INTEGRAL BINARY QUADRATIC FORMS

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An integral binary quadratic form is a polynomial $ax^2 + bxy + cy^2$ where a, b and c are integers. One of the simplest quadratic forms that comes to mind is $x^2 + y^2$. A classical problem in number theory is to describe which integers are sums of two squares. More generally, one can ask what are integral values of any quadratic form?

The aim of this workshop is to introduce binary quadratic forms following a book *The Sensual (Quadratic) Form* by Conway, and then go on to establish some deep properties, such as the Gauss group law, and a connection with continued fractions. More precisely we intend to cover the following topics:

- Integral values: In his book, Conway develops a method to visualize values of a quadratic binary form by introducing certain "topographic" objects: rivers, lakes etc.
- Gauss law: In 1801 Gauss defined a composition law of binary quadratic forms of a fixed discriminant $D = b^2 - 4ac$. In a modern language this group law corresponds to multiplication of ideals in the quadratic field of discriminant D. We shall take an approach to this topic from Manjul Bhargava's 2001 Ph. D. thesis where the Gauss law is defined using $2 \times 2 \times 2$ integer cubes.
- Reduction theory: Two quadratic forms can differ only by a change of coordinates. Such quadratic forms are called equivalent. We shall show that there are only finitely many equivalence classes of binary quadratic forms of a fixed discriminant D. If $D > 0$ the equivalence classes can be interpreted as cycles of purely periodic continued fractions. This interpretation allows us to make some unexpected discoveries!

Prerequisites for this course are rather minimal, as Bhargava thesis is written in elementary language (early 19-th century mathematics). Our main tool will be row/column reduction. In particular, linear algebra and some familiarity with groups, $\mathbb{Z}$-modules, rings, ideals, and fields, are required. Galois theory is not needed! Topics to be covered:

(I) The first part of this course will deal with some well known results in the theory of quadratic fields: orders, ideals and the ideal class group. The relation with quadratic binary forms and the Gauss' law. Reduction theory of quadratic forms, as an effective tool to calculate the ideal class group. A rough syllabus for this part is:

(1) $\mathbb{Z}$-modules.
(2) Quadratic fields over $\mathbb{Q}$: orders, modules, and class numbers.

(3) Binary quadratic forms.
(4) Reduction theory and calculating class numbers:
    - Upper half plane in the case of complex fields.
    - Continued fractions in the case of real fields.

(II) The second part will be based on Bhargava's thesis. We intend to cover several of his composition laws, such as: composition of $2 \times 2 \times 2$ integer cubes and composition of binary cubic forms. Cubic rings and their parameterization by cubic binary forms. The law of composition of $2 \times 3 \times 3$ integer boxes, and its relation to the ideal class group of a cubic ring.

(III) The third part will explore connections of Bhargava's composition laws with exceptional Lie groups: introduction to root systems through the example of $\mathfrak{sl}(n)$. Quick construction of simple split Lie algebras, and corresponding Chevalley groups. (In essence, row/column manipulations again.)