Recovering Exposure Coefficients in the SEIR Model in Two and Three Populations

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Abstract

The SEIR model, which is used by epidemiologists, requires knowing the values of multiple coefficients in order to correctly model the disease they are studying. The forward problem when using this model is to determine the spread of an epidemic over time. The goal of this research is to study the inverse problem and recover one of the coefficients in the SEIR model when applied to multiple populations.

1 Introduction

One of the models used in epidemiology is the SEIR model. The model is applied to diseases that people can be exposed to but not infected and cannot be infected after recovering. The name of the model refers to the four groups that the model accounts for: **S**usceptible, **E**xposed, **I**nfected, **R**ecovered. The people move through the groups as follows:

$$S \xrightarrow{\beta I} E \xrightarrow{\alpha} I \xrightarrow{\gamma} R$$
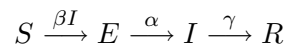
The standard one population formulas for this model are:

$$\frac{dS}{dt} = -\beta SI$$
$$\frac{dE}{dt} = \beta SI - \alpha E$$
$$\frac{dI}{dt} = \alpha E - \gamma I$$
$$\frac{dR}{dt} = \gamma I,$$

with the restriction: $S + E + I + R = 1$.

In this model, $\frac{1}{\beta}$ is the exposure rate, $\frac{1}{\alpha}$ is the infection rate, and $\frac{1}{\gamma}$ is the recovery rate. This same model can be expanded to multiple populations. Normally epidemiologists use these equations to

solve for the final infected population when they have the values for the coefficients for the disease. The model used in the research will be heavily simplified from the standard model for heterogeneous populations. It is assumed that there is no birth or death and therefore the population is constant. It is also assumed that all populations are suffering from the same infection so that the coefficients representing the recovery rate and the latency period before infection are equal in all populations. The populations interact with each other with different frequencies and therefore the contact rates between the populations are different. The equations for a heterogeneous population as suggested by [1] after setting μ to 0 are:

$$\lambda_i = \sum_{j=1}^{n} \beta_{ij} I_i,$$

$$\frac{dS_i}{dt} = -\lambda_i S_i$$

$$\frac{dE_i}{dt} = \lambda_i S_i - \alpha E_i$$

$$\frac{dI_i}{dt} = \alpha E_i - \gamma I_i,$$

where γ and α hold the same properties as above. i represents the population for each of the n populations. The value of what was formerly β is now dependent on all the surrounding populations. The equation $\frac{dR}{dt} = \gamma I$; is excluded because it is dependent on I and can be calculated given the value of I. This research is to determine if the exposure rates between the populations can be recovered given a set of data about an epidemic. The goal is to recover all β_{ij}'s given S_0, E_0, I_0, S_T, E_T, and I_T in two and three populations. Section 2 will discuss the computations required to recover the β_{ij} and the setup used. Section 3 will discuss the results found by estimating the calculations. The final section will talk about future work to be done on this problem.

2   Computation

In order to recover β, the equation $J(\beta) = \|F(\beta) - F(\widetilde{\beta})\|_2^2$ needs to be minimized. β represents the beta that is guessed to be the actual value and $\widetilde{\beta}$ represents the value of beta given from the data. The formula for the gradient is:

$$< \delta\beta, D_\beta f(x, \beta) y > = \sum_{i,j=1}^{n} \delta\beta_{ij}$$

where y is the adjoint given by the formula:

$$\begin{cases} \dot{y} = -D_x f(x, \beta) - r(t) \\ y(T) = 0. \end{cases}$$

$r(t)$ represents the difference between the model and the data.

2.1 Two populations

In the two population model, the formulas for each of the groups is:

$$\frac{dS_1}{dt} = -S_1(\beta_{11}I_1 + \beta_{12}I_2) \qquad\qquad \frac{dS_2}{dt} = -S_2(\beta_{21}I_1 + \beta_{22}I_2)$$

$$\frac{dE_1}{dt} = S_1(\beta_{11}I_1 + \beta_{12}I_2) - \alpha E_1 \qquad\qquad \frac{dE_2}{dt} = S_2(\beta_{21}I_1 + \beta_{22}I_2) - \alpha E_2$$

$$\frac{dI_1}{dt} = \alpha E_1 - \gamma I_1, \qquad\qquad \frac{dI_2}{dt} = \alpha E_2 - \gamma I_2.$$

The matrix to simulate the different values of β_{ij} between the populations has the formula:

$$\beta = \begin{bmatrix} \beta & \beta^2 \\ \beta r & \beta^2 r^2 \end{bmatrix}$$

$$\text{with } r \approx 0.13300190127129.$$

$\widetilde{\beta} = .166$ was randomly generated to act as the actual data value for the computer simulation. The values of α and γ were fixed at .11 and .12 respectively. The populations were fixed with the following initial values:

$$S_1(0) = .95 \qquad I_1(0) = .05 \qquad E_1(0) = 0 \qquad R_1(0) = 0$$
$$S_2(0) = .75 \qquad I_2(0) = .25 \qquad E_2(0) = 0 \qquad R_2(0) = 0.$$

In the two population model, the $\frac{\partial F}{\partial \beta}$ computation produces the following matrix:

$$\frac{\partial F}{\partial \beta} = \begin{bmatrix} -S_1 I_1 \delta\beta_{11} & -S_1 I_2 \delta\beta_{12} & 0 & 0 \\ S_1 I_1 \delta\beta_{11} & S_1 I_2 \delta\beta_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -S_2 I_1 \delta\beta_{21} & -S_2 I_2 \delta\beta_{22} \\ 0 & 0 & S_2 I_1 \delta\beta_{21} & S_2 I_2 \delta\beta_{22} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Gâteaux derivative is calculated to determine which direction to take the gradient in using the formula:

$$D_\beta f(x, \beta) = \delta\beta_{11} \int_0^T (-S_1 I_1 y_1 + S_1 I_1 y_2) + \delta\beta_{12} \int_0^T (-S_1 I_2 y_1 + S_1 I_2 y_2)$$

$$+ \delta\beta_{21} \int_0^T (-S_2 I_1 y_4 + S_2 I_1 y_5) + \delta\beta_{22} \int_0^T (-S_2 I_2 y_4 + S_2 I_2 y_5).$$

2.2 Three Populations

In three populations, the matrix to simulate the different values of β_{ij} between the populations has the formula:

$$\beta = \begin{bmatrix} \beta & \beta^2 & \beta^{2/3} \\ \beta r & \beta^2 r^2 & \beta^{2/3} r^{2/3} \\ \beta \hat{r} & \beta^2 \hat{r}^2 & \beta^{2/3} \hat{r}^{2/3} \end{bmatrix}$$

with $r \approx 0.13300190127129$ and $\hat{r} \approx 0.48669634316428$.

$\widetilde{\beta} = .166$ was randomly generated to act as the actual data value for the computer simulation. The values of α and γ were fixed at .11 and .12 respectively. The populations were fixed with the following initial values:

$$S_1(0) = .90 \qquad I_1(0) = .10 \qquad E_1(0) = 0 \qquad R_1(0) = 0$$
$$S_2(0) = .75 \qquad I_2(0) = .25 \qquad E_2(0) = 0 \qquad R_2(0) = 0$$
$$S_3(0) = .85 \qquad I_3(0) = .15 \qquad E_3(0) = 0 \qquad R_3(0) = 0.$$

In the three population model, the $\frac{\partial F}{\partial \beta}$ computation produces the following matrix:

$$\frac{\partial F}{\partial \beta} = \begin{bmatrix} -S_1 I_1 \delta\beta_{11} & -S_1 I_2 \delta\beta_{12} & -S_1 I_3 \delta\beta_{13} & 0 & 0 \\ S_1 I_1 \delta\beta_{11} & S_1 I_2 \delta\beta_{12} & S_1 I_3 \delta\beta_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S_2 I_1 \delta\beta_{21} & -S_2 I_2 \delta\beta_{22} \\ 0 & 0 & 0 & S_2 I_1 \delta\beta_{21} & S_2 I_2 \delta\beta_{22} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{(Columns 1-5)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -S_2 I_3 \delta\beta_{23} & 0 & 0 & 0 \\ S_2 I_3 \delta\beta_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -S_3 I_1 \delta\beta_{31} & -S_3 I_2 \delta\beta_{32} & -S_3 I_3 \delta\beta_{33} \\ 0 & S_3 I_1 \delta\beta_{31} & S_3 I_2 \delta\beta_{32} & S_3 I_3 \delta\beta_{33} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Columns 6-9).}$$

The Gâteaux derivative is calculated to determine which direction to take the gradient in using the formula:

$$D_\beta f(x, \beta) = \delta\beta_{11} \int_0^T (-S_1 I_1 y_1 + S_1 I_1 y_2) + \delta\beta_{12} \int_0^T (-S_1 I_2 y_1 + S_1 I_2 y_2) + \delta\beta_{13} \int_0^T (-S_1 I_3 y_1 + S_1 I_3 y_2)$$

$$+\delta\beta_{21} \int_0^T (-S_2 I_1 y_4 + S_2 I_1 y_5) + \delta\beta_{22} \int_0^T (-S_2 I_2 y_4 + S_2 I_2 y_5) + \delta\beta_{23} \int_0^T (-S_2 I_3 y_4 + S_2 I_3 y_5)$$

$$+\delta\beta_{31} \int_0^T (-S_3 I_1 y_7 + S_3 I_1 y_8) + \delta\beta_{32} \int_0^T (-S_3 I_2 y_7 + S_3 I_2 y_8) + \delta\beta_{33} \int_0^T (-S_3 I_3 y_7 + S_3 I_3 y_8).$$

3 Results

The value of β was recovered by calculating the finite difference:

$$\beta_{t+1} = \beta_t - \tau \bigtriangledown J(\beta_t)$$

where τ is the step size. The gradient was estimated by performing the calculation:

$$\bigtriangledown J(\beta) =< \frac{\partial J}{\partial \beta_{11}}, \frac{\partial J}{\partial \beta_{12}}, \frac{\partial J}{\partial \beta_{21}}, \frac{\partial J}{\partial \beta_{22}} >$$

where, with h being the step size, $\dfrac{\partial J}{\partial \beta_{ij}}(\beta) \approx \dfrac{1}{h}(J(\beta + he_{ij}) - J(\beta))$.

Given good values for τ and h and running the formula for a reasonable period of time, the difference between β_{ij} and $\widetilde{\beta_{ij}}$ is less than or equal to .01. If the experimental value for τ or h is too large, the error greatly increases. τ and h control the step size so large values will create greater fluxuations from the calculated derivative. If the calculations are run for too large a T, the actual β_{ij} will be overshot or undershot depending on the starting value of β. The smaller values of β have the largest errors while the larger values of β have the smallest errors as a result of the maximum difference being .01. If the value of $\beta_{ij} < .01$, then the percent error can be relatively large.

4 Future Work

The recovery of variables was done through estimations in these calculations. The computation of each of the β_{ij} needs to worked out computationally through the calculation of the adjoint. The actual calculations of the adjoint need to be performed and the directional derivative computed. Other possible continuations would be to include more population groups. The increased complexity with a large number of populations could create issues in recovering some of the β_{ij}'s. Also, the model can be desimplied to make the values more realistic. This should include using a realistic set of data instead of computer generated data to see if it recovery of β is possible outside of the idealized case.

5 References

1. Lloyd, A. L. and May, R. M. (1995) Spatial Heterogeneity in Epidemic Models *J. Theor. Biol.* **179**, 1-11