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Lecture 1

Passive Scalar in Turbulent Velocity Field
The Transport Equation

\[ \frac{\partial \theta(t, x)}{\partial t} = -\mathbf{v}(t, x) \cdot \nabla \theta(t, x), \quad t > 0, \ x \in \mathbb{R}^d; \]
\[ \theta(0, x) = \theta_0(x); \]
\[ \mathbf{v} = (v^1, \ldots, v^d) \in \mathbb{R}^d, \ d \geq 2. \]

If each \( v^i \) is Lipschitz continuous in \( x \), then
\[ \theta(t, x) = \theta_0(X^x_{t,0}); \]
\( X \) is the flow of \( \mathbf{v} \):
\[ \frac{dX^x_{s,t}}{dt} = \mathbf{v}(t, X^x_{s,t}), \ t > s, \ X^x_{s,s} = x. \]
What if $v$ is not Lipschitz continuous in $x$?

- Example — Kolmogorov’s theory: $v$ is Hölder $\approx 1/3$.
- Difficulty — Existence but no uniqueness for the flow equation.
  $$\frac{dX^x_{s,t}}{dt} = v(t, X^x_{s,t}), \ t > s, \ X^x_{s,s} = x.$$ 
- How to find $\theta$?
  $$\theta(t, x) = \theta_0(X^x_{t,0}) \text{ is not true}$$
Regularization

- Introducing viscosity ($\kappa$-limit):

\[
\frac{\partial \theta^\kappa(t, x)}{\partial t} = \kappa \Delta \theta^\kappa(t, x) - \mathbf{v}(t, x) \cdot \nabla \theta^\kappa(t, x), \quad t > 0, \quad x \in \mathbb{R}^d;
\]

\[
dX^{\kappa,x}_{s,t} = \mathbf{v}(t, X^{\kappa,x}_{s,t}) dt + \sqrt{2\kappa} dw(t), \quad t > s, \quad X^{\kappa,x}_{s,s} = x.
\]

- Smoothing out $\mathbf{v}$ ($\varepsilon$-limit):

\[
\mathbf{v}^\varepsilon(t, x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \mathbf{v}(t, y) \psi \left( \frac{x - y}{\varepsilon} \right) dy
\]

(Gawędzki and Vergassola (2000), E and Vanden Eijnden (2000))
Kraichnan’s Model of Turbulence

Physical Model for $v$:

- $v$ is a statistically homogeneous, isotropic, and stationary Gaussian vector field with zero mean and covariance
  \[ E(v^i(t, x)v^j(s, y)) = \delta(t - s)C^{ij}(x - y). \]

- For small $x$, $C^{ij}(x) \sim C^{ij}(0)(1 - |x|^\gamma)$, $0 < \gamma < 2$.

Mathematical Model for $v$ (Le Jan and Raimond (2002), Baxendale, Harris (1988)):

- The matrix $C$ is characterized by its Fourier transform:
  \[ \hat{C}(z) = \frac{A_0}{(1 + |z|^2)^{(d+\gamma)/2}} \left( a \frac{zz^*}{|z|^2} + \frac{b}{d - 1} \left( I - \frac{zz^*}{|z|^2} \right) \right), \]

- $a = 0 \Rightarrow \nabla \cdot v = 0$;
- $b = 0 \Rightarrow v = \nabla V$ for some scalar $V$.
- $\zeta = b/(a + b)$ — degree of incompressibility.
• Representation of \( \nu \):

\[
\nu^i(t, x) = \sum_{k \geq 1} \sigma_k^i(x) \dot{w}_k(t),
\]

\( \dot{w}_k(t) \) are independent standard Gaussian white noises; \( \{\sigma_k, \ k \geq 1\} \) is a CONS in \( H_C \).

• \( H_C = H^{(d+\gamma)/2}(\mathbb{R}^d; \mathbb{R}^d), \ a, b > 0; \)

• \( \sigma_k^i \) is Hölder \( \gamma/2 \).

\[
C^{ij}(x - y) = \sum_k \sigma_k^i(x) \sigma_k^j(y),
\]

\textbf{Thm} \ (Le Jan, Raimond, 2002) For a suitable class of initial conditions \( \theta_0, \theta(t, x) \)

\[
\theta(t, x) = \int \theta_0(y) P\left( X_{0,t}^x \in dy | \mathcal{F}_t^W \right)
\]

where

\[ X_{t,x}(s) = x + \int_s^t \sigma^k(X_{t,x}(r)) \circ d\dot{w}_k(r). \]

Le Jan and Raimond have also derived an equation for the measure \( P\left( X_{0,t}^x \in dy | \mathcal{F}_t^W \right) \), similar to the Zakai equation of nonlinear filtering.

• \textit{Statistical ("weak" in probabilistic sense) solution of the flow equation.}

• \textit{Still very little info about} \( \theta \) \textit{(Transport equation is solved in the space of measures, no uniqueness was established).}
Transport Equation as an SPDE

- $\mathcal{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a stochastic basis with the usual assumptions.
- $(w_k(t), k \geq 1, t \geq 0)$, independent standard Wiener processes on $\mathcal{F}$.
- $\mathbf{v}$ divergence-free (incompressible flow $\implies$ $\text{div}\sigma_k = 0$).

Since the Kraichnan velocity $\mathbf{V}$:

$$V^i(t, x) = \sum_{k \geq 1} \sigma^i_k(x) \dot{w}_k(t),$$

the transport equation is given by

$$d\theta(t, x) = -\sum_k \sigma_k(x) \cdot \nabla \theta(t, x) \circ dw_k(t).$$

or

$$d\theta(t, x) = \frac{1}{2} C^{ij}(0) D_i D_j \theta(t, x) dt - \sigma^i_k(x) D_i \theta(t, x) dw_k(t)$$

Notation: $D_i = \frac{\partial}{\partial x^i}$.

**Summation convention:** summation over a pair of repeating indices.
Consider a stochastic evolution equation
\[ du(t, x) = A(t, x) u(t, x) dt + \sum_{k=1}^{\infty} M^k(t, x) u(t, x) dw_k(t), \quad u(0, x) = u_0(x) \]

where \( A \) and \( M \) are differential operators, and \( w_k \) are independent standard Wiener processes.

\((H): A - \frac{1}{2} MM^* \) is elliptic.

If (H) does not hold, one could not guarantee that a solution of this equation is square integrable, i.e. \( E \| u(t, \cdot) \|^2_{L^2} < \infty \) for all \( t \);

Examples:

1. \( A = \frac{1}{2} \Delta, M = \varepsilon \nabla, \varepsilon < 1 \) - elliptic;
2. \( A = \frac{1}{2} \Delta, M = \nabla \) - degenerate elliptic;
3. \( A = \frac{1}{2} \Delta, M = \varepsilon \nabla, \varepsilon > 1 \) - non-elliptic;

The transport equation
\[ d\theta(t, x) = \frac{1}{2} C^{ij}(0) D_i D_j \theta(t, x) dt - \sigma^i_k(x) D_i \theta(t, x) dw_k(t) \]

is degenerate elliptic!

(Krylov-R., P. Chow, J. Potthoff, B. Øksendal, etc. \( \sigma \)-smooth)
A Wiener Chaos Approach to Solving the Stochastic Transport Equation

**Wiener chaos:**

\[ W(t) = (w_k(t), \ k \geq 1, \ 0 < t < T), \]

\( \{m_i(s), \ i \geq 1 \} \) — CONS in \( L_2([0, T]) \),

\[ \xi_i^k = \int_0^T m_i(s)dw_k(s). \]

\( \mathcal{J} = \left( \alpha = (\alpha_i^k, \ i, k \geq 1) \right | |\alpha| = \sum_{i,k} |\alpha_i^k| < \infty \);

\[ \xi_\alpha = \prod_{i,k} \left( \frac{H_{\alpha_i^k}(\xi_{ik})}{\sqrt{\alpha_i^k!}} \right), \text{ where} \]

\[ H_n(x) = (-1)^n \exp \left\{ \frac{x^2}{2} \right\} \frac{d^n}{dx^n} \exp \left\{ -\frac{x^2}{2} \right\}. \]

**Theorem.**  
(Cameron and Martin, 1947)

The collection \( \{\xi_\alpha, \ \alpha \in \mathcal{J}\} \) is an orthonormal basis in \( L_2(\Omega, \mathcal{F}_T^W, \mathbb{P}) \):

If \( \eta \in L_2(\Omega, \mathcal{F}_T^W, \mathbb{P}) \) and \( \eta_\alpha = \mathbb{E}(\eta \xi_\alpha) \), then

\[ \eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha \]

and

\[ \mathbb{E}|\eta|^2 = \sum_{\alpha \in \mathcal{J}} \eta_{\alpha}^2. \]
The Propagator System

\[ d\theta(t, x) = \frac{1}{2} C^{ij}(0) D_i D_j \theta(t, x) dt - \sigma_k^i(x) D_i \theta(t, x) dw_k(t) \]

If \( \sigma_k^i, \theta_0 \) are smooth, then the STE has a nice square integrable solution, moreover \( \theta(t, x) = \sum_{\alpha \in J} \theta_{\alpha}(t, x) \). Define: \( \xi_{\alpha}(t) = \mathbb{E}(\xi_{\alpha}|\mathcal{F}_t) \); \( \xi_{\alpha}(0) = I(|\alpha| = 0) \).

Fact: \( d\xi_{\alpha}(t) = D\xi_{\alpha}(t)dW(t) \),

where \( D\xi_{\alpha}(t) = m_i(t) \sqrt{\alpha_k^i} \xi_{\alpha^-(i,k)}(t) \ell_k \) is the Maliavin derivative, and \( \alpha^-(i, k) \) is the multi-index with the components

\[ (\alpha^-(i, k))_j^l = \begin{cases} \max(\alpha_i^k - 1, 0), & \text{if } i = j \text{ and } k = l, \\ \alpha_j^l, & \text{otherwise.} \end{cases} \]

By the Itô formula

\[ \frac{\partial \theta_{\alpha}(t, x)}{\partial t} = \frac{1}{2} C^{ij}(0) D_i D_j \theta_{\alpha}(t, x) \\
- \sum_{i, k} \sqrt{\alpha_i^k} \sigma_k^j(x) D_j \theta_{\alpha^-(i,j)}(t, x) m_i(t) ; \]

\( \theta_{\alpha}(0, x) = \theta_0(x) I(|\alpha| = 0) \)
Solving the Propagator

**Good news:** \(\sigma_k^i\) do not have to be smooth or even continuous.

\[|\alpha| = 0:\]

\[
\frac{\partial \theta_{(0)}(t, x)}{\partial t} = \frac{1}{2} C^{ij}(0) D_i D_j \theta_{(0)}(t, x)
\]

\(\theta_{(0)}(0, x) = \theta_0(x) \Rightarrow \theta_{(0)}(t, x) = T_t \theta_0(x)\).

\(\alpha = \delta_{ik}:\)

\[
\frac{\partial \theta_{(ik)}(t, x)}{\partial t} = \frac{1}{2} C^{ij}(0) D_i D_j \theta_{(ik)}(t, x) - \sigma_k^j(x) D_j \theta_{(0)}(t, x) m_i(t); \quad \theta_{(ik)}(0, x) = 0.
\]

\(\theta_{ik}(t, x) = - \int_0^t m_i(s) T_{t-s} \sigma_k^j D_j T_s \theta_0(x) ds.\)

In fact, with \(M_k = -\sigma_k^j D_j\),

\[
\sum_{|\alpha| = N} |\theta_\alpha(t, x)|^2 = \sum_{k_1, \ldots, k_N = 1}^\infty \int_0^t \int_0^{s_N} \cdots \int_0^{s_2} |T_{t-s_N} M_{k_N} \cdots T_{s_2-s_1} M_{k_1} T_{s_1} \theta_0(x)|^2 ds_1 \cdots ds_N.
\]
∃! in $L_2$

$$d\theta(t,x) = \frac{1}{2} C^{ij}(0) D_i D_j \theta(t,x) dt - \sigma^i_k(x) D_i \theta(t,x) dw_k(t)$$

**Thm** (Lototsky and R., Russian Math. Surveys, 2003)

If $\theta_0 \in L_2(\mathbb{R}^d)$, then:

- For every $\varphi \in C_0^\infty(\mathbb{R}^d)$, the random field $\theta(t,x) = \sum_{\alpha \in \mathcal{J}} \theta_\alpha(t,x) \xi_\alpha$ is a unique strong solution of the transport equation in that for any test-function $\varphi$,

$$\langle \theta, \varphi \rangle(t) = \langle \theta_0, \varphi \rangle + \frac{1}{2} \int_0^t C^{ij}(0) (\theta, D_i D_j \varphi)(s) ds$$

- For $t > 0$,

$$\|\theta(t)\|_{L_2(\mathbb{R}^d)}^2 = \sum_{\alpha \in \mathcal{J}} \|\theta_\alpha(t)\|_{L_2(\mathbb{R}^d)}^2 < \|\theta_0\|_{L_2(\mathbb{R}^d)}^2.$$

- For

$$d_s X^x_i(s) = -\sigma^i_k(X^x_i(s)) dw_k(s), s \in [0, t),$$

$$X^x_i(t) = x$$

(martingale solution), and

$$\theta(t,x) = \mathbb{E} \left( \theta_0(X^x_i(0)) \mid \mathcal{F}_t^W \right)$$
Remarks

- $E\theta(t, x) = \theta_0(t, x)$. ($\emptyset$ is a multi-index with zero entries)

- $E(\theta(t, x)\theta(s, y)) = \sum_{\alpha \in \mathcal{J}} \theta_{\alpha}(t, x)\theta_{\alpha}(s, y)$.

- By interpolation: $E\|\theta\|_{L^p(\mathbb{R}^d)}^p < \|\theta_0\|_{L^p(\mathbb{R}^d)}^p$, $2 < p < \infty$. Weighted $L_p$ (e.g. $\theta_0(x) = |x|$) are also OK.

- Conservation of energy, $E\|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 = \|\theta_0\|_{L^2(\mathbb{R}^d)}^2$.

- Pathwise solution of the flow equation.
Totally Turbulent Transport

\[ d\theta(t, x) = \nu \Delta \theta(t, x) dt - \sigma_k^i(x) D_i \theta(t, x) dw_k(t), \]

\[ \sigma_k, \ k \geq 1 \quad \text{CONS in } L_2(\mathbb{R}^d; \mathbb{R}^d) \leftrightarrow \dot{W} \text{ is space-time white noise} \]

**Note:** \( \sum_{k \geq 1} \sigma_k^i(x) \sigma_k^l(x) \) diverges.

S-system:

\[
\frac{\partial \theta_\alpha(t, x)}{\partial t} = \nu \Delta \theta_\alpha(t, x) - \sum_{i, k} \sqrt{\alpha_k^i \sigma_k^l(x) D_j \theta_{\alpha - (i, k)}(t, x) m_i(t)};
\]

Still solvable, but now

\[
\sum_{\alpha \in \mathcal{J}} \|\theta_\alpha\|_{L_2(\mathbb{R}^d)}^2 = \infty
\]
Weighted Wiener Chaos

Let $Q := \{q_1, q_2, \ldots \}$, $q_k > 0$, and $q^\alpha := \prod_{i,k} q_k^\alpha_i$.

**Definition.** The $Q$-weighted Wiener Chaos space $L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ is

$$L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d)) = \left\{ (u_\alpha) : \sum_{\alpha \in J} q^{-2\alpha} \| u_\alpha \|_{L_2(\mathbb{R}^d)}^2 < \infty \right\}.$$

Still write $u = \sum_{\alpha \in J} u_\alpha \xi_\alpha$

Where this series converges?

**Examples:**

1. (Obvious) If $u(t) = 1 + \sum_{k \geq 1} \int_0^t u(s) dw_k(s)$, then $u \in L_{2,Q}(\mathcal{F}_T^W; \mathbb{R})$ for every $Q = (q_1, q_2, \ldots)$ so that $\sum_{k \geq 1} q_k^2 < \infty$.

2. (Nualart-R., JFA, 1997) If

$$du(t, x) = \Delta u(t, x) dt + u(t, x) dw(t, x), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2,$$

then $u \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ for some $Q$. 

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Assume that $\theta_0 \in L_2(\mathbb{R}^d)$ and $|\sigma_k^i(x)| \leq C_k$. Let $Q$ be a sequence with $q_k = \sqrt{\frac{\delta}{d^2 C_k}}$ for some $0 < \delta < 2$. If

$$
\frac{\partial \theta_\alpha(t, x)}{\partial t} = \nu \Delta \theta_\alpha(t, x) - \sum_{i,k} \sqrt{\alpha_i^k} \sigma_k^i(x) D_j \theta_{\alpha^{-(i,k)}}(t, x) m_i(t);
$$

then $\sum_{\alpha \in J} q^{2\alpha} \| \theta_\alpha(t) \|^2_{L_2(\mathbb{R}^d)} < \infty$

and $\theta(t, x) = \sum_{\alpha \in J} \theta_\alpha(t, x) \xi_\alpha$ satisfies

$$
\theta \in L_{2,Q} \left( \mathcal{F}_T^W; C((0, T); L_2(\mathbb{R}^d)) \right).
$$

This $\theta$ is called the *Wiener Chaos solution* of the totally turbulent transport equation

$$
d\theta(t, x) = \nu \Delta \theta(t, x) dt - \sigma_k^i(x) D_i \theta(t, x) d\omega_k(t).
$$
Wiener Chaos Approach

- Computable expressions for the solution and its moments from the S-system.
- New regularity results.
- Possibilities for generalization.

Further Directions

- Anticipating equations.
- Elliptic equations.
- Nonlinear equations.

The main results can be found in: