The p-Laplacian & ∞-Laplacian

An Lê

Imagine a nonlinear elastic membrane, fixed on the boundary $\partial\Omega$ of a plane domain Ω. If $u(x)$ denotes its vertical displacement and if the deformation energy is given by $\int_\Omega |\nabla u|^p dx$, then a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{6}$$

on $W_0^{1,p}(\Omega)$ satisfies the Euler-Langrange equation:

$$\begin{cases} -\Delta_p u = \lambda_p |u|^{p-2} u \quad , \quad \Omega, \\ u = 0 \ , \quad \partial\Omega, \end{cases} \tag{7}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is called the p-Laplace operator.

Solutions of (7) are interpreted in the weak sense, i.e.,

$u_p \in W_0^{1,p}(\Omega)$ solves (7) if

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla v = \lambda_p \int_\Omega |u_p|^{p-2} u_p v, \quad \forall v \in W_0^{1,p}(\Omega). \tag{8}$$

If we write $\quad \Lambda_p = \lambda_p^{1/p} = \min\limits_{u \neq 0} \dfrac{\|\nabla u\|_{L^p(\Omega)}}{\|u\|_{L^p(\Omega)}}$

and let $p \to \infty$, then the limit equation reads

$$\min\left\{ |\nabla u| - \Lambda_\infty u \, , \, -\Delta_\infty u \right\} = 0 \, , \tag{9}$$

where $\Delta_\infty u = \langle D^2 u \cdot Du, Du \rangle = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i x_j}$ is the ∞-Laplacian,

and $\Lambda_\infty = \lim\limits_{p \to \infty} \Lambda_p$.

On the other hand, Λ_∞ has a geometric characterization:

$\Lambda_\infty = \dfrac{1}{R}$, where R is the radius of the largest ball inscribed in Ω.

Solution of (9) are understood in the viscosity sense.

Example Let $\Omega = B_{\mathbb{R}^N}(0,1)$. Then the function $\delta(x) = \text{dist}(x, \partial\Omega) = 1 - |x|$ is a viscosity solution of

$$\min\{|\nabla u| - u, -\Delta_\infty u\} = 0 \qquad (10)$$

(Note that in this case $\Lambda_\infty = 1$)

Proof.

We observe that $\delta(x) \in W_0^{1,\infty}(\Omega)$ is a Lipschitz function with $|\nabla \delta(x)| = 1$, a.e. Ω, for any bounded domain Ω.

In fact, if Ω is the unit ball then

$\delta \in C^\infty(\Omega \setminus \{0\})$ and $|\nabla\delta(x)| = 1$, $\delta(x) < 1$, $\Delta_\infty \delta(x) = 0$ for $x \neq 0$.

It remains to show (10) at the origin $x = 0$.

. $\delta(x)$ is a viscosity supersolution

Suppose $\varphi \in C^2$, $\varphi(0) = \delta(0) = 1$, $\delta(x) > \varphi(x)$, $x \neq 0$.

Then for any $h \in \Omega$, consider

$$\varphi(th) - \varphi(0) = \langle \nabla\varphi(0), th\rangle + o(th)$$
$$< \delta(th) - \delta(0) = -|th|$$

Dividing by t and letting $t \to 0^+$, we obtain

$$\left.\begin{array}{l} \langle \nabla\varphi(0), h\rangle < -|h| \\ \text{Replacing } h \text{ by } -h \text{ we get} \quad \langle \nabla\varphi(0), h\rangle > |h| \end{array}\right\} \text{contradiction.}$$

Thus, there is no such φ.

. $\delta(x)$ is a viscosity subsolution

Let $\varphi \in C^2$, $\varphi(0) = \delta(0) = 1$, $\delta(x) < \varphi(x)$, $x \neq 0$.

For any $h \in \Omega$ we have:

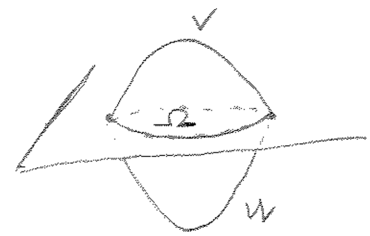
$$\varphi(th) - \varphi(0) = \langle \nabla\varphi(0), th\rangle + o(th) > \delta(th) - \delta(0) = -|th|$$

and thus $\quad |h| > \langle \nabla\varphi(0), h\rangle > -|h|$.

It follows that $\quad |\nabla\varphi(0)| \leq 1$.

Therefore $\min\{|\nabla\varphi(0)| - \varphi(0), -\Delta_\infty \phi(0)\} \leq 1 - 1 = 0$.

COMPARISON PRINCIPLE



Suppose $u \in USC(\overline{\Omega})$ is a vis. sol. of $F \leq 0$

$v \in LSC(\overline{\Omega})$ is a vis. sol. of $F \geq 0$

and $u \leq v$ on $\partial\Omega$

We want to show $u \leq v$ on $\overline{\Omega}$.

Suppose that u and v are classical sub/super-solution.

Let $w = u - v$ and let $\hat{x}$ be a maximizer of w.

If $\hat{x} \in \partial\Omega$, then $u(x) - v(x) \leq \max\limits_{\partial\Omega} (u - v) \leq 0, \quad \forall x \in \overline{\Omega}$.

If $\hat{x} \in \Omega$, then $Du(\hat{x}) = Dv(\hat{x})$ and $D^2u(\hat{x}) \leq D^2v(\hat{x})$.

It follows the degenerate ellipticity of F that

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0 \leq F(\hat{x}, v(\hat{x}), Dv(\hat{x}), D^2v(\hat{x}))$$
$$\leq F(\hat{x}, v(\hat{x}), Du(\hat{x}), D^2u(\hat{x})).$$

Assuming F is strictly increasing in r, we obtain

$$u(\hat{x}) \leq v(\hat{x})$$

Thus $\quad u(x) - v(x) \leq 0, \quad \forall x \in \overline{\Omega}$.

<u>Theorem 5</u>

Let F be degenerate elliptic, proper, continuous + some extra technical hypothesis.

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be vis. subsol. and vis. supersol. of $F = 0$, $u \leq v$ on $\partial\Omega$. Then

$$u \leq v \quad \text{on } \overline{\Omega}.$$

<u>Example</u>
$$\begin{cases} -\Delta u = \lambda_1 u, & \Omega \\ u = 0, & \partial\Omega \end{cases} \qquad (*)$$

Let u_1 be a positive first eigenfunction, then both u_1 and $-u_1$ satisfy $(*)$. However, $-u_1 < u_1$ in Ω.