J.evol.equ. 00 (2004) 000 – 000
1424–3199/04/040000 – 00
DOI 10.1007/s00028-004-0153-0

Journal of Evolution Equations

Weak and strong solvability of parabolic variational inequalities in Banach spaces

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Abstract. We consider parabolic variational inequalities having the strong formulation

$$\begin{cases} \langle u'(t),\ v - u(t)\rangle + \langle Au(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq 0, \\ \qquad\qquad \forall v \in V^{**},\ a.e.\ t \geq 0, \end{cases} \tag{1}$$

where $u(0) = u_0$ for some admissible initial datum, V is a separable Banach space with separable dual V^*, $A : V^{**} \to V^*$ is an appropriate monotone operator, and $\Phi : V^{**} \to \mathbb{R} \cup \{\infty\}$ is a convex, weak* lower semicontinuous functional. Well-posedness of (1) follows from an explicit construction of the related semigroup $\{S(t) : t \geq 0\}$. Illustrative applications to free boundary problems and to parabolic problems in Orlicz-Sobolev spaces are given.

1. Introduction

This paper examines constrained evolution problems which can be formulated as parabolic variational inequalities on Banach spaces. Our work relies on a version of the Crandall-Liggett exponential formula, developed in Section 3, which allows an explicit construction of the related semigroup. To motivate the framework described in Section 2, we begin with two examples.

Let Ω denote a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and consider the problem of minimizing

$$\mathcal{F}_p : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \mathcal{F}_p(v) := \int_\Omega |\nabla v|^p \, dx,$$

over a closed convex set $K \subset W_0^{1,p}(\Omega)$, where $1 < p < \infty$. This functional is strictly convex, weakly lower semicontinuous and coercive and therefore has a unique minimizer

Received: □□□□□; accepted: □□□□□.
Mathematics Subject Classifications (2000): □□□□□.
Key words and phrases: □□□□□.
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$u \in K$. In fact, u may be characterized as the solution of

$$\langle Au,\ v-u\rangle \geq 0,\ \forall\, v \in K, \tag{2}$$

where $A : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is the Fréchet derivative of $\mathcal{F}_p$,

$$\langle Au,\ v\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx,\ \forall\, v \in W_0^{1,p}(\Omega).$$

The elliptic variational inequality (2) may be rewritten in the form

$$\langle Au,\ v-u\rangle + \Phi(v) - \Phi(u) \geq 0,\ \forall\, v \in W_0^{1,p}(\Omega) \tag{3}$$

by introducing the indicator functional

$$\Phi : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}, \quad \Phi(v) := \begin{cases} 0, & \text{for } v \in K, \\ \infty, & \text{for } v \notin K. \end{cases}$$

The parabolic version of (3) is to determine $u : [0, T] \to W_0^{1,p}(\Omega)$ which satisfies

$$\langle u'(t),\ v-u(t)\rangle + \langle Au(t),\ v-u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq 0 \tag{4}$$

for all $v \in W_0^{1,p}(\Omega)$ and almost every $t \in (0, T)$, such that $u(0) = u_0$ for some admissible initial value u_0. The pairing involving $u'(t)$ will make sense if $u'(t) \in W^{-1,q}(\Omega)$ for a.e. t. Moreover, if $p \geq \frac{2N}{N+2}$, then

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,q}(\Omega),$$

and it follows that $u \in C(0, T; L^2(\Omega))$ if $u \in L^p(0, T; W_0^{1,p}(\Omega))$ and $u' \in L^q(0, T; W^{-1,q}(\Omega))$ ([18]). Under such assumptions, it is reasonable to prescribe $u_0 \in L^2(\Omega)$; we elaborate on the class of admissible initial data below.

Many choices are possible for the constraint functional Φ in (4). For example, if $p = 2$ and Φ is the indicator functional of the closed convex set

$$K := \{v \in H_0^1(\Omega) : v \geq 0 \quad \text{on} \quad \partial\Omega\},$$

then (4) models the filtration of liquid through an earthen dam ([11]). If K consists of functions which are nonnegative on $\partial\Omega$, we obtain a model of fluid flow in a domain with a semipermeable boundary ([2], [6]). We are interested in more general values of p (among other things), and we focus on mathematical issues rather than physical interpretations. Finally, note that Φ could be the trivial functional, in which case (4) is an equation.

As our second example, consider the minimization of

$$\mathcal{F}_M : W_0^1 L_M(\Omega) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{F}_M(v) := \int_\Omega M(|\nabla v|)\, dx,$$

where M is a Young function and $W_0^1 L_M(\Omega)$ is the corresponding Orlicz-Sobolev space ([1], [10], [12]). This function space is neither separable nor reflexive in general. If the conjugate $\overline{M}$ of M satisfies a Δ_2 condition, however, then $W_0^1 L_M(\Omega)$ is the double dual of the separable space $W_0^1 E_M(\Omega)$, whose dual is also separable ([1], [10]). A concrete example of such a Young function is

$$M(t) = \exp(t^p) - 1, \quad \text{for} \quad p > 1, \tag{5}$$

and problems in function spaces of this sort motivate the somewhat unusual structural framework described in Section 2.

In contrast to $\mathcal{F}_p$, $\mathcal{F}_M$ is not finite on all of its domain. From the variational point of view, though, $\mathcal{F}_M$ is still wonderful, as it is convex, coercive and lower semicontinuous with respect to the weak* topology of $W_0^1 L_M(\Omega)$ ([13]). $\mathcal{F}_M$ therefore has minimizers u which satisfy

$$0 \in \partial \mathcal{F}_M(u),$$

where $\partial \mathcal{F}_M$ denotes the subdifferential of $\mathcal{F}_M$. We may then consider the evolution problem

$$-u'(t) \in \partial \mathcal{F}_M(u(t)), \quad \text{for} \quad t > 0, \tag{6}$$

which is equivalent to finding $u : [0, T] \to W_0^1 L_M(\Omega)$ such that

$$\langle u'(t), v - u(t) \rangle + \mathcal{F}_M(v) - \mathcal{F}_M(u(t)) \geq 0 \tag{7}$$

for all $v \in W_0^1 L_M(\Omega)$ and a.e. $t > 0$. An initial condition $u(0) = u_0$ is also prescribed.

The parabolic variational inequalities (4) and (7) are special cases of (13) below. Section 2 clarifies the types of solutions we consider as well as the proper class of admissible initial data. Section 3 defines the associated resolvent J_λ and establishes the exponential formula which defines the map $S(t)$ for $t \geq 0$. This formula suggests that $\{S(t) : t \geq 0\}$ should be a semigroup and that $S(t)u_0$ should be the unique solution, in some sense, of the corresponding parabolic variational inequality with initial condition u_0. We address these issues in Section 4, where we show that $S(t)u_0$ is the unique strong solution if u_0 is sufficiently smooth and is the unique weak solution otherwise. The semigroup property follows from unique solvability, and we conclude with some applications in Section 5.

2. Preliminaries

We refer to, e.g., [7], [14], [16] and [18] for definitions of terms used here. Let V be a separable Banach space whose dual V^* is separable, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between V^* and V^{**}. Let H be a Hilbert space which embeds continuously into V^*; identifying H with its dual yields

$$V^{**} \hookrightarrow H \hookrightarrow V^*.$$

Let $A : V^{**} \to V^*$ be a bounded hemicontinuous operator which satisfies

$$\langle Au - Av,\ u - v\rangle \geq \omega \|u - v\|_H^2,\ \forall\, u, v \in V^{**}, \tag{8}$$

for some nonnegative monotonicity constant $\omega \geq 0$, let $\Phi : V^{**} \to \mathbb{R} \cup \{\infty\}$ be a convex, proper and weak* lower semicontinuous functional, and assume that A and Φ satisfy the coercivity condition

$$\lim_{\|v\|_{V^{**}} \to \infty} \frac{\langle Av,\ v - v_0\rangle + \Phi(v)}{\|v - v_0\|_{V^{**}}} = \infty, \tag{9}$$

for some $v_0 \in D(\Phi)$, the effective domain of Φ. Finally, for each $u \in D(\Phi)$, suppose that there exists a sequence $\{u_n\} \subseteq D(\Phi)$ such that

$$u_n \to u \quad \text{in} \quad H, \quad Au_n \in H, \quad \text{and} \quad \partial\Phi(u_n) \cap H \neq \emptyset. \tag{10}$$

For convenience, we therefore define the set

$$\mathfrak{D} := \{v \in D(\Phi) \ :\ Av \in H \quad \text{and} \quad \partial\Phi(v) \cap H \neq \emptyset\}; \tag{11}$$

note that $\overline{D(\Phi)}^{\|\cdot\|_H} = \overline{\mathfrak{D}}^{\|\cdot\|_H}$. We have the following basic existence result, a detailed proof of which is given in [16] (available upon request).

THEOREM 2.1. *For each $f \in V^*$, there exists $u \in V^{**}$ such that*

$$\langle Au,\ v - u\rangle + \Phi(v) - \Phi(u) \geq 0,\ \forall\, v \in V^{**}. \tag{12}$$

If A is strictly monotone, then the solution of (12) *is unique.*

For a prescribed initial condition $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$ and a given final time $T > 0$, we consider the following two types of solutions.

DEFINITION 2.2. A function $u : [0, T] \longrightarrow V^{**}$ is a *strong solution* of the parabolic variational inequality associated to A and Φ with initial value u_0 if

(i) $u \in L^\infty(0, T; V^{**}) \cap C^{0,1}(0, T; H)$,
(ii) $u(0) = u_0$, and
(iii) u satisfies

$$\langle u'(t),\ v - u(t)\rangle + \langle Au(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq 0, \quad \text{for all } v \in V^{**} \text{ and a.e. } t \in (0, T). \tag{13}$$

Strong solutions are Lipschitz continuous maps into H and are thus differentiable almost everywhere. We will find that strong solutions exist for initial data in $\mathfrak{D}$. For arbitrary initial data in $\overline{D(\Phi)}^{\|\cdot\|_H}$, we will need

DEFINITION 2.3. A function $u : [0, T] \to V^{**}$ is a *weak solution* of the parabolic variational inequality associated to A and Φ with initial value u_0 if

(i) $u \in C(0, T; H)$,
(ii) $u(0) = u_0$, and
(iii) There exists a sequence u_n of strong solutions of the parabolic variational inequality associated to A and Φ with initial values $u_{n,0}$ such that $u_{n,0} \to u_0$ in H and $u_n \to u$ in $C(0, T; H)$.

3. The exponential formula

Consider the strong formulation (13) with a fixed $t > 0$, and let $\lambda > 0$ be a time step such that $\lambda < t$. To approximate (13), replace the time derivative $u'(t)$ with the usual difference quotient and consider the elliptic inequality

$$\left\langle \frac{u(t) - u(t-\lambda)}{\lambda}, v - u(t) \right\rangle + \langle Au(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq 0,$$

which must hold for all $v \in V^{**}$ and in which $u(t-\lambda)$ is a known element of H. For a given $x \in H$, we therefore seek a solution $u \in V^{**}$ of

$$\left\langle \frac{u - x}{\lambda}, v - u \right\rangle + \langle Au,\ v - u\rangle, +\Phi(v) - \Phi(u)\ \geq 0,\ \forall\, v \in V^{**}. \tag{14}$$

PROPOSITION 3.1. *For each $x \in H$,* (14) *has a unique solution $u \in D(\Phi)$, and the resulting solution operator $J_\lambda : H \to D(\Phi)$ is Lipschitz with respect to the norm of H,*

$$\|J_\lambda x - J_\lambda y\|_H \leq \frac{1}{1 + \lambda\omega}\, \|x - y\|_H, \quad for \quad x, y \in H. \tag{15}$$

Proof. Direct calculations show that the operator $A_{x,\lambda} : V^{**} \to V^*$, defined by $A_{x,\lambda} u := \frac{1}{\lambda}(u - x) + Au$, satisfies the hypotheses of Theorem 2.1 and is strictly monotone. The Lipschitz continuity of J_λ follows from standard techniques ([11], [16]). □

The solution operator J_λ defined by Proposition 3.1 is the *resolvent map* associated to (13). The following lemma records the other basic properties of the resolvent; part (iii) is the *resolvent identity*.

LEMMA 3.2. *Let $\lambda > 0$ and $x \in H$.*

(i) *If $x \in \mathfrak{D}$, then* $\|J_\lambda x - x\|_H \leq \frac{\lambda}{1+\lambda\omega} (\|Ax\|_H + \inf_{y \in \partial\Phi(x) \cap H} \|y\|_H)$.
(ii) *If n is a positive integer, then* $\|J_\lambda^n x - x\|_H \leq n\, \|J_\lambda x - x\|_H$.
(iii) *For any μ such that $\lambda \geq \mu > 0$,* $J_\lambda x = J_\mu(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x)$.

Proof. By definition of the resolvent, $J_\lambda x$ satisfies

$$\left\langle \frac{J_\lambda x - x}{\lambda}, v - J_\lambda x \right\rangle + \langle A J_\lambda x,\ v - J_\lambda x\rangle + \Phi(v) - \Phi(J_\lambda x)\ \geq 0,$$

for all $v \in V^{**}$. Substitute $v = x$ into this inequality, then use the definition of $\mathfrak{D}$ and the monotonicity of A to obtain

$$\begin{aligned}\frac{1}{\lambda}\|J_\lambda x - x\|_H^2 &\leq \langle A J_\lambda x,\ x - J_\lambda x\rangle + \Phi(x) - \Phi(J_\lambda x)\\ &\leq \langle A J_\lambda x - Ax,\ x - J_\lambda x\rangle + \langle Ax,\ x - J_\lambda x\rangle + \langle y,\ x - J_\lambda x\rangle\\ &\leq -\omega\|J_\lambda x - x\|_H^2 + (\|Ax\|_H + \|y\|_H)\|x - J_\lambda x\|_H,\end{aligned}$$

where y is any element of $\partial\Phi(x) \cap H$. Estimate (i) follows.

Using the Lipschitz continuity of the resolvent, we have

$$\|J_\lambda^n x - x\|_H = \left\| \sum_{j=0}^{n-1} (J_\lambda^{n-j} x - J_\lambda^{n-j-1} x) \right\|_H \leq \sum_{j=0}^{n-1} \left(\frac{1}{1+\lambda\omega}\right)^{n-j-1} \|J_\lambda x - x\|_H\,,$$

and (ii) follows since $\frac{1}{1+\lambda\omega} \leq 1$.

To verify (iii), define $y := \frac{\lambda-\mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x$. By definition, $J_\mu y$ satisfies

$$\left\langle \frac{J_\mu y - y}{\mu},\ v - J_\mu y \right\rangle + \langle A J_\mu y,\ v - J_\mu y\rangle + \Phi(v) - \Phi(J_\mu y)\ \geq 0,\ \forall\, v \in V^{**}.$$

For any $v \in V^{**}$, we now calculate:

$$\begin{aligned}&\left\langle \frac{J_\lambda x - y}{\mu},\ v - J_\lambda x \right\rangle + \langle A J_\lambda x,\ v - J_\lambda x\rangle + \Phi(v) - \Phi(J_\lambda x)\\ &= \frac{1}{\mu}\left\langle J_\lambda x - \frac{\lambda-\mu}{\lambda} J_\lambda x - \frac{\mu}{\lambda} x,\ v - J_\lambda x \right\rangle + \langle A J_\lambda x,\ v - J_\lambda x\rangle + \ \Phi(v) - \Phi(J_\lambda x)\\ &= \left\langle \frac{J_\lambda x - x}{\lambda},\ v - J_\lambda x \right\rangle + \langle A J_\lambda x,\ v - J_\lambda x\rangle + \Phi(v) - \Phi(J_\lambda x)\ \geq 0,\end{aligned}$$

by definition of $J_\lambda x$. Consequently, $J_\lambda x$ solves the variational inequality which defines $J_\mu y$; by uniqueness, it follows that $J_\lambda x = J_\mu y$. □

Lemma 3.3 is a slight modification of a result in [4], and Lemma 3.4 comes directly from [4].

LEMMA 3.3. *Let $n \geq m > 0$ be integers, and let $\lambda \geq \mu > 0$. For $x \in H$,*

$$\|J_\mu^n x - J_\lambda^m x\|_H \leq (1+\mu\omega)^{-n} \sum_{j=0}^{m-1} \binom{n}{j} \alpha^j \beta^{n-j} \|J_\lambda^{m-j} x - x\|_H + \sum_{j=m}^{n} (1+\mu\omega)^{-j} \binom{j-1}{m-1} \alpha^m \beta^{j-m} \|J_\mu^{n-j} x - x\|_H,$$

where $\alpha = \dfrac{\mu}{\lambda}$ and $\beta = \frac{\lambda-\mu}{\lambda}$.

LEMMA 3.4. *Let $n \geq m > 0$ be integers, and let α and β be positive numbers such that $\alpha + \beta = 1$. Then*

$$\sum_{j=0}^{m} \binom{n}{j} \alpha^j \beta^{n-j} (m-j) \leq \sqrt{(n\alpha - m)^2 + n\alpha\beta}, \quad \text{and}$$

$$\sum_{j=m}^{n} \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n-j) \leq \sqrt{\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2}.$$

We can now establish the main results of this section, including the exponential formula (16).

THEOREM 3.5. *Let x and y be elements of $\mathfrak{D}$. For $t \geq 0$, the limit*

$$\tilde{S}(t)x := \lim_{n\to\infty} J_{t/n}^n x \tag{16}$$

exists, relative to the strong topology of H, and the map $\tilde{S} : [0,\infty) \times \mathfrak{D} \to H$ has the following properties:

(i) *$\tilde{S}(\cdot)x$ is Lipschitz continuous:*

$$\|\tilde{S}(t)x - \tilde{S}(\tau)x\|_H \leq 2C|t-\tau|,$$

where $C = C(x) = \|Ax\|_H + \inf_{y\in\partial\Phi(x)\cap H} \|y\|_H$.

(ii) *$\tilde{S}(t)$ is a contraction:*

$$\|\tilde{S}(t)x - \tilde{S}(t)y\|_H \leq e^{-\omega t}\|x-y\|_H.$$

Proof. Let $\lambda \geq \mu > 0$ and $n \geq m > 0$. Combining the estimates from Lemma 3.3 and Lemma 3.2 (ii), (iii), we obtain

$$\|J_\mu^n x - J_\lambda^m x\|_H \leq C\left((1+\mu\omega)^{-n}\frac{\lambda}{1+\lambda\omega}\sum_{j=0}^{m}\binom{n}{j}\alpha^j\beta^{n-j}(m-j)\right.$$
$$\left.+\frac{\mu}{1+\mu\omega}\sum_{j=m}^{n}(1+\mu\omega)^{-j}\binom{j-1}{m-1}\alpha^m\beta^{j-m}(n-j)\right),$$

where $C := \|Ax\|_H + \inf_{y\in\partial\Phi(x)\cap H}\|y\|_H$, $\alpha := \frac{\mu}{\lambda}$, and $\beta := \dfrac{\lambda-\mu}{\lambda}$. As μ and λ are positive, negative powers of $(1+\mu\omega)$ and $(1+\lambda\omega)$ are bounded above by 1. Thus, we obtain the simpler inequality

$$\|J_\mu^n x - J_\lambda^m x\|_H \;\leq\; C\left(\lambda\sum_{j=0}^{m}\binom{n}{j}\alpha^j\beta^{n-j}(m-j)\right.$$
$$\left.+\mu\sum_{j=m}^{n}\binom{j-1}{m-1}\alpha^m\beta^{j-m}(n-j)\right).$$

It then follows from Lemma 3.4 that

$$\|J_\mu^n x - J_\lambda^m x\|_H \;\leq\; C\left(\lambda[(n\alpha-m)^2+n\alpha\beta]^{1/2}\right.$$
$$\left.+\mu\left[\frac{m\beta}{\alpha^2}+\left(\frac{m\beta}{\alpha}+m-n\right)^2\right]^{1/2}\right),$$

and, after substituting the values of α and β, we see that the quantity in parentheses equals

$$[(n\mu-m\lambda)^2+n\mu(\lambda-\mu)]^{1/2}+[m\lambda(\lambda-\mu)+(m\lambda-n\mu)^2]^{1/2}. \tag{17}$$

For a given $t > 0$, we now put $\lambda = \frac{t}{m}$ and $\mu = \frac{t}{n}$ in (17) and obtain

$$\|J_{t/n}^n x - J_{t/m}^m x\|_H \leq 2Ct\left[\frac{1}{m}-\frac{1}{n}\right]^{1/2}. \tag{18}$$

Consequently, $\{J_{t/n}^n x\}$ is a Cauchy sequence in H, and we define

$$\tilde{S}(t)x := \lim_{n\to\infty} J_{t/n}^n x.$$

Since $(1+\omega t/n)^{-n}$ is the Lipschitz constant for $J_{t/n}^n$ and

$$\lim_{n\to\infty}\left(1+\frac{\omega t}{n}\right)^{-n} = e^{-\omega t},$$

$\tilde{S}(t)$ has $e^{-\omega t}$ as its Lipschitz constant, i.e.,

$$\|\tilde{S}(t)x - \tilde{S}(t)y\|_H \le e^{-\omega t}\|x - y\|_H.$$

To see that $\tilde{S}(\cdot)x$ is Lipschitz continuous, let $t > \tau > 0$, put $\lambda = \tau/n$ and $\mu = t/n$ in (17) with $m = n$, and let $n \to \infty$ to find that

$$\|\tilde{S}(t)x - \tilde{S}(\tau)x\|_H \le 2C(t - \tau).$$

□

The approximation hypothesis (10) allows the unique extension of $\tilde{S}$ to $S : [0, \infty) \times \overline{D(\Phi)}^{\|\cdot\|_H} \to H$. The next result shows that $S(\cdot)x$ is continuous for each $x \in \overline{D(\Phi)}^{\|\cdot\|_H}$ and that $S(t)$ is a contraction for each $t \ge 0$. However, $S(\cdot)x$ is not Lipschitz continuous for an arbitrary $x \in \overline{D(\Phi)}^{\|\cdot\|_H}$. This distinction between $\tilde{S}$ and S is the reason that we obtain strong solutions for smooth initial data but only have weak solutions for arbitrary initial data.

COROLLARY 3.6. *The map* $\tilde{S} : [0, \infty) \times \mathfrak{D} \to H$ *constructed in Theorem* 3.5 *extends uniquely to* $S : [0, \infty) \times \overline{D(\Phi)}^{\|\cdot\|_H} \to H$*, which satisfies:*

(i) *For each* $x \in \overline{D(\Phi)}^{\|\cdot\|_H}$*,* $S(\cdot)x$ *is continuous.*
(ii) *For* $x, y \in \overline{D(\Phi)}^{\|\cdot\|_H}$ *and* $t \ge 0$*,* $S(t)$ *is a contraction:*

$$\|S(t)x - S(t)y\|_H \le e^{-\omega t}\|x - y\|_H.$$

Proof. Let $x, y \in \overline{D(\Phi)}^{\|\cdot\|_H}$ be given, and let $\{x_n\}$ and $\{y_n\}$ be approximating sequences from $\mathfrak{D}$ for x and y, respectively, as in (10). As $\{x_n\}$ is a Cauchy sequence in H, it follows that $\{\tilde{S}(t)x_n\}$ is a Cauchy sequence in H, since

$$\|\tilde{S}(t)x_n - \tilde{S}(t)x_m\|_H \le \|x_n - x_m\|_H.$$

We then define $S(t)x$ by the strong H-limit

$$S(t)x := \lim_{n\to\infty} \tilde{S}(t)x_n.$$

This extension is clearly unique.

Let $\varepsilon > 0$ be given. For $t, \tau \ge 0$, we have

$$\begin{aligned}\|S(t)x - S(\tau)x\|_H &\le \|S(t)x - \tilde{S}(t)x_n\|_H + \|\tilde{S}(t)x_n - \tilde{S}(\tau)x_n\|_H \\ &\quad + \|\tilde{S}(\tau)x_n - S(\tau)x\|_H \\ &\le \|S(t)x - \tilde{S}(t)x_n\|_H + C|t - \tau| \\ &\quad + \|\tilde{S}(\tau)x_n - S(\tau)x\|_H,\end{aligned}$$

where $C = C(x_n)$ is the Lipschitz constant of $\tilde{S}(\cdot)x_n$. The right-hand side will clearly be less than ε for n sufficiently large and $|t - \tau|$ sufficiently small, verifying the continuity of $S(\cdot)x$. A similar estimate of $\|S(t)x - S(t)y\|_H$ shows that $S(t)$ is a contraction for $t \ge 0$.
□

4. Strong and weak solutions

To verify that $S(t)u_0$ belongs to $D(\Phi)$, this section analyzes the properties of the approximations $J^n_{t/n}u_0$ and the difference quotients appearing in (14). Along the way, we obtain the results needed to prove Theorem 4.5 and Corollary 4.6, which assert the existence of a unique strong or weak solution, respectively, of the parabolic variational inequality associated to A and Φ, depending on the smoothness of the initial datum u_0. These results imply that $\{S(t) : t \geq 0\}$ is a semigroup of nonlinear contractions on $\overline{D(\Phi)}^{\|\cdot\|_H}$.

LEMMA 4.1. *Let $u_0 \in \mathfrak{D}$. The function $u : [0, \infty) \to H$ defined by $u(t) := S(t)u_0$ is differentiable almost everywhere and satisfies $u' \in L^p(0, T; H)$ for $1 \leq p \leq \infty$ and any $T > 0$.*

Proof. Theorem 3.5 shows that $u(t)$ is Lipschitz continuous in t. Combine this with the results in Section 1.4 of [3]. □

As discussed in [3], it follows that, for almost every $t > 0$ and $u_0 \in \mathfrak{D}$,

$$u(t) = u_0 + \int_0^t u'(s)\, ds. \tag{19}$$

LEMMA 4.2. *Let $u_0 \in \mathfrak{D}$, and define $u(t) := S(t)u_0$. For almost every $t > 0$, the sequence $\{\frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n}\}$ has a subsequence which converges weakly in H to $u'(t)$.*

Proof. Let t^* be a point where $u(t)$ is differentiable. For each n, partition the interval $[0, 2t^*]$ with the points $t_i := it^*/n$, for $i = 0, 1, \ldots, 2n$. Associate to this partition the piecewise linear function

$$u_n(t) := \begin{cases} J^{i-1}_{t^*/n}u_0 + \left(\frac{t-t_{i-1}}{t^*/n}\right)(J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0), \\ \quad \text{for } t_{i-1} \leq t \leq t_i,\ 0 \leq i \leq n-1, \\ J^{n-1}_{t^*/n}u_0 + \left(\frac{t-t_{n-1}}{t^*/n}\right)(J^n_{t^*/n}u_0 - J^{n-1}_{t^*/n}u_0), \\ \quad \text{for } t_{n-1} \leq t \leq t_{n+1}, \\ J^{i-1}_{t^*/n}u_0 + \left(\frac{t-t_{i-1}}{t^*/n}\right)(J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0), \\ \quad \text{for } t_{i-1} \leq t \leq t_i,\ n+2 \leq i \leq 2n. \end{cases}$$

Note that, at the point t^*, we have

$$u_n'(t^*) = \frac{J^n_{t^*/n}u_0 - J^{n-1}_{t^*/n}u_0}{t^*/n}.$$

Applying (15) $i - 1$ times and putting $\lambda = t^*/n$ in Lemma 3.2(i), we see that

$$\left\| \frac{J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0}{t^*/n} \right\|_H \leq C \tag{20}$$

for any index i, where $C := \|Ax\|_H + \inf_{y \in \partial\Phi(x) \cap H} \|y\|_H$. The sequence $\{u_n'(t)\}$ is therefore bounded for each $t \in [0, 2t^*]$ and, after passing to a subsequence,

$$u_n'(t) \rightharpoonup w(t)$$

as $n \to \infty$, for some $w(t) \in H$. Note that

$$\langle w(t),\, u_n'(t)\rangle \le C\|w(t)\|_H,$$

from which it follows after letting $n \to \infty$ that $\|w(t)\|_H \le C$ for $t \in [0, 2t^*]$. Bochner's Theorem ([3]) shows that $w(t)$ is integrable.

To show that the sequence $\{u_n\}$ converges to u in $C(0, 2t^*; H)$, we consider several cases. First, suppose that $t \in [t_{i-1}, t_i]$ for some $i \le n-1$, so that $t < t^*$ and $(i-1)t^*/n \le t \le it^*/n$. We have

$$\begin{aligned}\|u_n(t) - u(t)\|_H &\le \left\| J^n_{t/n}u_0 - \left\{ J^{i-1}_{t^*/n}u_0 + \left(\frac{t - t_{i-1}}{t^*/n}\right)(J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0)\right\}\right\|_H \\ &\quad + \|u(t) - J^n_{t/n}u_0\|_H \\ &\le \|J^n_{t/n}u_0 - J^{i-1}_{t^*/n}u_0\|_H + \|J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0\|_H \\ &\quad + \|u(t) - J^n_{t/n}u_0\|_H,\end{aligned}$$

and we can estimate each of the summands on the right as follows. Let $\mu = t/n$, $\lambda = t^*/n$, and $m = i-1$ in (17) to obtain

$$\begin{aligned}\|J^n_{t/n}u_0 - J^{i-1}_{t^*/n}u_0\|_H &\le \left[\left(t - (i-1)\frac{t^*}{n}\right)^2 + \frac{t(t^*-t)}{n}\right]^{1/2} \\ &\quad + \left[(i-1)\frac{t^*}{n}\left(\frac{t^*-t}{n}\right) + \left(t - (i-1)\frac{t^*}{n}\right)^2\right]^{1/2}.\end{aligned}$$

For $t < t^*$, we have the elementary inequality $\frac{t(t^*-t)}{n} \le \frac{(t^*)^2}{4n}$, and, since $\frac{(i-1)t^*}{n} \le t \le \frac{it^*}{n}$, $(t - (i-1)\frac{t^*}{n})^2 \le (\frac{t^*}{n})^2$. Also, $(i-1)t^*/n \le t$. Thus,

$$\|J^n_{t/n}u_0 - J^{i-1}_{t^*/n}u_0\|_H \le 2\sqrt{\frac{(t^*)^2}{4n} + \frac{(t^*)^2}{n^2}} \le \frac{2t^*}{\sqrt{n}}.$$

For the remaining summands, Lemma 3.2 shows that

$$\|J^i_{t^*/n}u_0 - J^{i-1}_{t^*/n}u_0\|_H \le \frac{Ct^*}{n} \le \frac{Ct^*}{\sqrt{n}},$$

where C is the constant in (20), and the proof of Theorem 3.5 shows that

$$\|u(t) - J^n_{t/n}u_0\|_H \le \frac{2Ct}{\sqrt{n}} \le \frac{2Ct^*}{\sqrt{n}}.$$

Combining these bounds, we have

$$\|u(t) - u_n(t)\|_H \leq \frac{c}{\sqrt{n}}, \quad \text{for} \quad t \in [t_{i-1}, t_i], \quad i \leq n-1, \tag{21}$$

where c is a constant depending on t^* and u_0.

This argument requires only minor changes if $t \in [t_{i-1}, t_i]$ for some $i \geq n+2$, so that $t \geq t^*$. In this case, we must put $\lambda = t/n$, $\mu = t^*/n$, $m = n$, and $n = (i-1)$ in (17), which yields exactly the same estimate as above for the first summand. Since $t^* \leq t \leq 2t^*$, we have $\frac{t(t-t^*)}{n} \leq \frac{2(t^*)^2}{n}$, which changes the final bound on the first term slightly. As the bounds on the other terms are unaffected, we obtain a bound of the form (21) with a different constant c. For the remaining case, in which $t_{n-1} \leq t \leq t_{n+1}$, we have

$$\begin{aligned}\|u_n(t) - u(t)\|_H \leq \|J^n_{t/n}u_0 - J^{n-1}_{t^*/n}u_0\|_H + 2\|J^n_{t^*/n}u_0 - J^{n-1}_{t^*/n}u_0\|_H \\ + \|u(t) - J^n_{t/n}u_0\|_H,\end{aligned}$$

and arguments similar to those of the two previous cases provide bounds on each of the three right-hand terms. We thus obtain the uniform bound

$$\|u(t) - u_n(t)\|_H \leq \frac{c}{\sqrt{n}}, \quad t \in [0, 2t^*],$$

for a constant c depending only on t^* and u_0, verifying convergence.

For each n, the identity

$$\langle u_n(t),\ v\rangle = \langle u_0,\ v\rangle + \int_0^t \langle u_n'(s),\ v\rangle\, ds \tag{22}$$

holds for all $v \in H$. By construction, $\{u_n'\}$ is a sequence of bounded, integrable functions which converges weakly to the integrable function w. Consequently, by dominated convergence, letting $n \to \infty$ in (22) yields

$$\langle u(t),\ v\rangle = \langle u_0,\ v\rangle + \int_0^t \langle w(s),\ v\rangle\, ds. \tag{23}$$

It then follows from (19) that $u_n'(s) \rightharpoonup u'(s)$ for almost every $s \in (0, 2t^*)$. In particular, this holds for $s = t^*$, which proves that

$$\frac{J^n_{t^*/n}u_0 - J^{n-1}_{t^*/n}u_0}{t^*/n} \rightharpoonup u'(t^*). \tag{24}$$

□

$\{J^n_{t/n}u_0\}$ By definition of the resolvent, $\{J^n_{t/n}u_0\}$ is a sequence in $D(\Phi)$. We show next that this sequence is bounded in V^{**}, obtaining

LEMMA 4.3. *Let $u_0 \in \mathfrak{D}$ and $u(t) = S(t)u_0$. For each $t > 0$, the sequence $\{J^n_{t/n}u_0\}$ converges to $u(t)$ in the weak* topology $\sigma(V^{**}, V^*)$.*

Proof. For each integer n, $J^n_{t/n}u_0$ is the unique solution of

$$\left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle + \langle AJ^n_{t/n}u_0,\ v - J^n_{t/n}u_0\rangle$$
$$+\ \Phi(v) - \Phi(J^n_{t/n}u_0)\ \geq 0,\ \forall\, v \in V^{**},$$

which we rewrite as

$$\langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - v\rangle + \Phi(J^n_{t/n}u_0) - \Phi(v)$$
$$\leq \left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle,\ \forall\, v \in V^{**}.$$

From the proof of Lemma 4.2 and the embedding of V^{**} into H, we have

$$\langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - v\rangle + \Phi(J^n_{t/n}u_0) - \Phi(v)\ \leq c\|v - J^n_{t/n}u_0\|_{V^{**}},\ \forall\, v \in V^{**},$$

for a constant $c = c(u_0)$. Coercivity forces $\{J^n_{t/n}u_0\}$ to be bounded in V^{**}, and weak* precompactness follows from the Banach-Alaoglu Theorem.

Let $\tilde{u}(t) \in V^{**}$ be the weak* limit of a convergent subsequence $\{J^{n_i}_{t/n_i}u_0\}$ of the sequence $\{J^n_{t/n}u_0\}$. We have

$$\langle u(t) - \tilde{u}(t),\ u(t) - J^{n_i}_{t/n_i}u_0\rangle \leq \|u(t) - \tilde{u}(t)\|_H\|u(t) - J^{n_i}_{t/n_i}u_0\|_H,$$

from which it follows upon letting $n \to \infty$ that

$$\|u(t) - \tilde{u}(t)\|^2_H \leq 0,$$

since $u(t)$ is the strong limit in H of the original sequence. Consequently, $\tilde{u}(t) = u(t)$. As this holds for any convergent subsequence, the sequence $\{J^n_{t/n}u_0\}$ converges in the weak* topology to $u(t)$. □

Since $D(\Phi)$ is weak* closed, $u(t) = S(t)u_0 \in D(\Phi)$ for $u_0 \in \mathfrak{D}$. Furthermore, we conclude from Lemma 4.3 that $S(t)u_0$ belongs to $\overline{D(\Phi)}^{\|\cdot\|_H}$ for *any* $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$. Thus, for $t, \tau > 0$ and $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, the expression $S(t)\,(S(\tau)u_0)$ makes sense; we will see in Corollary 4.7 that this expression equals $S(t + \tau)u_0$.

The next lemma yields the uniqueness of solutions. The special case $s = 0$ reflects the fact that $S(t)$ is a contraction, although we have not yet verified that solutions are given by $S(t)u_0$.

LEMMA 4.4. *Let u_1 and u_2 be weak solutions of the parabolic variational inequality associated to A and Φ with the prescribed initial values $u_{1,0}$ and $u_{2,0}$ in $\overline{D(\Phi)}^{\|\cdot\|_H}$, respectively. Then, for $0 \le s \le t$,*

$$\|u_1(t) - u_2(t)\|_H \le e^{\omega(s-t)} \|u_1(s) - u_2(s)\|_H. \tag{25}$$

Proof. Suppose first that u_1 and u_2 are strong solutions. For almost every $\tau \in (s, t)$, the inequalities

$$\langle u_1'(\tau),\ v - u_1(\tau)\rangle + \langle Au_1(\tau),\ v - u_1(\tau)\rangle + \Phi(v) - \Phi(u_1(\tau)) \ge 0,$$
$$\langle u_2'(\tau),\ v - u_2(\tau)\rangle + \langle Au_2(\tau),\ v - u_2(\tau)\rangle + \Phi(v) - \Phi(u_2(\tau)) \ge 0$$

therefore hold for all $v \in V^{**}$. Inserting $v = u_2$ in the first inequality and $v = u_1$ in the second inequality and then adding yields

$$\langle u_2'(\tau) - u_1'(\tau),\ u_1(\tau) - u_2(\tau)\rangle + \langle Au_2(\tau) - Au_1(\tau),\ u_1(\tau) - u_2(\tau)\rangle \ge 0,$$

and the monotonicity of A immediately yields

$$\frac{1}{2}\frac{d}{d\tau}\|u_1(\tau) - u_2(\tau)\|_H^2 + \omega\|u_1(\tau) - u_2(\tau)\|_H^2 \le 0.$$

Multiplying through by $2e^{2\omega\tau}$, we have

$$\frac{d}{d\tau}(e^{2\omega\tau}\|u_1(\tau) - u_2(\tau)\|_H^2) \le 0,$$

which we integrate from s to t to obtain

$$e^{2\omega t}\|u_1(t) - u_2(t)\|_H^2 - e^{2\omega s}\|u_1(s) - u_2(s)\|_H^2 \le 0.$$

It follows immediately that

$$\|u_1(t) - u_2(t)\|_H \le e^{\omega(s-t)}\|u_1(s) - u_2(s)\|_H.$$

For the general case in which u_1 and u_2 are weak solutions, let u_1^n and u_2^n be sequences of strong solutions with initial values $u_{1,0}^n$ and $u_{2,0}^n$, respectively, such that

$$u_{i,0}^n \to u_{i,0} \text{ in } H \quad \text{and} \quad u_i^n \to u_i \text{ in } C(0, T; H),\ i = 1, 2.$$

We have

$$\begin{aligned}\|u_1(t) - u_2(t)\|_H &\le \|u_1(t) - u_1^n(t)\|_H + \|u_1^n(t) - u_2^n(t)\|_H \\ &\quad + \|u_2^n(t) - u_2(t)\|_H \\ &\le \|u_1 - u_1^n\|_{C(0,T;H)} + e^{\omega(s-t)}\|u_1^n(s) - u_2^n(s)\|_H \\ &\quad + \|u_2^n - u_2\|_{C(0,T;H)},\end{aligned}$$

since (25) holds for u_1^n and u_2^n. Letting $n \to \infty$ finishes the proof. □

THEOREM 4.5. *Let $u_0 \in \mathfrak{D}$. The function $u(t) := S(t)u_0$ is the unique strong solution of the parabolic variational inequality associated to A and Φ with the initial value u_0.*

Proof. Theorem 3.5 shows that $u(t) \in C^{0,1}(0, T; H)$ for any $T > 0$, and uniqueness follows from Lemma 4.4.

Let $t > 0$ be a point where u is differentiable, and let v be an arbitrary element of V^{**}. By definition, $J^n_{t/n}u_0$ satisfies

$$\left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle + \langle AJ^n_{t/n}u_0,\ v - J^n_{t/n}u_0\rangle$$
$$+ \Phi(v) - \Phi(J^n_{t/n}u_0) \geq 0, \tag{26}$$

which we rewrite in the form

$$\langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - u(t)\rangle \leq \left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle$$
$$+ \langle AJ^n_{t/n}u_0,\ v - u(t)\rangle + \Phi(v) - \Phi(J^n_{t/n}u_0).$$

As $\{J^n_{t/n}u_0\}$ is bounded in V^{**}, $\{AJ^n_{t/n}u_0\}$ is bounded in V^*. We thus obtain

$$\langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - u(t)\rangle \leq \left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle$$
$$+ c\,\|v - u(t)\|_{V^{**}} + \Phi(v) - \Phi(J^n_{t/n}u_0), \tag{27}$$

for some $c > 0$, and taking the upper limit of both sides of (27) yields

$$\limsup_{n\to\infty} \langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - u(t)\rangle \leq \langle u'(t),\ v - u(t)\rangle$$
$$+c\,\|v - u(t)\|_{V^{**}} + \Phi(v) - \Phi(u(t)),$$

by the previously established convergence results and the weak* lower semicontinuity of Φ. We now let $v = u(t)$ to obtain

$$\limsup_{n\to\infty} \langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - u(t)\rangle \leq 0. \tag{28}$$

Let v again represent an arbitrary element of V^{**}. Since $J^n_{t/n}u_0 \overset{*}{\rightharpoonup} u(t)$ in V^{**}, it follows from (28) and the pseudomonotonicity of A that

$$\langle Au(t),\ u(t) - v\rangle \leq \liminf_{n\to\infty} \langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - v\rangle. \tag{29}$$

However, by (26), we have

$$\langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - v\rangle \leq \left\langle \frac{J^n_{t/n}u_0 - J^{n-1}_{t/n}u_0}{t/n},\ v - J^n_{t/n}u_0 \right\rangle + \Phi(v) - \Phi(J^n_{t/n}u_0),$$

from which we see that

$$\limsup_{n\to\infty} \langle AJ^n_{t/n}u_0,\ J^n_{t/n}u_0 - v\rangle \le \langle u'(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)). \tag{30}$$

Concatenating (29) and (30), we conclude that

$$\langle u'(t),\ v - u(t)\rangle + \langle Au(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \ \ge 0.$$

This holds for almost every $t > 0$ and for any $v \in V^{**}$, verifying (13).

It remains to show that $u \in L^\infty(0, T; V^{**})$ for any $T > 0$. Suppose that this is false. Then there exists a sequence of points $\{t_k\}$ in $[0, T]$ such that u is differentiable at each t_k and

$$\lim_{k\to\infty} \|u(t_k)\|_{V^{**}} = \infty.$$

It then follows from the coercivity condition (9) that there exists an unbounded, increasing sequence $\{C_k\}$ of constants such that

$$\langle Au(t_k),\ u(t_k) - v_0\rangle + \Phi(u(t_k)) \ge C_k\|u(t_k) - v_0\|_{V^{**}}.$$

As $u(t_k)$ satisfies (13), we have

$$\langle u'(t_k),\ v_0 - u(t_k)\rangle + \langle Au(t_k),\ v_0 - u(t_k)\rangle + \Phi(v_0) - \Phi(u(t_k)) \ge 0,$$

which, combined with the previous bound, yields

$$\langle u'(t_k),\ v_0 - u(t_k)\rangle + \Phi(v_0) \ge \langle Au(t_k),\ u(t_k) - v_0\rangle + \Phi(u(t_k)) \ge C_k\|u(t_k) - v_0\|_{V^{**}}.$$

Since

$$\langle u'(t_k),\ v_0 - u(t_k)\rangle \le \|u'(t_k)\|_H\|v_0 - u(t_k)\|_H \le \|u'(t_k)\|_H\|v_0 - u(t_k)\|_{V^{**}},$$

we thus have

$$\|u'(t_k)\|_H + \frac{\Phi(v_0)}{\|u(t_k) - v_0\|_{V^{**}}} \ge C_k,$$

which forces

$$\lim_{k\to\infty} \|u'(t_k)\|_H = \infty,$$

contradicting the fact that $u' \in L^\infty(0, T; H)$.

□

The existence and uniqueness of weak solutions corresponding to arbitrary initial data in $\overline{D(\Phi)}^{\|\cdot\|_H}$ now follow easily.

COROLLARY 4.6. *Let* $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$*. The function* $u(t) := S(t)u_0$ *is the unique weak solution of the parabolic variational inequality associated to* A *and* Φ *with the initial value* u_0.

Proof. Corollary 3.6 shows that u is *a* weak solution, and Lemma 4.4 verifies its uniqueness. □

Corollary 4.6 reveals that $\{S(t) : t \geq 0\}$ is a semigroup of (nonlinear) contractions on $\overline{D(\Phi)}^{\|\cdot\|_H}$, as suggested earlier.

COROLLARY 4.7. *For* $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$ *and all* $t, \tau \geq 0$*, the semigroup identity,*

$$S(t+\tau)u_0 = S(t)(S(\tau)u_0),$$

holds.

Proof. For each $\tau \geq 0$, we simply observe that $u_1(t) = S(t+\tau)u_0$ and $u_2(t) = S(t)\,(S(\tau)u_0)$ are weak solutions with the same initial value, $S(\tau)u_0$. □

5. Applications

Our first applications concern parabolic versions of some well-known elliptic free boundary problems ([2], [11]). Ω denotes an open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

5.1. *Signorini problems*

Let $p > \frac{2N}{N+2}$ be given, and define $V := W^{1,p}(\Omega)$, which embeds continuously into $H := L^2(\Omega)$. Define $A : W^{1,p}(\Omega) \to [W^{1,p}(\Omega)]^*$ by

$$Au := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u,$$

and let $\Phi : W^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the indicator functional of the set

$$K := \{v \in W^{1,p}(\Omega) \ : \ v \geq 0 \text{ on } \partial\Omega\}.$$

Since K is closed and convex, Φ is lower semicontinuous and convex, hence weakly lower semicontinuous. It is well-known that the operator A is strictly monotone, hemicontinuous, coercive and bounded.

In this case, the approximation hypothesis (10) follows directly from the definition of inequality in $W^{1,p}(\Omega)$ ([11], [16], [19]). If u is an arbitrary element of K, then there exists a sequence $\{u_n\} \subset C^\infty(\overline{\Omega})$ such that $u_n \geq 0$ on $\partial\Omega$ for each n and $u_n \to u$ in $W^{1,p}(\Omega)$. Consequently, $Au_n \in L^2(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$. Finally, $0 \in \partial\Phi(u_n)$, so $\partial\Phi(u_n) \cap L^2(\Omega)$ is not empty.

We can now apply Theorem 4.5 and Corollary 4.6. Given $u_0 \in \mathfrak{D}$ and a final time $T > 0$, Theorem 4.5 provides the unique strong solution $u \in L^\infty(0, T; W^{1,p}(\Omega)) \cap C^{0,1}(0, T; L^2(\Omega))$ which satisfies $u(0) = u_0$ and

$$\langle u'(t),\ v - u(t)\rangle + \langle Au(t),\ v - u(t)\rangle \geq 0,\ \forall\, v \in K,$$

for almost every $t \geq 0$. Corollary 4.6 yields the unique weak solution corresponding to a given initial value $u_0 \in L^2(\Omega)$.

Given an initial value $u_0 \in \mathfrak{D}$, arguments similar to those used in the elliptic case ([2]) show that the strong solution just obtained solves the following initial-boundary-value problem with nonlinear boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = f & \text{for} \quad x \in \Omega,\ t > 0 \\ u(x, 0) = u_0(x) & \text{for} \quad x \in \overline{\Omega}, \\ u \geq 0, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad \text{and} \quad u\,\frac{\partial u}{\partial \nu} = 0 & \text{for} \quad x \in \partial\Omega,\ t \geq 0, \end{cases}$$

where $\frac{\partial}{\partial \nu}$ denotes the conormal derivative associated to A. These boundary conditions model nonlinear diffusion in a domain Ω with a semi-permeable boundary ([6], [17]).

5.2. *Obstacle problems*

Let $p > \frac{2N}{N+2}$ be given, and define $V := W_0^{1,p}(\Omega)$, which embeds continuously into $H := L^2(\Omega)$. Define $A : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ by

$$Au := -\operatorname{div}(|\nabla u|^{p-2}\nabla u);$$

this operator is strictly monotone, hemicontinuous, bounded, and coercive on $W_0^{1,p}(\Omega)$. Let $\psi \in W^{1,p}(\Omega)$ be a given obstacle such that

(i) $\psi \leq 0$ on $\partial\Omega$ in $W^{1,p}(\Omega)$, and
(ii) there exists a sequence $\{\psi_n\} \subset C_0^\infty(\Omega)$ which converges to ψ in $W^{1,p}(\Omega)$ and satisfies $\psi_n \geq \psi$ on Ω in $W^{1,p}(\Omega)$ for each n.

Let $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$ be the indicator of the closed convex set

$$K := \{v \in W_0^{1,p}(\Omega)\ :\ v \geq \psi \text{ on } \Omega \text{ in } W^{1,p}(\Omega)\}.$$

As in the previous example, Φ is weakly lower semicontinuous. Condition (i) guarantees that K is not empty, and we will see that condition (ii) implies the necessary approximation hypothesis (10).

Given an element u of K, we must find a sequence u_n in K such that

$$u_n \to u \text{ in } H, \ Au_n \in H, \text{ and } \partial\Phi(u_n) \cap H \neq \emptyset.$$

Note that $0 \in \partial\Phi(v)$ for any $v \in K$, so the last of these conditions holds. Since $u \geq \psi$ on Ω in $W^{1,p}(\Omega)$, there exists a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that $v_n \geq 0$ on Ω and $v_n \to u - \psi$ in $W^{1,p}(\Omega)$. For each n, define $u_n := v_n + \psi_n \in C_0^\infty(\Omega)$, which clearly belongs to K. Moreover, $u_n \to u$ in $W^{1,p}(\Omega)$, hence also in $L^2(\Omega)$, and $Au_n \in L^2(\Omega)$. The approximation hypothesis therefore holds.

Our results provide the semigroup $\{S(t) : t \geq 0\}$ on $\overline{K}^{\|\cdot\|_H}$ corresponding to the parabolic variational inequality associated to A and Φ. The function $u(t) := S(t)u_0$ is the unique strong solution of this problem if $u_0 \in \mathfrak{D}$ and is its unique weak solution if $u_0 \in \overline{K}^{\|\cdot\|_H}$. For a strong solution ([16]),

$$u'(t) + Au(t) = 0, \quad \text{on} \quad \Omega_t^+, \quad u(t) = \psi \quad \text{otherwise},$$

where $\Omega_t^+ := \{x \in \Omega : u(t) > \psi\}$ and its complement in Ω is the *coincidence set* ([11]).

5.3. *Evolution in Orlicz-Sobolev spaces*

Our final application treats parabolic problems in Orlicz-Sobolev spaces; see [1], [10], [12], and [16] for more details concerning such spaces. There are surprisingly few results for such problems, but some related references include [5], [8], and [15]. In [8], Elmahi considers Young functions with controlled growth, whereas very rapidly growing Young functions are our primary concern (as indicated by the prototype defined above in equation (5)). In addition, these other works consider explicit operators which generalize Leray-Lions operators to an Orlicz-Sobolev space setting. Since the functional Φ defined below is not necessarily differentiable (even at a minimizer), we cannot write down the associated Euler-Lagrange equation, as would be needed to apply the results of Donaldson ([5]) or Robert ([15]). Moreover, taking advantage of the special variational structure of the problems considered here seems simpler than the rather technical methods used by Donaldson and Robert.

Let M be a Young function whose complementary Young function $\overline{M}$ satisfies the Δ_2 condition, and define $V := W_0^1 E_M(\Omega)$, so that

$$V^* = W^{-1}E_{\overline{M}}(\Omega) = W^{-1}L_{\overline{M}}(\Omega) \quad \text{and} \quad V^{**} = W_0^1 L_M(\Omega).$$

Suppose that the desired pivot space structure $V^{**} \hookrightarrow H \hookrightarrow V^*$ holds with $H := L^2(\Omega)$; this is a constraint on the Sobolev conjugate M^* of M which is analogous to our restriction on p in the previous applications.

Define the functional $\Phi : V^{**} \to \mathbb{R} \cup \{\infty\}$ by

$$\Phi(v) := \int_\Omega M(|\nabla v|)dx. \tag{31}$$

Given $f \in H$, define $A : V^{**} \to V^*$ by $Au := -f$. Lemma 3.14 of [10] verifies the needed coercivity condition, and Le and Schmitt have shown that Φ is weak* lower semicontinuous ([13]). Thus, it only remains to check the approximation hypothesis (10).

Let u be an arbitrary element of $D(\Phi)$, and let u_ε be the corresponding mollified sequence. Gossez proved that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in V^{**} ([10]), and the compact embedding of V^{**} into H implies that $u_\varepsilon \to u$ in H. The first condition of (10) is satisfied, and the second condition is vacuous by our choice of A. The fact that the intersection $\partial\Phi(u_\varepsilon) \cap H$ is not empty follows from the differentiability of Φ on the (generally proper) subset $W_0^1 E_M(\Omega)$ of V^{**} ([9]), which in turn relies on the fact that $\overline{M} \in \Delta_2$ ([16]).

We may now apply Theorem 4.5 and Corollary 4.6 to the parabolic variational inequality associated to A and Φ. Given $u_0 \in \mathfrak{D}$, Theorem 4.5 establishes the existence of a unique strong solution $u \in L^\infty(0, T; V^{**}) \cap C^{0,1}(0, T; H)$ such that $u(0) = u_0$ and

$$\langle u'(t),\ v - u(t)\rangle + \Phi(v) - \Phi(u(t)) \geq \langle f,\ v - u(t)\rangle,\ \forall\, v \in V^{**},$$

for almost every $t \geq 0$. Equivalently, $u(t)$ satisfies the subdifferential inclusion

$$f - u'(t) \in \partial\Phi(u(t)), \tag{32}$$

for almost every $t \geq 0$. By analogy with the arguments of [13], we consider (32) to be the appropriate interpretation of the parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\right) = f, \tag{33}$$

with the initial condition $u(0) = u_0 \in \mathfrak{D}$ (and null Dirichlet boundary conditions included in the definition of the space V^{**}). The function m in equation (33) is the generator of the Young function Φ,

$$M(t) = \int_0^t m(s)ds.$$

If the prescribed initial value belongs to the set $\overline{D(\Phi)}^{\|\cdot\|_H}$, Corollary 4.6 yields the unique weak solution $u \in C(0, T; H)$ of the associated problem.

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