nielsen, 1924)

$Aut(F_n)$ (and $Out(F_n)$) are finitely presented. A generating set consists of the automorphisms σ: $a_1 \mapsto a_1 a_2$, $a_i \mapsto a_i$ for $i > 1$ plus the signed permutations of the a_i's.

Outer space

Definition

- graph: finite 1-dimensional cell complex Γ, all vertices have valence ≥ 3.
- rose $R = R_n$: wedge of n circles.



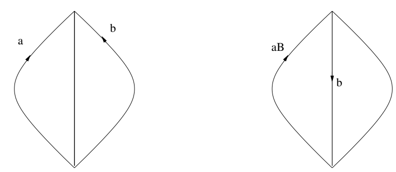
- marking: homotopy equivalence $g : \Gamma \to R$.
- metric on Γ: assignment of positive lengths to the edges of Γ so that the sum is 1.

Outer space

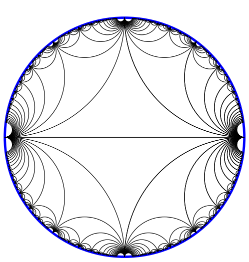
Definition (Culler-Vogtmann, 1986)

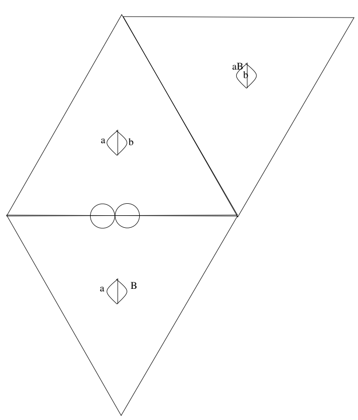
Outer space X_n is the space of equivalence classes of marked metric graphs (g, Γ) where $(g, \Gamma) \sim (g', \Gamma')$ if there is an isometry $\phi : \Gamma \to \Gamma'$ so that $g'\phi \simeq g$.

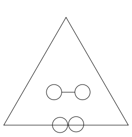
$$\begin{array}{ccc} \Gamma & & \\ & \overset{g}{\searrow} & \\ \phi \downarrow & & R \\ & \overset{g'}{\nearrow} & \\ \Gamma' & & \end{array}$$



Outer space in rank 2







Triangles have to be added to edges along the base.

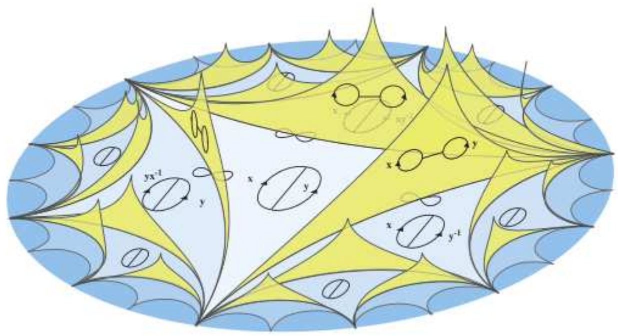


Figure 2: Outer space in rank 2

Outer space

Topology (3 approaches, all equivalent):

- simplicial, with respect to the obvious decomposition into "simplices with missing faces".
- (g, Γ) is close to (g', Γ') if there is a $(1+\epsilon)$-Lipschitz map $f : \Gamma \to \Gamma'$ with $g'f \simeq g$.
- via length functions: if α is a conjugacy class in F_n let $\ell_{(g,\Gamma)}(\alpha)$ be the length in Γ of the unique immersed curve a such that $g(a)$ represents α. Then, for S =set of conjugacy classes

$$X_n \to [0, \infty)^S$$

$$(g, \Gamma) \mapsto (\alpha \mapsto \ell_{(g,\Gamma)}(\alpha))$$

is injective – take the induced topology.

Theorem (Culler-Vogtmann, 1986)

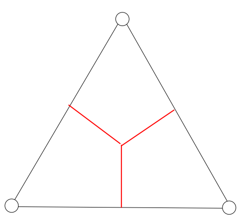
X_n is contractible.

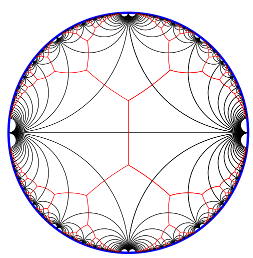
Action

If $\phi \in Out(F_n)$ let $f : R \to R$ be a h.e. with $\pi_1(f) = \phi$ and define

$$\phi(g, \Gamma) = (fg, \Gamma) \qquad \Gamma \xrightarrow{g} R_n \xrightarrow{f} R_n$$

- action is simplicial,
- point stabilizers are finite.
- there are finitely many orbits of simplices (but the quotient is not compact).
- the action is cocompact on the spine $SX_n \subset X_n$.





Topological properties

- Virtually finite $K(G, 1)$ (Culler-Vogtmann 1986).
- $vcd(Out(F_n)) = 2n - 3$ $(n \geq 2)$ (Culler-Vogtmann 1986).
- every finite subgroup fixes a point of X_n.
- every solvable subgroup is finitely generated and virtually abelian (Alibegović 2002)
- Tits alternative: every subgroup $H \subset Out(F_n)$ either contains a free group or is virtually abelian (B-Feighn-Handel, 2000, 2005)
- Bieri-Eckmann duality (B-Feighn 2000)

$$H^i(G; M) \cong H_{d-i}(G; M \otimes D)$$

- Homological stability (Hatcher 1995, Hatcher-Vogtmann 2004)

$$H_i(Aut(F_n)) \cong H_i(Aut(F_{n+1})) \text{ for } n >> i$$

- Computation of stable homology (Galatius, to appear)

Dictionary 1

$SL_2(\mathbb{Z})$	$SL_n(\mathbb{Z})$	$MCG(S)$	$Out(F_n)$
Trace	Jordan normal form	Nielsen-Thurston theory	train-tracks
$\mathbb{H}^2$	symmetric space	Teichmüller space	Outer space
hyperbolic (Anosov) element	semi-simple (diagonalizable)	pseudo-Anosov mapping class	fully irreducible automorphism
shear	parabolic	Dehn twist	polynomially growing automorphism

Links

An n-complex is Cohen-Macauley if the link of every k-cell is homotopy equivalent to a wedge of $(n-k-1)$-spheres (for every k).

Examples

- manifolds,
- buildings.

Theorem (Vogtmann 1990)

Both Outer space and its spine are Cohen-Macauley.

Homological stability

Some groups come in natural sequences

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

Examples

- permutation groups S_n,
- signed permutation groups $S_n^{\pm}$,
- braid groups,
- $SL_n(\mathbb{Z})$, $O_{n,n}(\mathbb{Z})$,
- mapping class groups of surfaces with one boundary component,
- $Aut(F_n)$.

The sequence satisfies homological stability if for every k for sufficiently large n

$$H_k(G_n) \xrightarrow{\cong} H_k(G_{n+1})$$

is an isomorphism.

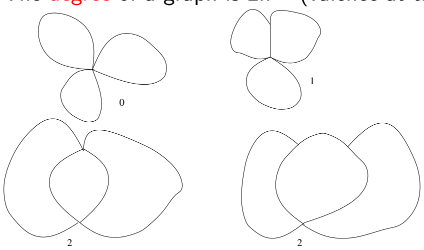
Homological stability

Theorem
$Aut(F_n)$ satisfies homological stability.

Hatcher 1995, Hatcher-Vogtmann 1998, Hatcher-Vogtmann 2004, Hatcher-Vogtmann-Wahl 2006.
The proof over $\mathbb{Q}$ is particularly striking, from [Hatcher-Vogtmann 1998].

Homological stability

$Aut(F_n)$ acts on Autre espace: this is the space of marked metric graphs with a basepoint.
The degree of a graph is $2n -$ (valence at the basepoint).



Degree Theorem. The space $A_{n,k}$ of graphs in the spine of degree $\leq k$ is $(k-1)$-connected.
[H-V 1998], [Bux-McEwen 2009]

Homological stability

We are interested in $H_{k-1}(Aut(F_n), \mathbb{Q}) = H_{k-1}(A_{n,k}/Aut(F_n), \mathbb{Q})$.
Homological stability follows from:

$$A_{n,k}/Aut(F_n) = A_{n+1,k}/Aut(F_{n+1})$$

for $n \geq 2k$.

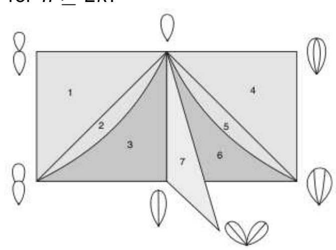


Figure 2. The degree 2 quotient.

$A_{n,k}/Aut(F_n)$ is the space of unmarked metric graphs.
Increasing n amounts to wedging a loop at the basepoint. When n is large every degree k graph has a loop at the basepoint.

Homological stability

One can give a proof over $\mathbb{Z}$ along the same lines.
Stability holds for

- symmetric groups S_n (Nakaoka 1960, Maazen 1979 simpler proof),
- signed symmetric groups $S_n^{\pm}$ (Maazen's proof easily modifies.)
- $H \times S_n^{\pm}$ for a fixed group H (Künneth formula).

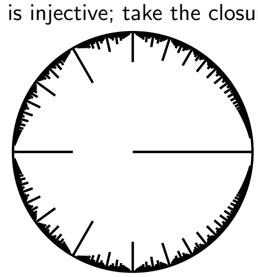
View $A_{n,k}/Aut(F_n)$ as an "orbihedron"; pay attention to the (finite) stabilizers.
E.g. at the rose, the stabilizers are signed permutation groups $S_n^{\pm} = S_n \ltimes \mathbb{Z}_2^n$. Stability holds for this sequence.
In fact, stability holds for the sequence of stabilizers at every point of the orbihedron. They all have the form

$$H \times S_n^{\pm}$$

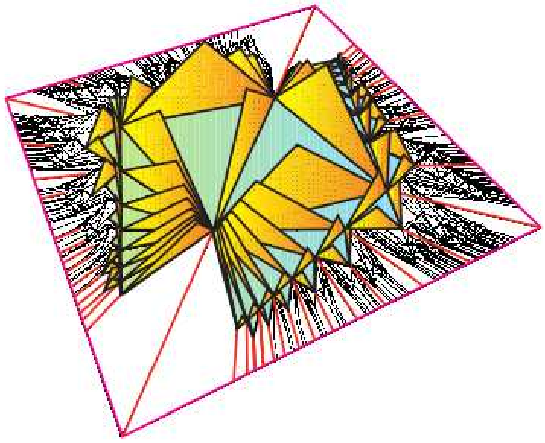
for a fixed group H.

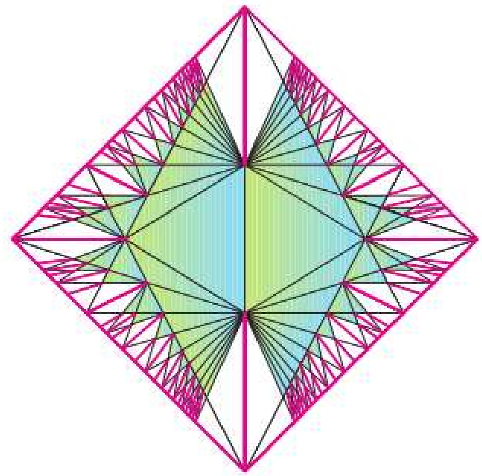
Dynamical properties

- X_n can be equivariantly compactified to $\overline{X_n}$ (Culler-Morgan, 1987), analogous to Thurston's compactification of Teichmüller space via projective measured laminations.
- $$X_n \subset [0,\infty)^S \to P[0,\infty)^S$$
  is injective; take the closure.



- A point of X_n can be viewed as a free simplicial F_n-tree; a point in $\partial X_n = \overline{X_n} \setminus X_n$ is an F_n-tree (not necessarily free nor simplicial).





Dynamical properties

- Points in ∂X_n can be studied using the Rips machine.
- Guirardel (2000): action on ∂X_n does not have dense orbits. He also conjecturally identified the minimal closed invariant set.
- North-South dynamics for fully irreducible elements (Levitt-Lustig, 2003)

Dictionary 2

$MCG(S)$	$Out(F_n)$
simple closed curve	primitive conjugacy class
incompressible subsurface	free factor splitting of F_n
measured lamination	$\mathbb{R}$-tree
Thurston's boundary	Culler-Morgan's boundary
attracting lamination for a pseudo-Anosov	attracting tree for a fully irreducible automorphism
measured geodesic current	measured geodesic current
intersection number between measured laminations	length of a current in an $\mathbb{R}$-tree
curve complex	free factor complex splitting complex

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