REU Project
Asymptotic Properties of GARCH

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1 Introduction and the existence of GJR–GARCH

It has been observed that the variance of log returns of stock values is not constant and it describes the volatility of the market. Engle (1982) and Bollerslev (1986) introduced the GARCH(1,1) model to model the volatility in time series data. They assume that the log returns satisfy the equations

$$y_k = \sigma_k \epsilon_k, \quad -\infty < k < \infty \tag{1.1}$$

and

$$\sigma_k^2 = \omega + \alpha y_{k-1}^2 + \beta \sigma_{k-1}^2, \quad -\infty < k < \infty, \tag{1.2}$$

where (ω, α, β) is the parameter of the process. It is assumed that the errors (innovations) $\epsilon_k, -\infty < k < \infty$ are independent identically distributed random variables. Nelson (1990) found the necessary and sufficient condition for the existence of $(y_k, \sigma_k^2), -\infty < k < \infty$. Lumsdaine (1996) used the quasi-maximum likelihood method to estimate the parameters.

Glosten, Jagannathan and Runke (1993) modified the GARCH(1,1), giving larger weight to negative returns in the volatility. In the GJR-GARCH(1,1), (1.2) is replaced by

$$\sigma_k^2 = \omega + \alpha_1 y_{k-1}^2 I\{y_{k-1} < 0\} + \alpha_2 y_{k-1}^2 I\{y_{k-1} \geq 0\} + \beta \sigma_{k-1}^2, \tag{1.3}$$

$-\infty < k < \infty$,where $\boldsymbol{\theta} = (\omega, \alpha_1, \alpha_2, \beta)$ is the parameter of the process.

We assme that

$$\omega > 0, \beta \geq 0, \alpha_1 \geq 0 \text{ and } \alpha_2 \geq 0 \tag{1.4}$$

and

$$\epsilon_k \quad -\infty < k < \infty \text{ are independent identically distributed random variables.} \tag{1.5}$$

Using the recursion in (1.3) we get

$$\begin{aligned} \sigma_k^2 &= \omega + \alpha_1 y_{k-1}^2 I\{y_{k-1} < 0\} + \alpha_2 y_{k-1}^2 I\{y_{k-1} \geq 0\} + \beta \sigma_{k-1}^2 \\ &= \omega + \alpha_1 \epsilon_{k-1}^2 \sigma_{k-1}^2 I\{\epsilon_{k-1} < 0\} + \alpha_2 \epsilon_{k-1}^2 \sigma_{k-1}^2 I\{\epsilon_{k-1} \geq 0\} + \beta \sigma_{k-1}^2 \\ &= \omega + \beta \sigma_{k-1}^2 + \eta_{k-1} \sigma_{k-1}^2, \end{aligned} \tag{1.6}$$

where

$$\eta_{k-1} = \alpha_1 \epsilon_{k-1}^2 I\{\epsilon_{k-1} < 0\} + \alpha_2 \epsilon_{k-1}^2 I\{\epsilon_{k-1} \geq 0\}.$$

Using the recursion in (1.3) backwards we get

$$
\begin{aligned}
\sigma_k^2 &= \omega + \eta_{k-1}\sigma_{k-1}^2 \\
&= \omega + \eta_{k-1}(\omega + \eta_{k-2}\sigma_{k-2}^2) \\
&= \omega + \omega\eta_{k-1} + \eta_{k-1}\eta_{k-2}\sigma_{k-2}^2 \\
&= \omega + \omega\eta_{k-1} + \eta_{k-1}\eta_{k-2}(\omega + \eta_{k-3})\sigma_{k-3}^2 \qquad (1.7)\\
&= \omega + \omega\eta_{k-1} + \omega\eta_{k-1}\eta_{k-2} + \eta_{k-1}\eta_{k-2}\eta_{k-3}\sigma_{k-3}^2 \\
&\;\vdots \\
&= \omega + \omega\eta_{k-1} + \omega\eta_{k-1}\eta_{k-2} + \cdots + \eta_{k-1}\cdots\eta_{k-N}\sigma_{k-N}^2.
\end{aligned}
$$

If there is a solution it must be in the form of

$$\sigma_k^2 = \omega\left[1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{k-i}\right].$$

Our first result gives a condition for the existence of σ_k^2.

Theorem 1.1 *If $E\log\eta_0 < 0$, then*

$$P\left\{1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i} < \infty\right\} = 1.$$

PROOF: Let $\gamma = E\log\eta_0$. By the Law of Large Numbers (cf. Durrett (1996)) we have

$$\frac{1}{j}\sum_{i=1}^{j}\log\eta_{-i} \longrightarrow \gamma \text{ almost surely.}$$

The Strong Law of Large Numbers means that there exists a random variable j_0 such that

$$\sum_{i=1}^{j}\log\eta_{-i} < \frac{\gamma}{2}j, \text{ if } j \geq j_0.$$

This yields

$$
\begin{aligned}
\sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i} &= \sum_{j=1}^{\infty}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) \\
&= \sum_{j=1}^{j_0}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) + \sum_{j=j_0}^{\infty}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) \\
&\leq \sum_{j=1}^{j_0}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) + \sum_{j=j_0}^{\infty}\exp\left(\frac{\gamma}{2}j\right).
\end{aligned}
$$

Since $0 < e^{\gamma/2} < 1$ by the convergence of the geometric series we conclude

$$\sum_{j=1}^{\infty}\exp\left(\frac{\gamma}{2}j\right) < \infty,$$

completing the proof of Theorem 1.1.

Using Theorem 1.1 we have a necessary condition for the existence of a uniqe solution of (1.3).

Theorem 1.2 *If* $\gamma = E \log \eta_0 < 0$, *then*

$$\sigma_k^2 = \omega \left[1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{k-i} \right]$$

is the unique stationary solution of GJR–GARCH.

PROOF: We showed that σ_k^2 exists. It is clear that it is a stationary sequence, since it is composed of independent identically distributed random variables. The argument before the proof of the existence of σ_0^2 gives that it is a solution. Alternately, just plug into the equation.

Next we prove the uniqueness. Assume that there exists another solution, τ_k^2 satisfying

$$y_k = \tau_k \epsilon_k$$

and

$$\tau_k^2 = \omega + \eta_{k-1} \tau_{k-1}^2.$$

Using the recursions again we conclude

$$\sigma_0^2 - \tau_0^2 = \eta_{-1}\eta_{-2} \cdots \eta_{-N}(\sigma_{-N}^2 - \tau_{-N}^2).$$

Using again the Law of Large Numbers and the condition $\gamma < 0$ we obtain

$$\eta_{-1}\eta_{-2} \cdots \eta_{-N} \longrightarrow 0 \text{ a.s.}$$

as $N \to \infty$. Since σ_k^2 and τ_k^2 are stationary sequences, σ_{-N}^2 and τ_{-N}^2 are bounded sequences in probability, we get

$$\sigma_0^2 - \tau_0^2 \longrightarrow 0$$

in probability, as $N \to \infty$. This implies $P\{\sigma_0^2 = \tau_0^2\} = 1$.

Conjecture If GJR–GARCH has a unique stationary solution, then $\gamma = E \log \eta_0 < 0$. It follows from our proof of Theorem 1.1 that there is no solution if $\gamma > 0$. We have to consider the case of $\gamma = E \log \eta_0 = 0$ only.

2 The moments of GJR–GARCH

By Theorem 1.2 it is enough to study the moments of

$$X = \sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{-i}.$$

We will use the following well-known inequality (cf. Hardy, Lettlewood and Pólya (1959)):

Minkowski's inequality *Let* $X_1, X_2, \dots$ *be non-negative random variables. If* $EX_i^\nu < \infty$, $1 \le \nu$, *then*

$$\sum_{i=1}^{n} EX_i^\nu \le E\left(\sum_{i=1}^{n} X_i\right)^\nu \le \left(\sum_{i=1}^{n} (EX_i^\nu)^{1/\nu}\right)^\nu$$

and

$$\sum_{i=1}^{\infty} EX_i^\nu \le E\left(\sum_{i=1}^{\infty} X_i\right)^\nu \le \left(\sum_{i=1}^{\infty} (EX_i^\nu)^{1/\nu}\right)^\nu .$$

Using Minkowski's inequality, we find the necessary and sufficient condition for the existence of EX^ν.

Theorem 2.1 *We assume that (1.4) and (1.5) hold and* $E \log \eta_0 < 0$.
(i) If $E\eta_0^\nu < 1$, *then* $EX^\nu < \infty$.
(ii) If $E\eta_0^\nu \ge 1$, *then* $EX^\nu = \infty$.

Proof: By Minkowski's inequality we have

$$E\left(\sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{-i}\right)^\nu \le \left(\sum_{j=1}^{\infty} \left(E\left(\prod_{i=1}^{j} \eta_{-i}\right)^\nu\right)^{1/\nu}\right)^\nu .$$

By condition (1.5) we get

$$E\left(\prod_{i=1}^{j} \eta_{-i}\right)^\nu = E\prod_{i=1}^{j} \eta_{-i}^\nu = \prod_{i=1}^{j} E\eta_{-i}^\nu = (E\eta_0^\nu)^j .$$

Using again the properties of the geometric series we conclude that

$$\sum_{j=1}^{\infty} \left(E\left(\prod_{i=1}^{j} \eta_{-i}\right)^\nu\right)^{1/\nu} = \sum_{j=1}^{\infty} \left((E\eta_0^\nu)^{1/\nu}\right)^j < \infty,$$

completing the proof of (i).

Using the other half of Minowski's inequality, similar arguments yield

$$E\left(\sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{-i}\right)^\nu \ge \sum_{i=1}^{\infty} E\left(\prod_{i=1}^{j}\right)^\nu = \sum_{i=1}^{\infty} (E\eta_0^\nu)^j = \infty,$$

since by assumption $E\eta_0^\nu \ge 1$. Hence (ii) is also proven.

3 Stability of the models

I investigated the stability of the GARCH(1,1) model. If there is no change in the values of the parameters or GARCH(1,1) does not change to a different model at an unknown time, according to the law of large numbers for dependent variables

$$Z_k = \frac{1}{k} \sum_{1 \le i \le k} y_i^2 \to E\sigma_0^2 \text{ a.s.}$$

According to the CUSUM (Cumulative Sums) principle, Z_k should be compared to

$$L_k = \frac{1}{n-k} \sum_{k+1 \le i \le n} y_i^2.$$

Hence I plotted the sequence

$$S_k = |Z_k - L_k|.$$

Figure 7 shows a sample path of S_k. It is easy to see that S_k takes very large values if k is small or large. Let $T_n = \max_{1 \le k < n} S_k$ and s_n be the point where S_k reaches its maximum. I repeated the simulations 250 times resulting in $T_n^{(1)}, \ldots, T_n^{(250)}$ and $s_n^{(1)}, \ldots, s_n^{(250)}$. I also computed the empirical distribution function of $T_n^{(1)}, \ldots, T_n^{(250)}$ defined as

$$F_{250}(t) = \frac{1}{250} \sum_{1 \le i \le 250} I\{T_n^{(i)} \le t\}, \quad 0 \le t < \infty.$$

Figure 8 contains the graph of $F_{250}(t)$. It shows that all $T_n^{(i)}$'s are large and there is little variation between them. I tried other sample sizes and the shape of the empirical distribution function changed very little when 250 was replaced with 50, 100, 150, 200 and 300. Figure 9 shows the clusters of $s_n^{(1)}, \ldots, s_n^{(250)}$. According to the picture the the largest values are reached at the beginning or at the end of the data. Essentially half of the maximum occured at the beginning and the other half at the end. I conjecture that

$$s_n/n \to \xi \text{ in distribution},$$

where

$$P\{\xi = 0\} = P\{\xi = 1\} = \frac{1}{2}.$$

The argument is the following: according to Aue *et al* (2005)

$$T_n = \max_{1 \le k < n} S_k \to \infty \text{ in probability}$$

and for any $\delta > 0$

$$\max_{n\delta \le k < n - n\delta} S_k \to \eta_\delta,$$

where η_δ is some random variable. Hence for any $\delta > 0$

$$P\{s_n < n\delta \text{ or } s_n > n - n\delta\} \to 1.$$

By symmetry

$$P\{s_n < n\delta\} \approx P\{s_n > n - n\delta\}.$$

Thus we have

$$\lim_{n\to\infty} P\{s_n < n\delta\} = \lim_{n\to\infty} P\{s_n > n - n\delta\} = \frac{1}{2}.$$

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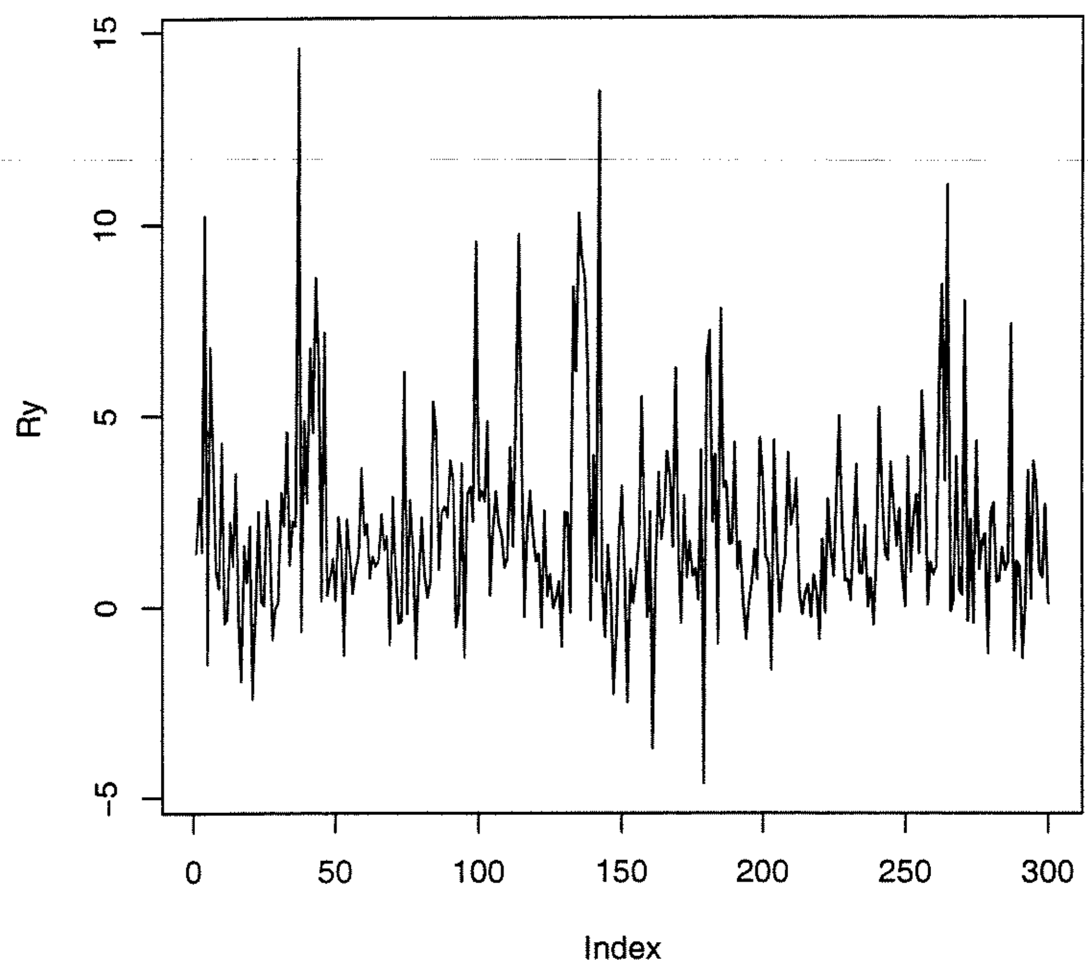

Figure 1

Graph of $y_k, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .2, \alpha_2 = .5, \beta = .3$

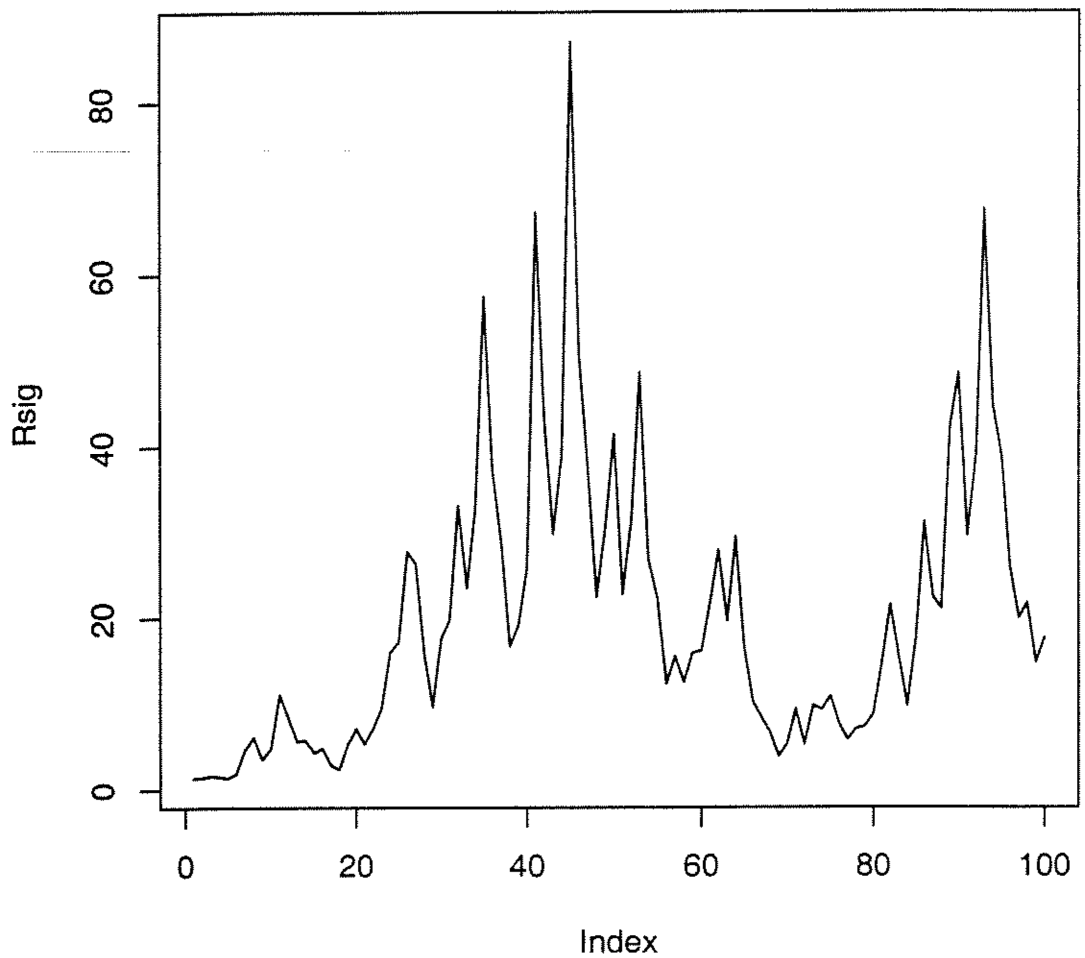

Figure 2

Graph of $\sigma_k^2, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .2, \alpha_2 = .5, \beta = .3$

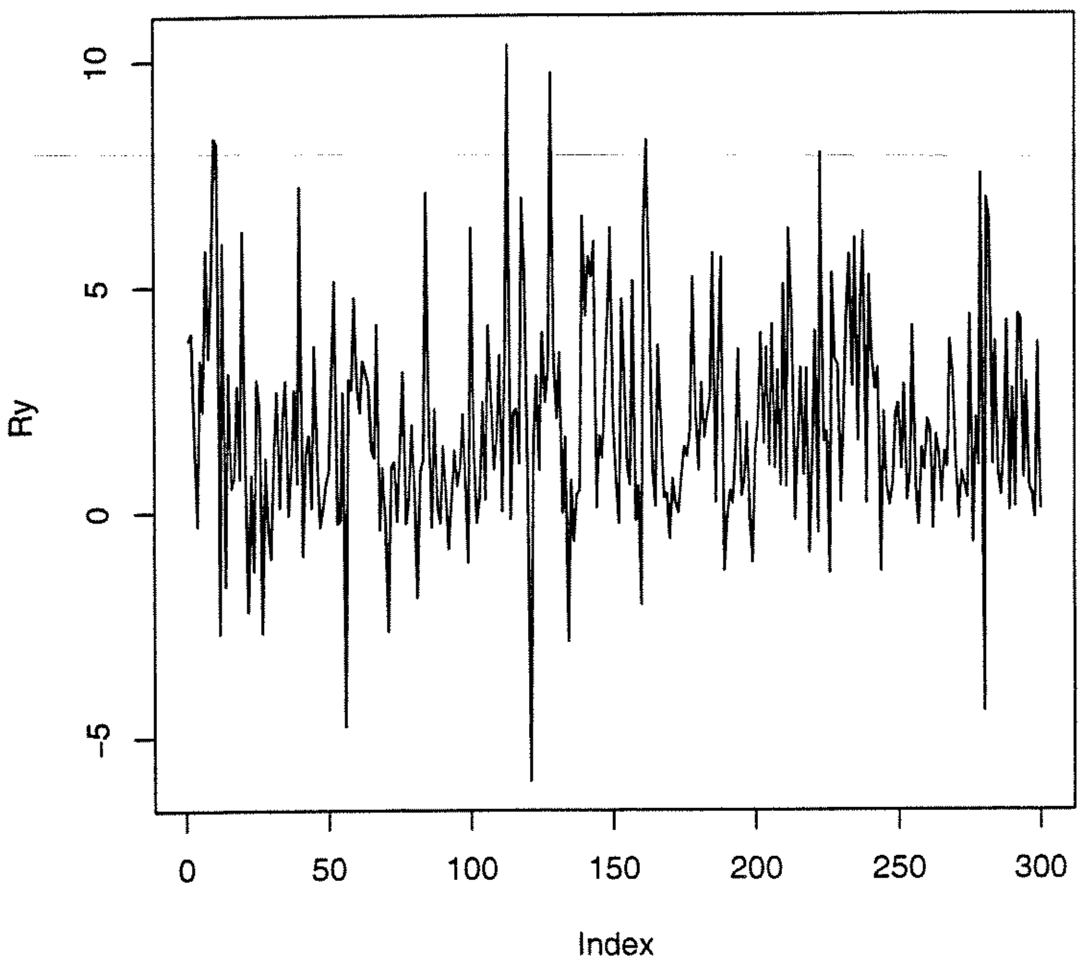

Figure 3

Graph of $y_k, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .5, \alpha_2 = .2, \beta = .3$

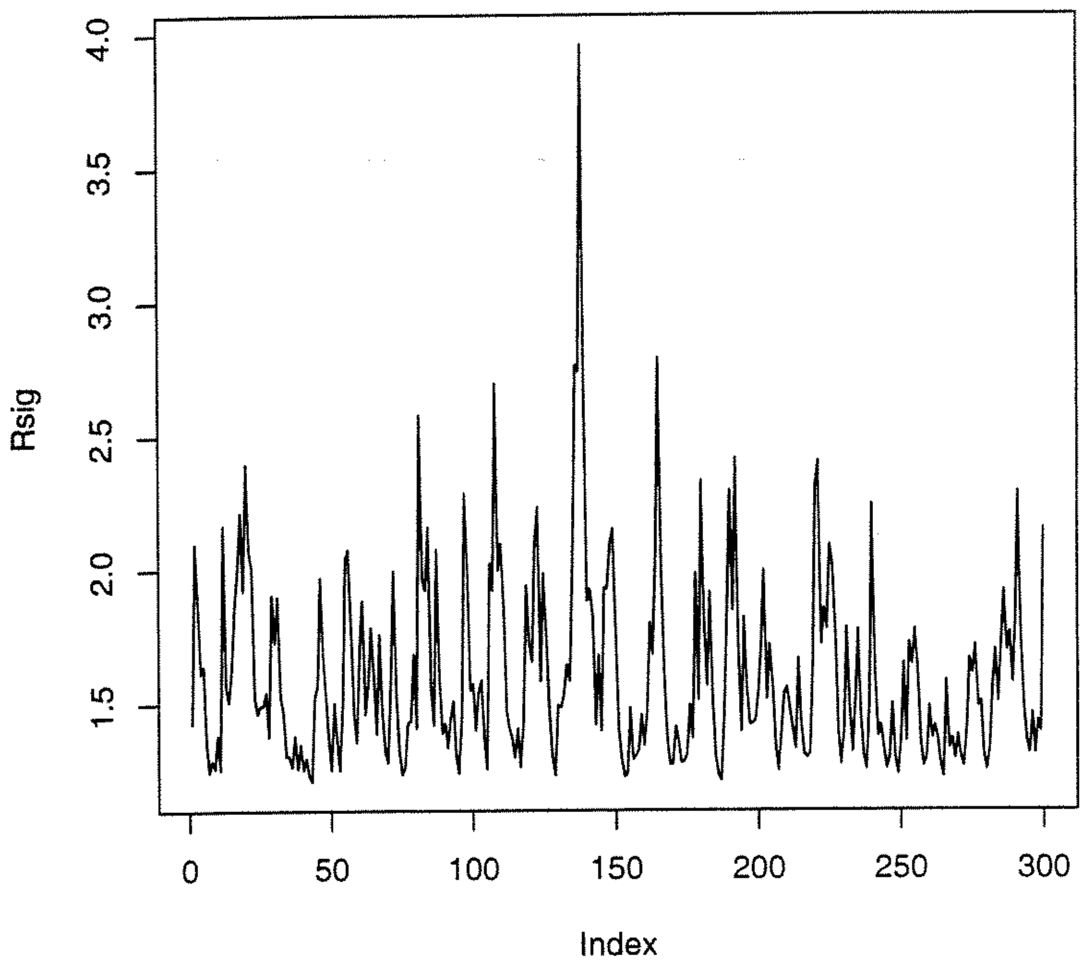

Figure 4

Graph of $\sigma_k^2, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .5, \alpha_2 = .2, \beta = .3$

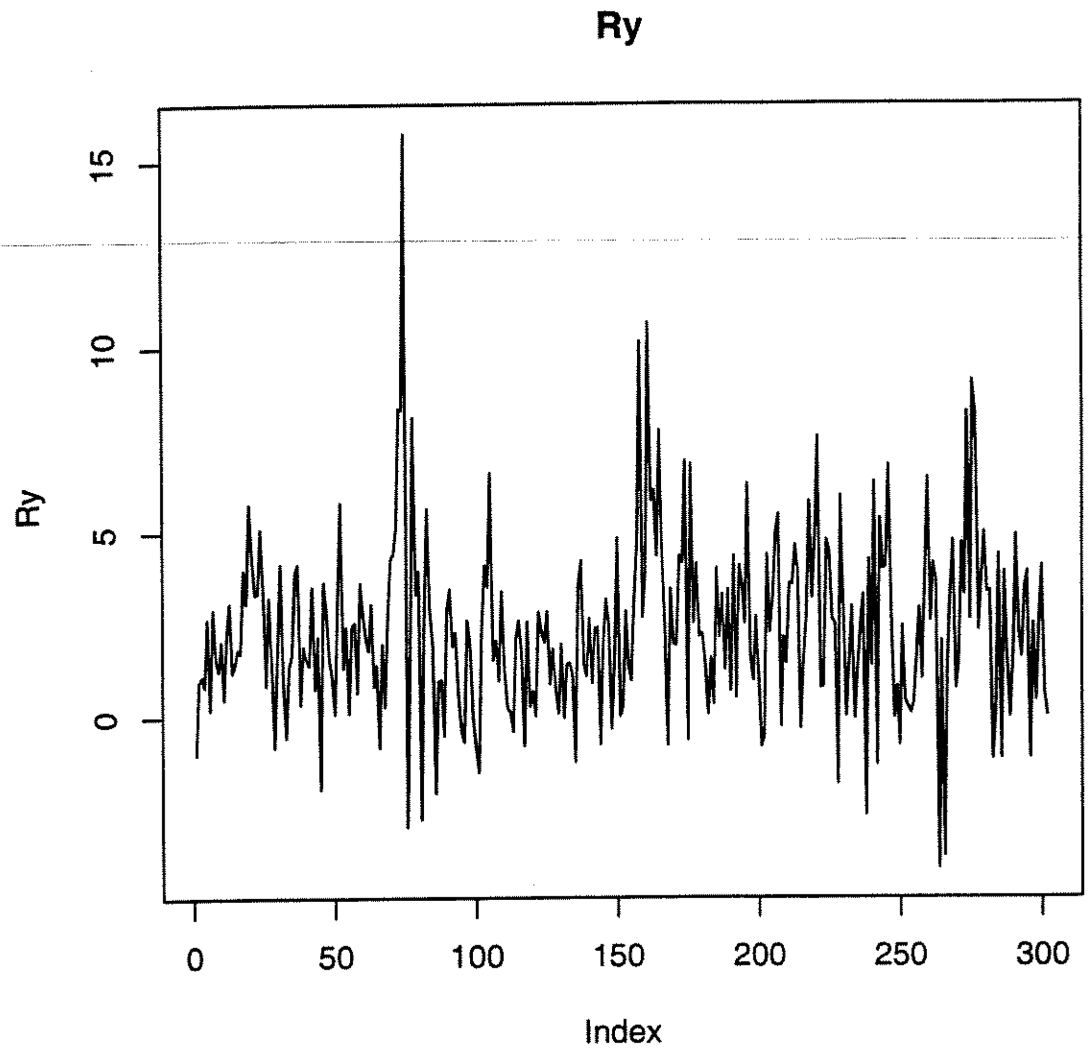

Figure 5

Graph of $y_k, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .5, \alpha_2 = .2, \beta = .3$ in the first 150 observations and $\beta = .5$ in the last 150 observations

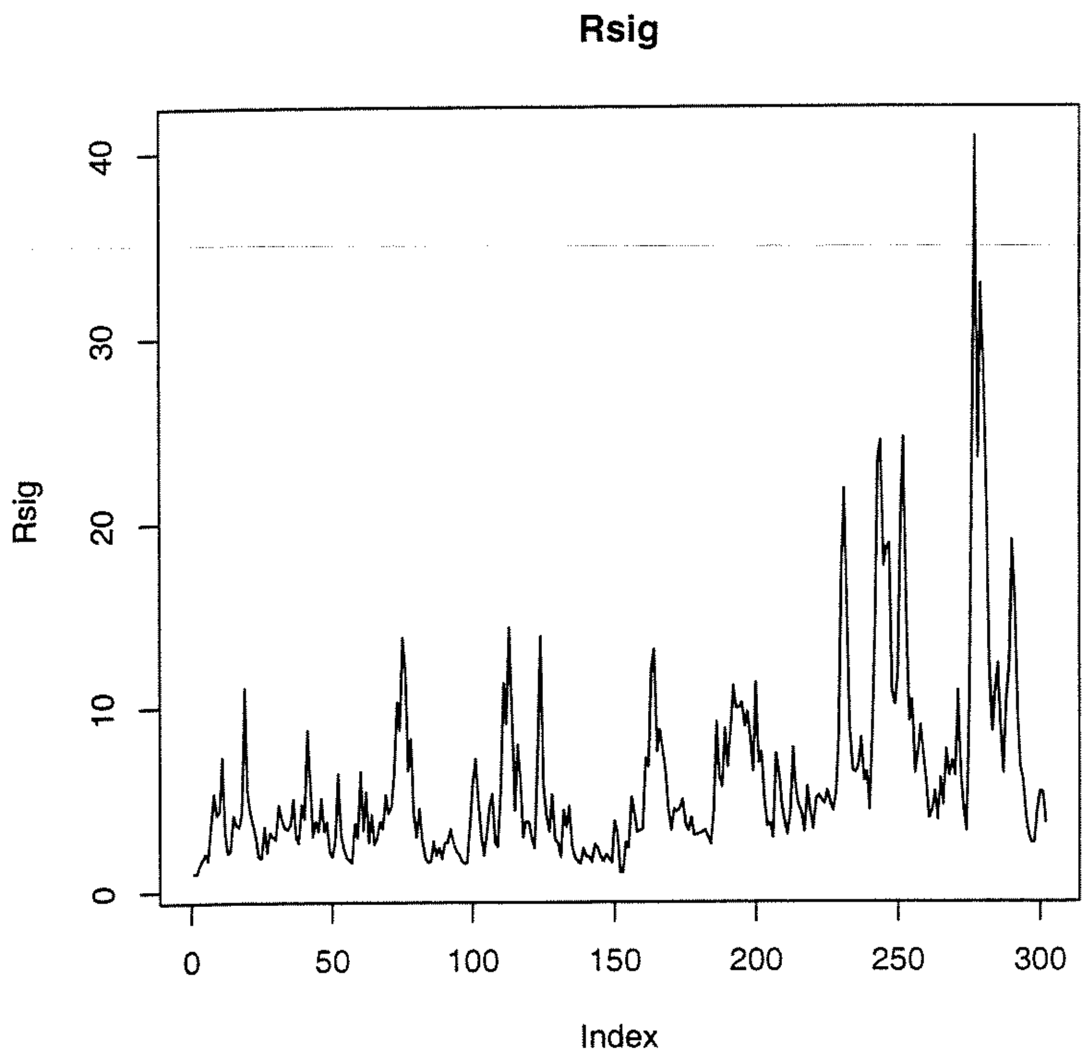

Figure 6

Graph of $\sigma_k^2, 1 \leq k \leq 300$, when $\omega = 1, \alpha_1 = .5, \alpha_2 = .2, \beta = .3$ in the first 150 observations and $\beta = .5$ in the last 150 observations

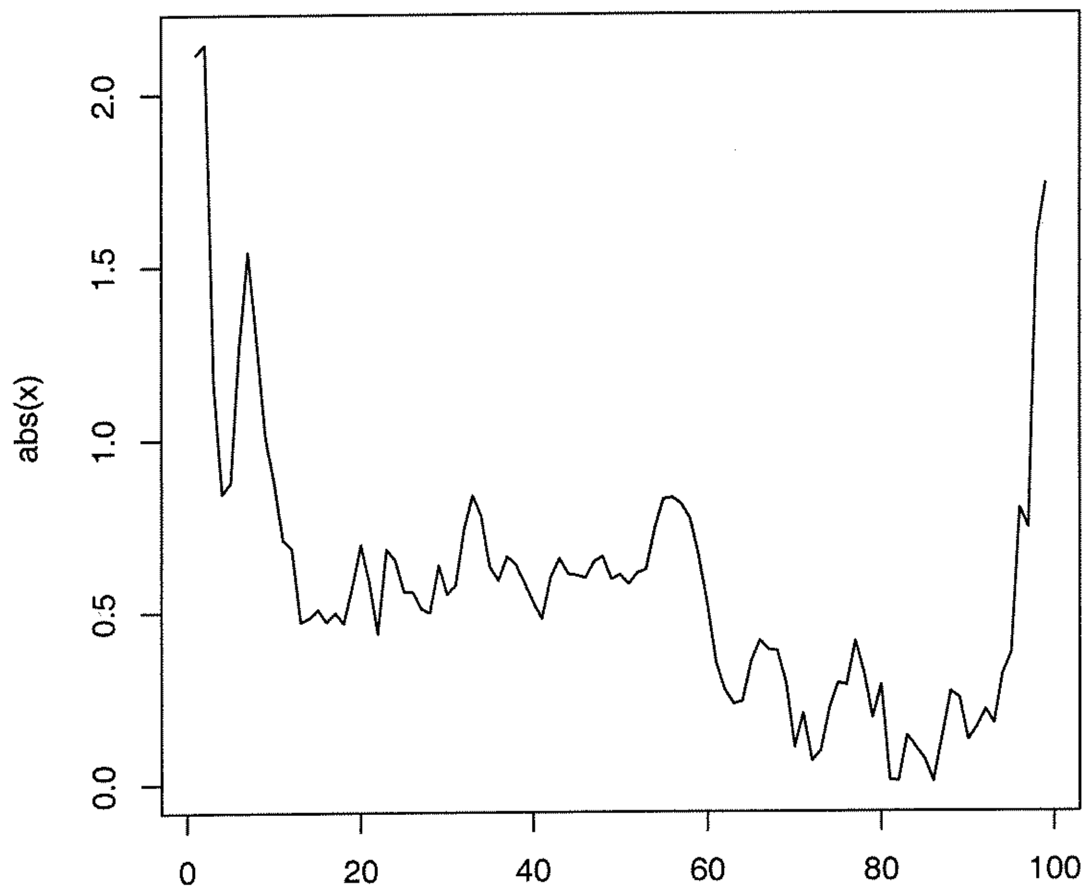

Figure 7

Graph of S_k, $1 \leq k \leq 100$ in case of GARCH(1,1) with $\omega = 1, \alpha = .02$ and $\beta = 0.3$

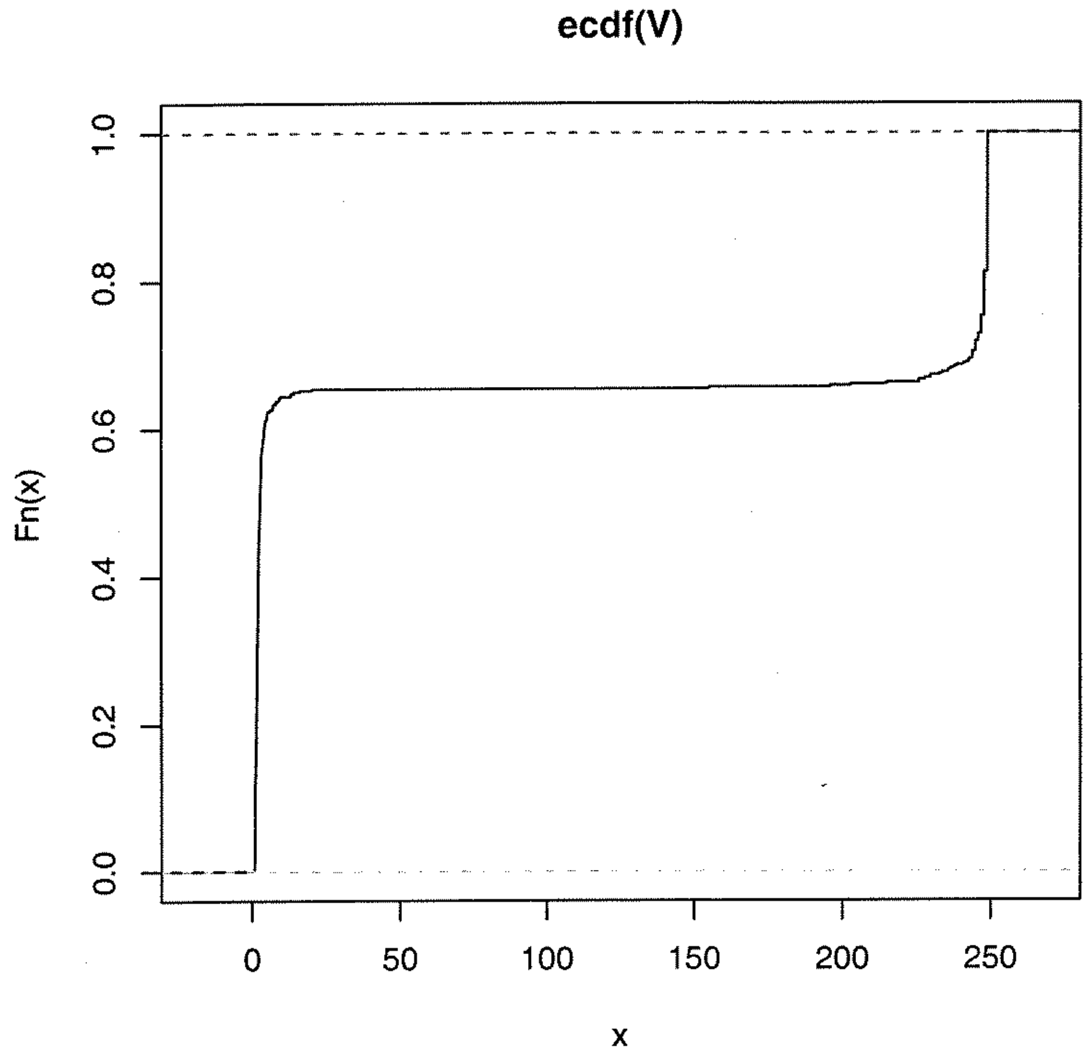

Figure 8

Graph of $F_{250}(t)$ in case of GARCH(1,1) with $\omega = 1, \alpha = .02$ and $\beta = 0.3$

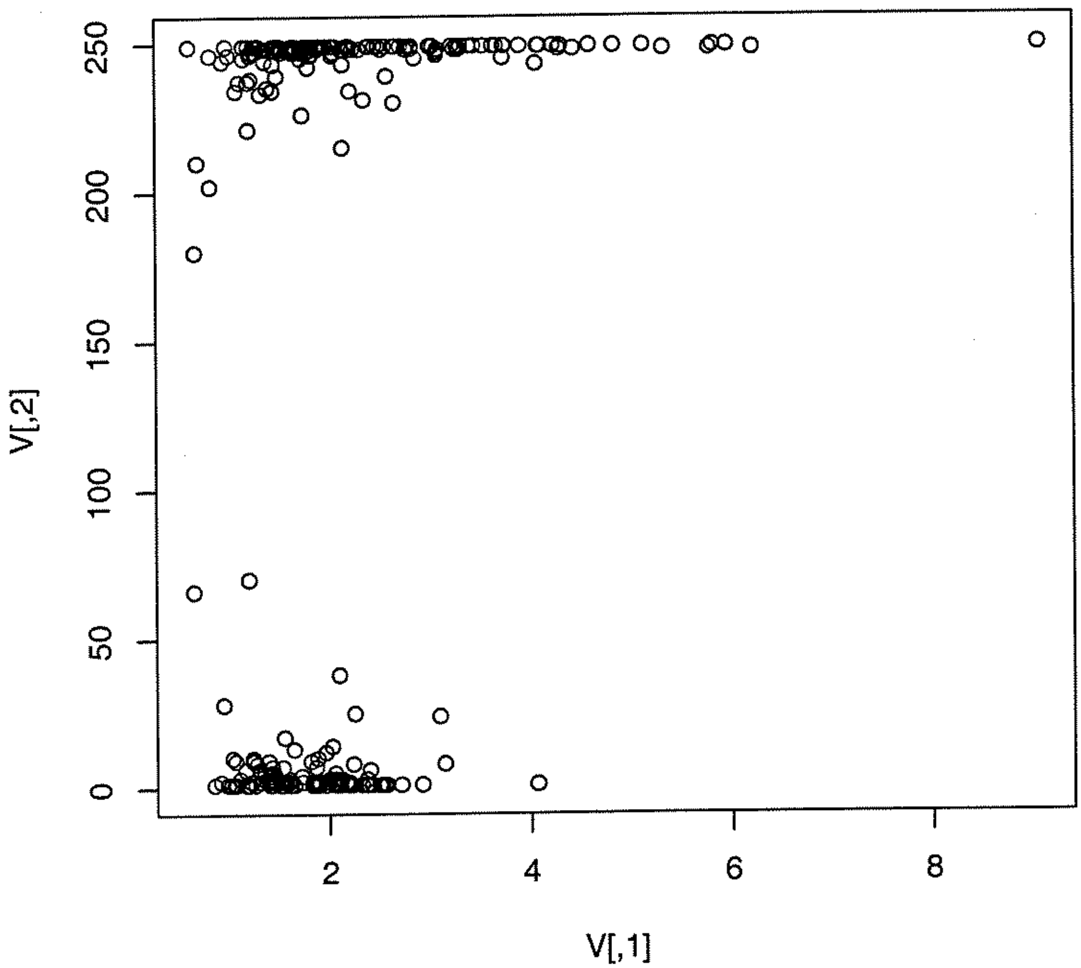

Figure 9

Graph of $s_n(i),\ 1 \leq i \leq 250$ in case of GARCH(1,1) with $\omega = 1, \alpha = .02$ and $\beta = 0.3$