Sample Path Properties of Anisotropic Gaussian Random Fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$. For simplicity, we assume the coordinate process $X_1, \ldots, X_d$ are i.i.d. Then many sample path properties of X such as continuity and Hausdorff dim- can be determined by the following function

$$\sigma^2(s,t) = \mathbb{E}\left[\left(X_1(t) - X(s)\right)^2\right].$$

If we have knowledge ~~about~~ on the dependence structure of X, then more precise information about the sample functions of X can be derived.

In these lectures, We will consider three conditions on X:

(C1)
$$K_1 \sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell} \leq \sigma^2(s,t) \leq K_2 \sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell}$$

(C2) $\quad \mathrm{Var}\left(X(t) \mid X(s)\right) \geq K_3 \sum_{\ell=1}^N |t_\ell - s_\ell|^{2H_\ell}.$

(C3) $\forall\, \varepsilon > 0$ $\exists\, K_4 > 0$ s.t. $\forall\, n \geq 1$, $u \in [\varepsilon, +\infty)^N$, $t^1, \ldots, t^n \in (\varepsilon, +\infty)^N$,
$$\mathrm{Var}\left(X(u) \mid X(t^1), \ldots, X(t^n)\right) \geq K_4 \sum_{\ell=1}^N \min\left\{ |u_\ell - t_\ell^j|^{2H_\ell} \quad j = 0, 1, \ldots, n\right\}$$

where $t^0 = 0$

We show: (C1) $\Rightarrow$ sample path continuity and Hausdorff dimension

(C1) + (C2) $\Rightarrow$ level sets and existence of Local times

(C1) + (C3) $\Rightarrow$ Joint continuity of the local times

Examples of Anisotropic Gaussian random fields

1. Fractional Brownian sheet: $B^H = \{B^H(t),\ t \in \mathbb{R}^N_*\}$ is $\mathbb{R}^d$-valued, with mean zero and covariance function given by

$$\mathbb{E}\left(B^H_j(t)\, B^H_k(s)\right) = \delta_{jk} \prod_{\ell=1}^N \frac{1}{2}\left(|t_\ell|^{2H_\ell} + |s_\ell|^{2H_\ell} - |t_\ell - s_\ell|^{2H_\ell}\right) \qquad \forall\, j, k = 1, \dots, d$$

so the coordinate processes are i.i.d.

$\{B^H_1(t),\ t \in \mathbb{R}^N_*\}$ has the following two stochastic integral representations

(i) $$B^H_1(t) = C_H \int_{\mathbb{R}^N} \prod_{\ell=1}^N \left[(t_\ell - r_\ell)_+^{H_\ell - \frac{1}{2}} - (-r_\ell)_+^{H_\ell - \frac{1}{2}}\right] W(dr)$$

where $a_+ = \max\{a, 0\}$, and W is the standard Brownian sheet.

(ii) $$B^H_1(t) = C'_H \int_{\mathbb{R}^N} \prod_{\ell=1}^N \frac{e^{it_\ell\lambda_\ell} - 1}{|\lambda_\ell|^{H_\ell + \frac{1}{2}}}\, \widetilde{W}(d\lambda)$$

where $\widetilde{W}$ is a complex Gaussian measure with λ_N (Leb.) as its control measure.

Exercise ① verify the following scaling property: $\forall\, c > 0$

$$\{B^H(ct),\ t \in \mathbb{R}^N\} \overset{d}{=} \{c^{\sum_{\ell=1}^N H_\ell} B^H(t),\ t \in \mathbb{R}^N\} \Rightarrow \text{OSS}$$

② verify (i) and (ii) above.

- All three conditions (C1), (C2) and (C3) hold for B^H; see Ayache and Xiao (2005), Wu and Xiao (2006), Xiao and Zhang (2002)

{ Gaussian random fields with stationary increments,

2. Additive fractional Brownian motion

Let $X_1, \dots, X_N = \{X_N(t),\ t \in \mathbb{R}^1\}$ be independent fractional Brownian motions in $\mathbb{R}^d$ with Hurst indices $H_1, \dots, H_N$, respective. Then the (N, d)-random field $\{X(t),\ t \in \mathbb{R}^N\}$ defined by

$$X(t) = X_1(t_1) + \cdots + X_N(t_N)$$

is an operator self-similar Gaussian random field with stationary increments. Kôno (1975) studied some asymptotic properties of such X.

~~3. Gaussian random fields with stationary increments~~

3. Random string processes cf. Muller and Tribe (2002).

Let $\{u_t(x),\ t \geq 0,\ x \in \mathbb{R}\}$ be the solution of the SPDE:

$$\frac{\partial u_t(x)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{W},$$

where $\dot{W}(x,t)$ is a space-time white noise, with appropriate initial conditions.

Let $X = \{X(t,x),\ t \geq 0,\ x \in \mathbb{R}\}$ be defined by

$$X(t,x) = (X_1(t,x), \cdots, X_d(t,x)),$$

where $X_1, \cdots, X_d$ are independent copies of $\{u_t(x),\ t \geq 0,\ x \in \mathbb{R}\}$.

$$\mathbb{E}\left[\left(u_t(x) - u_s(y)\right)^2\right] \asymp |x-y| + |t-s|^{1/2}$$

It can be shown that conditions (C1) and (C2) hold for X.

§ 1. Modulus of continuity of Anisotropic Gaussian Random Fields

- For fractional Brownian sheet, see Ayache and Xiao (2005), using wavelet expansion.

- More generally, we may apply the following extension of Garsia's lemma due to Arnold and Imkeller (1996), Funaki, Kikuchi and Potthoff (2006), Khoshnevisan (pri-c)

To state their theorem, we need some notation: Let $I \subseteq \mathbb{R}^N$ be a fixed interval. For any $s, t \in I$, let

$$q(s,t) = \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{2H_\ell},$$

(where $0 < H_1 \le \cdots \le H_N \le 1$ are constants) For any $s \in I$, denote

$$V_s(r) = \{ t \in I : q(s,t) \le r \}$$

Theorem 1.1. Suppose that X is a continuous mapping from I to $\mathbb{R}$. If there exist two strictly increasing functions ψ and p on $\mathbb{R}_+$ with $\psi(0) = p(0) = 0$ and $\lim_{u \to \infty} \psi(u) = \infty$ such that

$$B := \int_I \int_I \psi\left(\frac{|X(t) - X(s)|}{p(q(s,t))} \right) ds\, dt < \infty$$

Then for all $s, t \in I$, we have

$$|X(t) - X(s)| \le K_1 \max_{z \in \{s,t\}} \int_0^{q(s,t)} \psi^{-1}\left(\frac{4B}{\lambda_N(V_z(u))^2} \right) p(du),$$

where $K_1 > 0$ is a constant depending on N and $(H_1, \cdots, H_N)$ only.

Corollary 1.2. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian r.f. satisfying

$$\cancel{K_1 \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell} \le}\ \mathbb{E}(X(t) - X(s))^2 \le K_2 \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell} \quad \text{for all } s,t \in I$$

Then there exist a random variable A ~~and a constant $K_3 > 0$~~ such that

$$\sup_{(s,t) \in I \times I} |X(t) - X(s)| \le A \cdot \sum_{\ell=1}^{N} |t$$

Corollary 1.2 Let $X = \{X(t),\ t \in I\}$ be a centered Gaussian random field. If there exist a constant $K_2 > 0$ and $(H_1, \cdots, H_N) \in (0,1]^N$ such that

$$\mathbb{E}\left[(X(s)-X(t))^2\right] \leq K_2 \sum_{l=1}^{N} |s_l - t_l|^{2H_l}, \qquad \forall\ s, t \in I$$

Then there exists a random variable A with finite moments such that

$$|X(s) - X(t)| \leq A \left(\sum_{l=1}^{N} |s_l - t_l|^{H_l}\right) \sqrt{\log\left(1 + \left(\sum_{l=1}^{N} |t_l - s_l|^{H_l}\right)^{-1}\right)}, \qquad \forall\ s, t \in I.$$

Proof. We choose $\psi(x) = e^{\frac{x^2}{4K_2}} - 1$, $p(x) = \sqrt{x}$, then

$$\mathbb{E} \int_I \int_I \psi\left(\frac{|X(s)-X(t)|}{\sqrt{q(s,t)}}\right) ds\,dt \leq \int_I \int_I \left(\int_{\mathbb{R}} e^{\frac{x^2}{4\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\, dx - 1\right) ds\,dt$$

$$:= K < \infty, \qquad \text{where } \sigma^2 = \mathbb{E}(X(s) - X(t))^2.$$

Hence, Theorem 1.1 implies that $\forall\ s, t \in I$,

$$|X(t) - X(s)| \leq A \cdot \max_{z \in \{s,t\}} \int_0^{q(s,t)} \sqrt{\log(1+u^{-1})} \frac{du}{\sqrt{u}}$$

$$\leq A \sqrt{q(s,t) \cdot \log\left(1 + q(s,t)^{-1}\right)} \qquad \#.$$

Note $\lambda_N(V_z(u)) = \lambda_N\{t \in I : q(t,z) \leq u\} \asymp u^{\sum_{l=1}^{N} \frac{1}{2H_l}}$.

§2. Hausdorff dimension of the range and graph

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$. We assume the coordinate processes $X_1, \cdots, X_d$ are i.i.d and, $\forall\, \varepsilon \in (0,1)$, there are constants $K_4, K_5 > 0$ and $(H_1, \cdots, H_N) \in (0,1)^N$ such that

$$K_4 \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{2H_\ell} \le \mathbb{E}\left[X(t) - X(s)\right]^2 \le K_5 \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{2H_\ell}$$

for all $s, t \in [\varepsilon, 1]^N$. For convenience, we assume $0 < H_1 \le H_2 \le \cdots \le H_N < 1$.

We now determine the Hausdorff dimensions of the random sets:

$$X([0,1]^N) = \{X(t) : t \in [0,1]^N\} \subseteq \mathbb{R}^d,$$

$$\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\} \subseteq \mathbb{R}^{N+d}.$$

<u>Theorem 2.1</u> Under the above assumptions, we have

$$\dim_H X([0,1]^N) = \min\left\{d;\ \sum_{\ell=1}^{N} \frac{1}{H_\ell}\right\} \quad \text{a.s.} \tag{2.1}$$

$$\dim_H \mathrm{Gr}X([0,1]^N) = \min\left\{\sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d,\ 1 \le k \le N;\ \sum_{\ell=1}^{N} \frac{1}{H_\ell}\right\}$$

$$= \begin{cases} \sum_{\ell=1}^{N} \frac{1}{H_\ell} & \text{if } \sum_{\ell=1}^{N} \frac{1}{H_\ell} \le d \\ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d & \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j} \end{cases}$$

We divide the proof into two parts: (i) proof of the upper bounds (ii) Proof of the lower bounds

For any $F \subseteq \mathbb{R}^d$,

(i) In order to prove $\dim_H F \le a$, it is sufficient to show that for any $\gamma > a$, $\forall\, \varepsilon > 0$, there exists a sequence of balls (cubes) $\{C_n\}$ with $\mathrm{diam}\, C_n \le \varepsilon$ such that

$$F \subseteq \bigcup_n C_n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\mathrm{diam}(C_n)\right)^{\gamma} \le K, \tag{2.3}$$

where $K > 0$ does not depend on ε

The upper bounds in Theorem 2.1 follow from Lemma 2.3 below and Corollary 1.2.

→ We assume $H_1 \le H_2 \le \cdots \le H_N$

Lemma 2.2. Let $f : [0,1]^N \to \mathbb{R}^d$ be a function satisfying the following conditions: there exist $(H_1, \cdots, H_N) \in (0,1)$ and a constant $K > 0$ such that

$$|f(s) - f(t)| \le K \sum_{\ell=1}^{N} |s_\ell - t_\ell|^{H_\ell} \quad \text{for all } s, t \in [0,1]^N. \tag{2.4}$$

Then

$$\dim_H f([0,1]^N) \le \min\left\{ d, \sum_{\ell=1}^{N} \frac{1}{H_\ell} \right\} \tag{2.5}$$

$$\dim_H \operatorname{Gr} f([0,1]^N) \le \min\left\{ \sum_{\ell=1}^{N} \frac{1}{H_\ell};\ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d,\ 1 \le k \le N \right\} \tag{2.6}$$

Proof. ~~I will only~~ first prove (2.5). Since $f([0,1]^N) \subseteq \mathbb{R}^d$, we have $\dim_H f([0,1]^N) \le d$. Hence it remains to prove

$$\dim_H f([0,1]^N) \le \sum_{j=1}^{N} \frac{1}{H_j}. \tag{2.7}$$

~~$\left[\forall\ \gamma > \sum_{j=1}^{N} \frac{1}{H_j}\right]$~~ For any $n \ge 2$, we divide $[0,1]^N$ into m_n sub-rectangles $\{R_{n,i}\}$ with sides parallel to the axes and side-lengths $n^{-\frac{1}{H_\ell}}$ $(\ell = 1, 2, \cdots, N)$ respectively. Then

$$m_n \le K\, n^{\sum_{\ell=1}^{N} \frac{1}{H_\ell}} \tag{2.8}$$

and $f([0,1]^N)$ can be covered by $f(R_{n,i})$ $(1 \le i \le m_n)$. By (2.5), we have

$$\operatorname{diam} f(R_{n,i}) \le K n^{-1}. \tag{2.9}$$

For any $\gamma > \sum_{j=1}^{N} \frac{1}{H_j}$, we have

$$\sum_{i=1}^{m_n} \left(\operatorname{diam} f(R_{n,i})\right)^{\gamma} \le K\, n^{\sum_{\ell=1}^{N} \frac{1}{H_\ell}} \cdot n^{-\gamma} \le K$$

this implies $\dim_H f([0,1]^N) \le \gamma$. Since $\gamma > \sum_{j=1}^{N} \frac{1}{H_j}$ is arbitrary, (2.7) follows.

Now we turn to the proof of (2.6). We will show that there are several different ways to cover $\operatorname{Gr} f([0,1]^N)$, each of which leads to an upper bound for $\dim_H \operatorname{Gr} f([0,1]^N)$.

For each fixed integer $n \ge 2$, we have

$$\operatorname{Gr} f([0,1]^N) \subseteq \bigcup_{i=1}^{m_n} R_{n,i} \times f(R_{n,i})$$

and

$$\operatorname{diam}\left(R_{n,i} \times f(R_{n,i})\right) \le K n^{-1}$$

This implies $$\dim_H Grf([0,1]^N) \le \sum_{\ell=1}^{N} \frac{1}{H_\ell}. \tag{2.10}$$

We fix an integer $1 \le k \le N$. Observe that each $R_{ni} \times f(R_{ni})$ can be covered by $l_{n,k}$ cubes in $\mathbb{R}^{N+d}$ of sides $n^{-\frac{1}{H_k}}$, where

$$l_{n,k} \le K \cdot n^{\sum_{j=k}^{N}\left(\frac{1}{H_k} - \frac{1}{H_j}\right)} \cdot n^{\left(\frac{1}{H_k} - 1\right)d} \tag{2.11}$$

Hence $Grf([0,1]^N)$ can be covered by $m_n \times l_{n,k}$ cubes in $\mathbb{R}^{N+d}$ with sides $n^{-\frac{1}{H_k}}$. ~~Denote~~ Since $\forall\, \gamma > \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d$, (2.8) and (2.11) imply

$$\sum_{i=1}^{m_k} \sum_{j=1}^{l_{nk}} \left(n^{-\frac{1}{H_k}}\right)^{\gamma} \le K\, n^{\frac{1}{H_k}\left(\sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d - \gamma\right)} \to 0$$

This yields $$\dim_H Grf([0,1]^N) \le \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d \tag{2.12}$$

It is clear that (2.6) follows from (2.10) and (2.12). #

The condition of Lemma 2.2 can be relaxed.

~~Lemma 2.2'. Let $f: [0,1]^N \to \mathbb{R}^d$ be a function satisfying the following conditi~~
~~there exists $(H_1, \cdots, H_N) \in (0,1)^N$~~

Lemma 2.3' Let $f: [0,1]^N \to \mathbb{R}^d$ be a function and $(H_1, \cdots, H_N) \in (0,1)^N$ satisfying $H_1 \le \cdots \le H_N$. If for any $\varepsilon > 0$, there exists a constant $K > 0$ such that

$$|f(s) - f(t)| \le K \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{H_\ell - \varepsilon}, \quad \forall\, s, t \in [0,1]^N$$

Then (2.5) and (2.6) still hold.

(ii) Proof of the lower bounds.

<u>Method</u> $\forall$ Borel set $F \subseteq \mathbb{R}^d$, in order to prove $\dim_H F \ge \gamma$, it is sufficient to find a ~~probability~~ Borel measure σ on F such that

$$\int_F \int_F \frac{\sigma(dx)\,\sigma(dy)}{|x-y|^{\gamma}} < \infty$$

this follows from Frostman's theorem.

Exercises: · Let C be the tenery Cantor set. Show that $\dim_H C = \frac{\log 2}{\log 3}$
· prove (2.17) (a simple substitution)
· prove Lemma 2.5.

The lower bounds in Theorem 2.1 follow from the following proposition

Proposition 2.4. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}^d$ with i.i.d. components and $(H_1, \cdots, H_N) \in (0,1)^N$ satisfy $H_1 \le \cdots \le H_N$. Assume for some $\varepsilon \in (0,1)$, there exist a constant $K > 0$ such that

$$\mathbb{E}\left[\left(X_1(t) - X_1(s)\right)^2\right] \geq K \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell} \quad \text{for all } s, t \in [\varepsilon, 1]^N \tag{2.13}$$

then almost surely

$$\dim_H X([0,1]^N) \geq \min\left\{d;\ \sum_{\ell=1}^{N} \frac{1}{H_\ell}\right\} \tag{2.14}$$

$$\dim_H \mathrm{Gr}X([0,1]^N) \geq \min\left\{\sum_{\ell=1}^{N} \frac{1}{H_\ell};\ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d,\ 1 \le k \le N\right\} \tag{2.15}$$

Proof. Since, for any $\varepsilon > 0$, $\dim_H X([0,1]^N) \geq \dim_H X([\varepsilon,1]^N)$, it is sufficient to show $\dim_H X([\varepsilon,1]^N) \geq \min\{d, \sum_{\ell=1}^{N} \frac{1}{H_\ell}\}$ a.s.

Let σ be the image measure of $\lambda_N \llcorner [\varepsilon,1]^N$ under the mapping $t \to X(t)$. Then σ is a random Borel measure on $X([\varepsilon,1]^N)$. We only need to show

$$\mathcal{E}_\gamma := \mathbb{E} \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{1}{|X(s) - X(t)|^\gamma}\, ds\, dt < \infty \tag{2.16}$$

for all $0 < \gamma < \min\{d, \sum_{\ell=1}^{N} \frac{1}{H_\ell}\}$.

By Fubini's theorem and (2.13), ~~the left-hand side of (2.16) is at most~~

$$\mathcal{E}_\gamma \le K \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{1}{\left(\sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell}\right)^{\gamma/2}} \cdot \mathbb{E}\left(\frac{1}{|\xi|^\gamma}\right) ds\, dt$$

$$\le K \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \frac{1}{\left(\sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell}\right)^{\gamma/2}}\, ds\, dt,$$

where we have used the fact that, for any standard d-dimensional normal vector ξ and any $\gamma \in (0,d)$, $\mathbb{E}(|\xi|^{-\gamma}) < \infty$.

To continue, we will make use of the following fact: $\forall\, \alpha \in (0,1)$ and $p > \frac{1}{2\alpha}$,

$$\int_0^1 \frac{dx}{(A + x^{2\alpha})^p} \le K A^{-(p - \frac{1}{2\alpha})} \tag{2.17}$$

Using (2.17) repeatedly, we have $\mathcal{E}_\gamma < \infty$. This proves (2.16).

In order to prove (2.15), we need the following lemma.

Lemma 2.5. Let α, β and η be positive constants. For $A>0$ and $B>0$, let

$$J := J(A,B) = \int_0^1 \frac{dx}{(A+x^\alpha)^\beta (B+x)^\eta}.$$

Then there exists a finite constant $K>0$, depending on α, β, η only, such that the following holds for all $A, B>0$ satisfying $A^{1/\alpha} \le K\cdot B$:

(i) If $\alpha\beta > 1$, then

$$J \le K \cdot \frac{1}{A^{\beta - 1/\alpha} B^\eta}$$

(ii) If $\alpha\beta = 1$, then

$$J \le K \frac{1}{B^\eta} \log(1 + BA^{-\frac{1}{\alpha}})$$

(iii) If $0<\alpha\beta<1$, and $\alpha\beta+\eta \ne 1$, then

$$J \le K\left(\frac{1}{B^{\alpha\beta+\eta-1}} + 1\right)$$

Proof. Omitted. Cf. Ayache and Xiao (2005)

Now we prove the lower bound in (2.15). Since $\dim_H \mathrm{Gr}X([0,1]^N) \ge \dim_H X([0,1]^N)$ always holds, we only need to consider the case when

$$\sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j} \quad \text{for some } 1 \le k \le N$$

Here and in the sequel, $\sum_{j=1}^{0} \frac{1}{H_j} = 0$.

Let $0<\varepsilon<1$ and $0<\gamma< \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k + (1-H_k)d$ be fixed, but arbitrary constants. We assume γ is close to the right-end point so that $\gamma \in (N-k+d, N-k+d+1)$. In order to show $\dim_H \mathrm{Gr}X([0,1]^N) \ge \gamma$ a.s., it is sufficient to verify

$$\mathcal{G}_\gamma = \int_{[\varepsilon,1]^N} \int_{[\varepsilon,1]^N} \mathbb{E}\left[\frac{1}{(|s-t|^2 + |X(t)-X(s)|^2)^{\gamma/2}}\right] ds\,dt < \infty \quad (2.18)$$

Denote $\sigma^2(s,t) = \mathbb{E}\{(X_1(s)-X_1(t))^2\}$. We write for all $s \neq t$,

$$\mathbb{E}\left[\frac{1}{\left(|s-t|^2 + |X(t)-X(s)|^2\right)^{\gamma/2}}\right] = \mathbb{E}\left[\frac{1}{\sigma(s,t)^{\gamma}\left(\frac{|s-t|^2}{\sigma^2(s,t)} + |\xi|^2\right)^{\gamma/2}}\right]$$

$$\leq K \frac{1}{\sigma(s,t)^{d}\,|s-t|^{\gamma-d}},$$

where the last inequality follows from the fact that $\forall\ \gamma > d$, $a \in \mathbb{R}$ and $\xi \sim N(0, I)$,

$$\mathbb{E}\left[\frac{1}{(a^2+|\xi|^2)^{\gamma/2}}\right] \leq K\, a^{-(\gamma-d)}. \qquad \text{(verify it!)}$$

Hence $\quad \mathcal{G}_\gamma \leq K \int_0^1 dt_N \cdots \int_0^1 \frac{1}{\left(\sum_{l=1}^N t_l^{H_l}\right)^d \left(\sum_{l=1}^N t_l\right)^{\gamma-d}}\, dt_1.$

Applying Lemma 2.5 repeatedly, one can verify $\mathcal{G}_\gamma < \infty$. This finishes the proof of (2.18), and, ~~the~~ hence, Proposition 2.4.

<u>Exercise</u> ~~[illegible]~~ Finish the last part of the proof.

Further questions arise, for example,

(1) What if we replace $[0,1]^N$ by an arbitrary Borel set $E \subseteq (0,\infty)^N$?

(2) What about the packing dimension of $X(E)$ and $\mathrm{Gr}X(E)$?

(3) Instead of Hausdorff dimension, can we consider the exact Hausdorff and packing measures of $X([0,1]^N)$ and $\mathrm{Gr}X([0,1]^N)$?

Answering these questions requires better understanding of the random field X.

§3. Level sets and Local times

For any $x \in \mathbb{R}^d$, let $L_x = \{ t \in (0,\infty)^N : X(t) = x \}$ be the x-level set of $X = \{X(t), t \in \mathbb{R}^N\}$. We are interested in the following two questions

(i) when is $L_x \neq \phi$ with positive probability? (X hits points)

(ii) when $L_x \neq \phi$, what is $\dim_H L_x = ?$

We assume the following conditions:

~~(C1)~~ $\forall \varepsilon > 0$, $\exists$ K_1, K_2 such that

(3.1) $$\mathbb{E}\left(X_1(t) - X_1(s)\right)^2 \leq K_2 \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell} \qquad \forall s, t \in [\varepsilon, 1]^N$$

~~(C2)~~ $\mathbb{E}(X_1(t)^2) \geq K$ for all $t \in [\varepsilon, 1]^N$ and

(3.2) $$\mathrm{Var}(X_1(t) \mid X_1(s)) \geq K_1 \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell} \qquad \forall s, t \in [\varepsilon, 1].$$

Remarks.

① Note that $\mathbb{E}(X_1(t) - X_1(s))^2 \geq \mathrm{Var}(X_1(t) \mid X_1(s))$, so (3.2) implies

$$\mathbb{E}\left(X_1(t) - X_1(s)\right)^2 \geq K_1 \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell}.$$

② Since $\det\mathrm{cov}(X_1(s), X_1(t)) = \mathrm{Var}(X_1(s)) \cdot \mathrm{Var}(X_1(t) \mid X_1(s))$, (3.1) and (3.2) imply that

$$\det\mathrm{cov}(X_1(s), X_1(t)) \geq K \cdot \sum_{\ell=1}^{N} |t_\ell - s_\ell|^{2H_\ell}, \qquad \forall s, t \in [\varepsilon, 1]^N$$

Theorem 3.1 Assume the above conditions hold. We have

(i) If $\sum_{\ell=1}^{N} \frac{1}{H_\ell} < d$, then for every $x \in \mathbb{R}^d$, $L_x = \phi$ a.s.

(ii) If $\sum_{\ell=1}^{N} \frac{1}{H_\ell} > d$, then for every $x \in \mathbb{R}^d$ and $0 < \varepsilon < 1$, with positive probability,

$$\dim_H (L_x \cap [\varepsilon, 1]^N) = \min\left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \ 1 \leq k \leq N \right\}$$

$$= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d \qquad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^{k} \frac{1}{H_j}. \qquad (3.3)$$

Remark The case $\sum_{\ell=1}^{N} \frac{1}{H_\ell} = d$ is unsolved in general. For X being the Brownian sheet or (N,d) fractional Brownian motion, Orey and Pruitt (1973), Talagrand (1998) proved $L_X = \phi$ a.s., respectively.

Proof of theorem 3.1 For any $n \geq 2$, we divide the interval $[\varepsilon, 1]^N$ into $m_n \leq n^{\sum_{\ell=1}^{N} \frac{1}{H_\ell}}$ sub-rectangles $R_{n,i}$ of side-lengths $n^{-\frac{1}{H_\ell}}$, $\ell = 1, 2, \cdots, N$. Let $\delta \in (0,1)$ be fixed, and let $\tau_{n,i}$ be the lower-left vertex of $R_{n,i}$. Then

$$\begin{aligned} \mathbb{P}\{x \in X(R_{n,i})\} &\leq \mathbb{P}\Big\{ \max_{s,t \in R_{n,i}} |X(s) - X(t)| \leq n^{-(1-\delta)}, \ x \in X(R_{n,i}) \Big\} \\ &\quad + \mathbb{P}\Big\{ \max_{s,t \in R_{n,i}} |X(s) - X(t)| > n^{-(1-\delta)} \Big\} \\ &\leq \mathbb{P}\big(|X(\tau_{n,i}) - x| \leq n^{-(1-\delta)} \big) + e^{-c\, n^{2\delta}} \\ &\leq K \cdot n^{-(1-\delta) d}, \end{aligned} \tag{3.4}$$

where, to derive the second inequality, we have used an estimate of the tail probability of the supremum of Gaussian process (Cf. e.g. Talagrand, 1995). Hence, if $\sum_{\ell=1}^{N} \frac{1}{H_\ell} < d$, we can choose $\delta > 0$ such that $(1-\delta) d > \sum_{\ell=1}^{N} \frac{1}{H_\ell}$. Let $N_n = \#$ of rectangles $R_{n,i}$ such that $x \in X(R_{n,i})$. It follows from (3.4) that

$$\mathbb{E}(N_n) \leq K\, n^{\sum_{\ell=1}^{N} \frac{1}{H_\ell}} \cdot n^{-(1-\delta) d} \to 0 \quad \text{as } n \to \infty$$

Since N_n is integer-valued, Fatou's lemma implies that a.s. $N_n = 0$ for infinitely many integers $n \geq 1$. This proves $L_X = \phi$ a.s.

Now we assume $\sum_{\ell=1}^{N} \frac{1}{H_\ell} > d$ and define a covering $\{R'_{n,i}\}$ of $L_X \cap [\varepsilon, 1]$ by

$$R'_{n,i} = \begin{cases} R_{n,i} & \text{if } x \in X(R_{n,i}) \\ \phi & \text{otherwise} \end{cases}$$

~~Hence~~, For every $k \in \{1, 2, \cdots, N\}$, ~~[illegible]~~ $R'_{n,i}$ can be covered by

$\leq n^{\sum_{j=k+1}^{N}(H_k^{-1}-H_j^{-1})}$ cubes of side-length $n^{-H_k^{-1}}$. Thus, we can cover the level set $L_x \cap [\varepsilon,1]^N$ by a sequence of cubes of side-length $n^{-\frac{1}{H_k}}$. Let $\delta \in (0,1)$ be an arbitrary constant and let

$$\gamma = \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(1-\delta)d$$

It follows from (3.4) that

$$\mathbb{E}\Big[\sum_{i=1}^{m_n} n^{\sum_{j=k+1}^{N}(H_k^{-1}-H_j^{-1})} \cdot \left(n^{-H_k^{-1}}\right)^{\gamma} \cdot \mathbb{1}\{x \in X(R_{n,i})\}\Big]$$

$$\leq K\, n^{\sum_{j=1}^{N} H_k^{-1} + \sum_{j=k+1}^{N}(H_k^{-1}-H_j^{-1}) - \gamma H_k^{-1} - (1-\delta)d} = K$$

Using Fatou's lemma again, we obtain that the γ-dimensional Hausdorff measure of $L_x \cap [\varepsilon,1]^N$ is finite a.s. This implies $\dim_H(L_x \cap [\varepsilon,1]^N) \leq \gamma$ a.s. Letting $\delta \downarrow 0$, we obtain

$$\dim_H\left(L_x \cap [\varepsilon,1]^N\right) \leq \min_{1\leq k\leq N}\left\{\sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d\right\} \quad \text{a.s.}$$

To prove the reverse inequality, we assume $\sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^{k} \frac{1}{H_j}$ for some $1 \leq k \leq N$. Let $\delta > 0$ be a small constant such that

$$\gamma := \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(1+\delta)d > N-k$$

If we can prove that there is a constant $K>0$, independent of δ, such that

$$\mathbb{P}\left\{\dim_H\left(L_x \cap [\varepsilon,1]^N\right) \geq \gamma\right\} \geq K \qquad (3.5)$$

Then, by letting $\delta \downarrow 0$, we have

$$\mathbb{P}\left\{\dim_H\left(L_x \cap [\varepsilon,1]^N\right) \geq \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d\right\} \geq K > 0.$$

The proof of (3.5) is based on the capacity argument due to Kahane (1985). Similar methods have been applied by Adler (1981), Testard (1986) and Xiao (1995).

Let M_γ^+ be the space of all non-negative measures on $\mathbb{R}_+^N$ with finite check

γ- energy. i.e $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^{-\gamma}\,\mu(dx)\,\mu(dy) < \infty$. It is known that M_γ^+ is a complete metric space under the metric

$$\|\mu\|_\gamma = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mu(dt)\,\mu(ds)}{|s-t|^\gamma} \qquad \text{(verify this!)}$$

see Adler (1981). Exercise Convergence in M_γ^+ implies weak convergence ?

We define a sequence of random positive measures μ_n on the Borel sets C of $[\varepsilon,1]^N$ by

$$\mu_n(C) = (2\pi n)^{d/2} \cdot \int_C \exp\left(-\frac{n|X(t)-x|^2}{2}\right) dt$$

$$= \int_C \int_{\mathbb{R}^d} \exp\left(-\frac{|\xi|^2}{2n} + i\langle X(t)-x, \xi\rangle\right) d\xi\, dt \qquad (3.6)$$

We will show the sequence $\{\mu_n\}$ has a subsequence that converges in M_γ^+ ~~weakly~~ to a Borel measure μ on $[\varepsilon,1]^N$, with positive probability. The measure μ must be supported on $L_x \cap [\varepsilon,1]^N$.

It follows from Kahan (1985), Testard (1986) that, if there are constant $K_1>0$, $K_2>0$ such that

(3.7) $\qquad \mathbb{E}(\|\mu_n\|) \geq K_1, \qquad \mathbb{E}(\|\mu_n\|^2) \leq K_2$, where $\|\mu_n\| = \mu([\varepsilon,1]^N)$

(3.8) $\qquad \mathbb{E}(\|\mu_n\|_\gamma) < \infty$

then there is a subsequence of $\{\mu_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu$ in M_γ^+ and μ is strictly positive with probability $\geq K_1^2/(2K_2)$. In this case, it follows (3.6) that the measure μ has its support in $L_x \cap [\varepsilon,1]^N$ almost surely. Hence Frostman's theorem yields (3.5) with $K = K_1^2/(2K_2)$.

It remains to verify (3.7) and (3.8). By Fubini's theorem we have

$$\mathbb{E}(\|\mu_n\|) = \int_{[\varepsilon,1]^N}\int_{\mathbb{R}^d} \exp\left(-\frac{|\xi|^2}{2n}\right) \cdot \mathbb{E}\exp(i\langle X(t)-x,\xi\rangle)\, d\xi\, dt$$

$$= \int_{[\varepsilon,1]^N}\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}(n^{-1}+\sigma^2(t))|\xi|^2\right)\cdot e^{-ix\xi}\, d\xi\, dt$$

$$= \int_{[\varepsilon,1]^N} \left(\frac{2\pi}{n^{-1}+\sigma^2(t)}\right)^{d/2} \exp\left(-\frac{|x|^2}{2(n^{-1}+\sigma^2(t))}\right) dt$$

$$\geq \int_{[\varepsilon,1]^N} \left(\frac{2\pi}{1+\sigma^2(t)}\right)^{d/2} \exp\left(-\frac{|x|^2}{2\sigma^2(t)}\right) dt \;=\; K_1$$

Denote by I_{2d} the identity matrix of order $2d$, and by $\mathrm{cov}(X(s), X(t))$ the covariance matrix of $(X(s), X(t))$. Let $\Gamma = n^{-1} I_{2d} + \mathrm{Cov}(X(s), X(t))$. Then

$$\mathbb{E}(\|\mu_n\|^2) = \int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle \xi+\eta, x\rangle} \exp\left(-\frac{1}{2}(\xi,\eta)\Gamma^{-1}(\xi,\eta)'\right) d\xi\, d\eta\, ds\, dt$$

$$= \int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N} \frac{(2\pi)^d}{\sqrt{\det\Gamma}} \exp\left(-\frac{1}{2}(x,x)\Gamma^{-1}(x,x)\right) ds\, dt$$

$$\leq \int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N} \frac{(2\pi)^d}{\sqrt{\det\mathrm{cov}(X(s), X(t))}}\, ds\, dt$$

Note that

$$\det\mathrm{cov}(X(s), X(t)) = \det\mathrm{cov}(X_1(s), X_1(t))^d = \left[\mathrm{Var}(X(s))\cdot \mathrm{Var}(X(t)\,|\,X(s))\right]^d$$

$$\geq K\cdot\left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell}\right)^d$$

Therefore

$$\mathbb{E}(\|\mu_n\|^2) \leq K\int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N} \frac{ds\, dt}{\left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell}\right)^{d/2}} = K_2 < \infty$$

since $d < \sum_{\ell=1}^N \frac{1}{H_\ell}$.

Similarly we can verify that

$$\mathbb{E}(\|\mu_n\|_\gamma) = \int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N} \frac{ds\, dt}{|s-t|^\gamma}\overbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle \xi+\eta, x\rangle} \exp\left(-\frac{1}{2}(\xi,\eta)\Gamma(\xi,\eta)'\right)}^{d\xi\, d\eta}$$

$$\leq K\int_{[\varepsilon,1]^N}\int_{[\varepsilon,1]^N} \frac{1}{\left(\sum_{\ell=1}^N |s_\ell - t_\ell|\right)^\gamma \left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell}\right)^{d/2}}\, ds\, dt$$

$$\leq K\int_0^1 \cdots \int_0^1 \frac{1}{\left(\sum_{\ell=1}^N t_\ell\right)^\gamma \left(\sum_{\ell=1}^N t_\ell^{2H_\ell}\right)^{d/2}}\, dt.$$

By using Lemma 2.5 repeatedly, we can show the last integral is convergent. This finishes the proof of theorem 3.1. #

Now we turn to the existence of local times of X. As a matter of fact, the random measure μ constructed in the proof of theorem 3.1 is a local time $L(x,\cdot)$.

Let us recall ~~the~~ the definition of Local time of a function $X: I \to \mathbb{R}^d$ where $I \subseteq \mathbb{R}^N$ is a Borel set. The occupation μ_X is defined by

$$\mu_X(C) = \lambda_N\{t \in I: X(t) \in C\} \qquad \forall\, C \in \mathcal{B}(\mathbb{R}^d)$$

If $\mu_X \ll \lambda_d$, then we say X has a local time on I and denote by

$$L(x, I) = \frac{d\mu_X}{d\lambda_d}(x)$$

The local time theory for Gaussian processes ~~was~~ initiated by Berman 69-7 the local time. See Geman and Horowitz (1980) for more information

<u>Theorem 3.2</u> Under the conditions of Theorem 3.1, for any rectangle $I \subseteq (0,\infty)^N$, X has a local time $L(\cdot, I) \in L^2(\mathbb{R}^d \times \Omega)$ if and only if $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$. In the latter case, $L(x,I)$ admits the following L^2-representation

$$L(x, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y, x\rangle} \int_I e^{i\langle y, X(s)\rangle}\, ds\, dy \qquad (3.9)$$

Proof Let μ_X be the occupation measure of X defined above. It follows from theorem 21.9 in Geman and Horowitz (1980) that X has a local time $L(\cdot, I) \in L^2(\lambda_d \times \mathbb{P})$ if and only if

$$\int_{\mathbb{R}^d} \int_I \int_I \mathbb{E}\, e^{i\langle \xi, X(s) - X(t)\rangle}\, ds\, dt\, d\xi < \infty$$

Note that, the left hand side equals

$$\int_I \int_I \frac{ds\,dt}{\left(\mathbb{E}(X_1(s) - X_1(t))^2\right)^{d/2}} \leq K \int_I \int_I \frac{ds\,dt}{\left(\sum_{\ell=1}^N |s_\ell - t_\ell|^{2H_\ell}\right)^{d/2}}$$

which is finite if and only if $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$.

Finally, when $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$, (3.9) follows from the Plancherel thm. #

§4. Joint continuity of the Local times

In order to study the regularity of the local times of X, we further assume X_1 has the following "sectorial local nondeterminism".

(C3). $\forall \varepsilon > 0$, there exists a constant $K > 0$ such that for any integer $n \geq 1$, $u \in [\varepsilon, 1]^N$, $t^1, \cdots, t^n \in [\varepsilon, 1]^N$, we have

$$\operatorname{Var}\left(X_1(u) \mid X_1(t^1), \cdots, X_1(t^n)\right) \geq K \sum_{l=1}^{N} \min\left\{ |u_l - t_l^j|^{2H_l} : 0 \leq j \leq n \right\}$$

where $t_l^0 = 0$ for $l = 1, 2, \cdots, N$.

Under this condition, we say X is sectorially locally nondeterministic with index $\vec{H} = (H_1, \cdots, H_N)$.

We will not deal with the question whether or when a Gaussian random field is "sectorially locally nondeterministic". Khoshnevisan and Xiao (2004) established this property for the Brownian sheet, Wu and Xiao (2006) proved that fractional Brownian sheets have this property.

__Theorem 4.1.__ Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d) Gaussian random field with i.i.d. components. If X_1 is sectorially locally nondeterministic with index $(H_1, \cdots, H_N)$ such that $0 < H_1 \leq \cdots \leq H_N < 1$ and $\sum_{l=1}^{N} \frac{1}{H_l} > d$. Then X has a jointly continuous local time. That is, the function

$$(x, t) \longmapsto L(x, [\langle\varepsilon\rangle, \langle\varepsilon\rangle + t])$$ is continuous.

The proof of theorem 4.1 is based on moment estimates for $L(x, B)$, $L(x+y, B) - L(x, B)$ and the multiparameter version of the Kolmogorov's continuity lemma.

__Lemma 4.2__ Suppose the conditions of Theorem 4.1 hold. There is a constant $K_1 > 0$ such that for all $B = [a, a + \langle r \rangle]$, where $a \in (\varepsilon, \infty)^N$, all $x \in \mathbb{R}^d$ and $n \geq 1$,

$$\mathbb{E}\left[L(x, B)^n\right] \leq K_1^n (n!)^N \, r^{nH_1\left(\sum_{l=1}^{N} \frac{1}{H_l} - d\right)} \tag{4.1}$$

Proof. It follows from [see also (25.5) in Geman and Horowitz (1980)] (3.9) that

$$\mathbb{E}\left[L(x,B)^n\right] = (2\pi)^{-nd}\int_{B^n}\int_{\mathbb{R}^{nd}} \exp\left(-i\sum_{j=1}^n \langle u^j, x\rangle\right)\cdot \mathbb{E}\exp\left(i\sum_{j=1}^n \langle u^j, X(t^j)\rangle\right) d\bar{u}\, d\bar{t}$$

$$\leq (2\pi)^{-nd}\int_{B^n}\int_{\mathbb{R}^{nd}} \exp\left[-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^n \langle u^j, X(t^j)\rangle\right)\right] d\bar{u}\, d\bar{t}$$

$$= \int_{B^n} \frac{(2\pi)^{-nd/2}}{\sqrt{\det\mathrm{Cov}(X(t^1),\cdots,X(t^n))}}\, d\bar{t}$$

$$= (2\pi)^{-\frac{nd}{2}}\int_{B^n} \frac{d\bar{t}}{\left[\det\mathrm{Cov}(X_1(t^1),\cdots,X_1(t^n))\right]^{d/2}}$$

For any Gaussian vector $(Z_1,\cdots,Z_n)$, we have

$$\det\mathrm{Cov}(Z_1,\cdots,Z_n) = \mathrm{Var}(Z_1)\prod_{j=2}^n \mathrm{Var}(Z_j \mid Z_1\cdots,Z_{j-1}).$$

It follows from this identity and the sectorial local nondeterminism that

$$\det\mathrm{Cov}(X_1(t^1),\cdots,X_1(t^n)) \geq \prod_{j=2}^n \mathrm{Var}\left(X(t^j)\mid X(t^1)\cdots X(t^{j-1})\right)$$

$$\geq K^n \prod_{j=2}^n\left(\sum_{\ell=1}^N \min\left\{|t^j_\ell - t^i_\ell|^{2H_\ell} : 0\leq i\leq j-1\right\}\right)$$

Hence $$\mathbb{E}\left[L(x,B)^n\right] \leq K^n\int_{B^n}\prod_{j=2}^n\left(\sum_{\ell=1}^N \min\left\{|t^j_\ell - t^i_\ell|^{2H_\ell} : 0\leq i\leq j-1\right\}\right)^{-d/2} d\bar{t}$$

We integrate in the order $dt^n\, dt^{n-1}\cdots dt^2\, dt^1$. First consider the integral

$$\mathcal{I}_n = \int_{B} \frac{dt^n}{\left(\sum_{\ell=1}^N \min\left\{|t^n_\ell - t^i_\ell|^{2H_\ell} : 0\leq i\leq n-1\right\}\right)^{\frac{d}{2}}},$$

where $t^1,\cdots,t^{n-1}$ are fixed, with distinct coordinates (the set of such points has full ~~(nN)~~ $((n-1)N)$-dimensional Lebesgue measure)

We introduce N permutations $\Gamma_1, \cdots, \Gamma_N$ of $\{1, 2, \cdots, n-1\}$ such that for every $l = 1, \cdots, N$,

$$t_l^{\Gamma_l(1)} < t_l^{\Gamma_l(2)} < \cdots < t_l^{\Gamma_l(n-1)} .$$

For convinence, we denote $t_l^{\Gamma_l(0)} = a_l$ and $t_l^{\Gamma_l(n)} = a_l + r$ [Recall $B = [a, a+\langle r\rangle]$]

For every sequence $(i_1, \cdots, i_N) \in \{1, \cdots, n-1\}^N$, let $\tau(i_1 \cdots i_N) = (t_1^{\Gamma_1(i_1)}, \cdots, t_N^{\Gamma_N(i_N)})$ be the "center" of the rectangle

$$I_{i_1 \cdots i_N} = \prod_{l=1}^{N} \left[t_l^{\Gamma_l(i_l)} - \frac{1}{2}\left(t_l^{\Gamma_l(i_l)} - t_l^{\Gamma_l(i_l-1)}\right),\ t_l^{\Gamma_l(i_l)} + \frac{1}{2}\left(t_l^{\Gamma_l(i_l)} - t_l^{\Gamma_l(i_l-1)}\right) \right)$$

~~where~~ with the convention that the left-end point of the interval is a_l whenever $i_l = 1$ and the interval is closed and its right-end is $a_l + r$ whenever $i_l = n-1$. Thus the rectangles $\{I_{i_1 \cdots i_N}\}$ form a partition of $B = [a, a+\langle r\rangle]$.

For every $t^n \in B$, let $I_{i_1 \cdots i_N}$ be the unique rectangle containing t^n. Then the sectorial local non-determinism of X yields

$$\mathrm{Var}\left(X_1(t^n) \mid X_1(t^i), i \le n-1\right) \ge K \sum_{l=1}^{N} \left| t_l^n - t_l^{\Gamma_l(i_l)} \right|^{2H_l}$$

Hence we have

$$\mathcal{I}_n \le K \sum_{(i_1 \cdots i_N)} \int_{I_{i_1 \cdots i_N}} \frac{dt^n}{\left(\sum_{l=1}^{N} \left| t_l^n - \tau_l(i_1 \cdots i_N) \right|^{2H_l} \right)^{d/2}}$$

$(n-1)^N$ does not matter

$$\le K n^N \int_0^r \cdots \int_0^r \frac{ds_1 \cdots ds_N}{\left(\sum_{l=1}^{N} s_l^{2H_l} \right)^{d/2}}$$

$$= K n^N \cdot r^N \int_0^1 \cdots \int_0^1 \frac{dt_1 \cdots dt_N}{\left(\sum_{l=1}^{N} r^{2H_l} t_l^{2H_l} \right)^{d/2}}$$

$$\le K n^N r^{H_1\left(\sum_{j=1}^{N} \frac{1}{H_j} - d\right)}$$

Repeating this procedure, we derive

$$\mathbb{E}\left(L(x,B)^n\right) \le K^n (n!)^N \cdot r^{nH_1\left(\sum_{j=1}^{N} \frac{1}{H_j} - d\right)}$$

#

Lemma 4.2. Assume the conditions of theorem 4.1 hold. Then, for any $\gamma \in (0,1)$ small, there exists a positive and finite constant K_2 such that for all $B = [a, a+\langle r\rangle]$, $x, y \in \mathbb{R}^d$ and all even numbers $n \geq 2$,

$$\mathbb{E}\left[\left(L(x,B) - L(y,B)\right)^n\right] \leq K_2^n (n!)^N |x-y|^{\frac{n}{NH}\gamma} \cdot r^{nH_1\left(\sum_{j=1}^N \frac{1}{H_j} - d - \gamma\right)}$$

Proof. It is known that for every even integer $n \geq 2$

$$\mathbb{E}\left[\left(L(x,B) - L(y,B)\right)^n\right] = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \left(e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle}\right)$$

$$\cdot \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u^j, X(t^j)\rangle\right) d\bar{u}\, d\bar{t} \qquad (4.2)$$

where $\bar{u} = (u^1, \dots, u^n)$, $\bar{t} = (t^1, \dots, t^n)$ and each $u^j \in \mathbb{R}^d$, $t^j \in (\varepsilon, \infty)^N$. The details that lead to (4.2) are explained in Geman and Horowitz (1980); see also Pitt (1978).

Consider the non-decreasing function $\Lambda(u) = \min\{1, |u|^\gamma\}$ $(0<\gamma<1)$ and the elementary inequality:

$$|e^{iu} - 1| \leq 2\Lambda(u), \qquad \forall u \in \mathbb{R}$$

By the triangle inequality,

$$\left|e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle}\right| \leq \sum_{k=1}^d \left|e^{-iu_k^j(x_k - y_k)} - 1\right|$$

$$\leq 2\sum_{k=1}^d \Lambda\left(u_k^j(x_k - y_k)\right).$$

It follows that

$$\mathbb{E}\left[\left(L(x,B) - L(y,B)\right)^n\right] \leq (2\pi)^{-nd} \cdot 2^n \sum{}' \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{m=1}^n \Lambda\left(u_{k_m}^m (x_{k_m} - y_{k_m})\right) \cdot$$

$$\cdot \exp\left(-\frac{1}{2} \operatorname{Var}\left(\sum_{j=1}^n \langle u^j, X(t^j)\rangle\right)\right) d\bar{u}\, d\bar{t}$$

$$\leq K^n \sum{}' \int_{B^n} d\bar{t} \prod_{m=1}^n \left\{\int_{\mathbb{R}^{nd}} \Lambda\left(u_{k_m}^m (x_{k_m} - y_{k_m})\right)^n \cdot \exp\left(-\frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^n \langle u^j, X(t^j)\rangle\right)\right) d\bar{u}\right\}^{1/n},$$

where the last inequality follows from Hölder's inequality. The summation $\sum'$ is taken over all sequences $(k_1, \dots, k_n) \in \{1, 2, \dots, d\}^n$ and

Fix $\bar{k} = (k_1, \cdots, k_n) \in \{1, \cdots, d\}^n$, and $t^1, \cdots, t^n \in B$, we proceed to estimate the integral

$$\mathcal{M} = \int_{\mathbb{R}^{nd}} \prod^n \Delta(u^m_{k_m}(x_m - y_m)) \exp\left[-\frac{1}{2} \mathrm{Var}\left(\sum_{j=1}^n \langle u^j, X(t^j)\rangle\right)\right] d\bar{u}$$

By a lemma of Cuzick and Dupreez (1982),

$$\mathcal{M} = \frac{(2\pi)^{(nd-1)/2}}{\left(\det\mathrm{cov}(X(t^1), \cdots, X(t^n))\right)^{1/2}} \int_{\mathbb{R}} \Delta^n\left(\frac{x_m - y_m}{\sigma_j(\bar{t})} z\right) e^{-\frac{z^2}{2}} dz,$$

where $\sigma_j^2(\bar{t})$ is the conditional Variance of $X_{k_m}(t^j)$ given $X_\ell(t^i)$ ($\ell \neq k_m$ and $1 \le i \le n$, or $\ell = k_m$ and $i \neq j$).

Since $X_1, \cdots, X_d$ are i.i.d., We have

$$\sigma_j^2(\bar{t}) = \mathrm{Var}(X_1(t^j) \mid X_1(t^i) : i \neq j) \geq K \cdot \sum_{\ell=1}^N \min\left\{|t^j_\ell - t^i_\ell| : i \neq j\right\}$$

The rest of the proof follow similar arguments in Khoshnevisan and Xiao (2004)

We omit the details