Let $A$ be a Noetherian ring and $X = \text{Spec}(A)$.

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**REMARK:** Geometers index the other way (by codimension)

**Example:** Let $A = k[x, y]$, where $k$ is a field. Since $\dim A = 2$, $Z_i(A) = 0$ except for possibly $i = 0, 1, 2$.

- $Z_0(A)$ consists of the free Abelian group on the set of maximal ideals of $A$.
- $Z_1(A)$ consists of the free Abelian group on the set of primes of height one
- $Z_2(A)$ is free of rank one on the class $[A]$ since $(0)$ is the only height zero prime of $A$

**Definition:** For an $A$-module $M$ with $\dim M \leq i$, let the **cycle of dim $i$ associated to** $M$ be $\sum_{\dim(A/\mathfrak{p}) = i} \text{length}(M_\mathfrak{p})[A/\mathfrak{p}]$, where $\text{length}(M_\mathfrak{p})$ is the length of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Denote this sum by $[M]_i$.

(Recall, $\dim M = \dim A/\text{ann}(M)$, and $\mathfrak{p} \in \text{Supp}(M) \Leftrightarrow \text{ann}(M) \subset \mathfrak{p}$.)

**REMARKS:** Assume $\dim M \leq i$.

(1.) If $\mathfrak{p} \in \text{Spec}(A)$ with $\dim(A/\mathfrak{p}) = i$, then $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module of finite length (possibly zero).

(2.) If $M = A/\mathfrak{p}$ and $\dim(A/\mathfrak{p}) = i$, then $[A/\mathfrak{p}]_i = [A/\mathfrak{p}]$.

(3.) If $\dim M < i$, then $[M]_i = 0$ (since no prime of dimension $i$ contains $\text{ann}M$).
**Definition:** Let \( p \) be a prime such that \( \dim(A/p) = i + 1 \) and let \( x \) be an element of \( A \) that is not in \( p \). Then \( p \notin \text{Supp}(\frac{A/p}{x(A/p)}) \), so \( \dim(\frac{A/p}{x(A/p)}) \leq i \) (since \( x \) is not in the unique minimal prime of the domain \( A/p \), which is \( p \)).

Denote the cycle \([\frac{A/p}{x(A/p)}]\) by \( \text{div}(p, x) \).

**Definition:** **Rational equivalence** is the equivalence relation on \( A \) generated by setting \( \text{div}(p, x) = 0 \) for all primes \( p \) of \( \dim i + 1 \) and all the elements \( x \notin p \). (In other words, two cycles are rationally equivalent if their difference lies in the subgroup generated by the cycles of the form \( \text{div}(p, x) \).)

**Definition:** The **Chow group of** \( A \) is the direct sum of the groups \( \text{CH}_i(A) \), where \( \text{CH}_i(A) \) is the group of cycles \( Z_i(A) \) modulo rational equivalence.

Denote the Chow group by \( \text{CH}_*(A) \). (Also, sometimes the notations \( A_i(A) \) and \( A_*(A) \) are used instead of the CH.)

**Example:** Set \( A = k[x, y] \), where \( k \) is algebraically closed.

- Since \( k \) is algebraically closed, every maximal ideal \( m \) is generated by two elements \( x - a \) and \( y - b \), with \( a, b \in k \), by the Nullstellensatz. We have \( \text{div}((x - a), y - b) = \left[ \frac{A/(x-a)}{y-b(A/(x-a))} \right] = [A/m] \), so we have \( \text{CH}_0(A) = 0 \).

- Since \( A \) is a UFD, every height one prime is principal. Thus, every generator \( [A/p] \) of \( Z_1(A) \) is of the form \( [A/fA] = \text{div}((0), f) \); so \( \text{CH}_1(A) = 0 \).

- Recall that \( Z_2(A) \cong \mathbb{Z} \). Since there are no prime ideals of \( \dim 3 \) (i.e., \( i + 1 \), where \( i = 2 \)), there are no relations in dimension 2. Therefore, \( \text{CH}_2(A) \cong \mathbb{Z} \).

Thus, \( \text{CH}_*(A) \cong \mathbb{Z} \).

**Remark 1:** Any ring has non-zero Chow group because \([A] \neq 0\) since there are no relations in dimension \( d + 1 \), where \( d = \dim A \).

**Remark 2:** In general, the Chow group is very difficult to compute!

**Remark 3:** To check that an element of the Chow group is zero is often do-able, but to show that an element is non-zero is more difficult. The fact that the divisor class group is isomorphic to the \( d - 1^{text}text \) component of the Chow group allows us to obtain non-trivial examples of Chow groups.
Chow Group Problem: If $A$ is a regular local ring, then $\text{CH}_i(A) = 0$ for $i \neq d$, and $\text{CH}_d(A) \cong \mathbb{Z}$ (where the generator is $[A]$).

Interlude on Divisor Class Groups

In fact, when $A$ is an integrally closed domain of dimension $d$, $\text{CH}_{d-1}(A)$ can be expressed in terms of the divisor class group. Recall that an integrally closed (or normal) domain has the properties $S_2$ and $R_1$. In particular, for any height 1 prime $p$ of $A$, $A_p$ is a regular local ring of dimension 1. In other words, $A_p$ is a discrete valuation ring. Consequently, any ideal of $A_p$ is a power of the maximal ideal $pA_p$. We use the notation $v_p(a)$ to denote this power, where $v_p$ is the valuation on $A_p$. Note that $v_p(a) = \text{length of } (A/a)_p$ as an $A_p$-module.

Let $K$ be the quotient field of $A$.

Definition: A nonzero finitely-generated $A$-submodule of $K$ is called a fractional ideal.

Note that a nonzero ideal of $A$ is just a special case of this; if $a$ is a fractional ideal of $A$, but not necessarily an ideal of $A$, then there exists a nonzero element $x \in A$ such that $xa$ is an ideal of $A$. Therefore, one can think of fractional ideals as being ideals of $A$.

Definition: A fractional ideal is called divisorial if $\text{Ass}(K/a)$ consists only of height 1 primes.

Definition: Let $D(A)$ be the set of divisorial ideals and $P(A)$ the set of principal fractional ideals. (In fact, every principal fractional ideal is divisorial.) Then the divisor class group of $A$, denoted $\text{Cl}(A)$, is the quotient $D(A)/P(A)$.

REMARK: The best way to think of the divisor class group of a normal domain $A$ is that it is a measure of the extent to which $A$ fails to be a Unique Factorization Domain. Recall that a Noetherian integral domain $A$ is a UFD if and only if every height 1 prime ideal is principal. This result is the reason that $A$ is a UFD if and only if $\text{Cl}(A) = 0$.

Discussion: Recall that $Z_{d-1}(A)$ denotes the free Abelian group $\bigoplus_{\dim(A/p)=d-1} \mathbb{Z}$; in other words, we’re summing over the set of height one primes. Elements
of \( Z_{d-1}(A) \) are formal sums \( \sum_{\dim(A/p) = d-1} n_p \cdot [A/p] \), where \( n_p \in \mathbb{Z} \) and all but finitely many of the \( n_p \) are 0. There is a bijection \( \phi : D(A) \to Z_{d-1}(A) \) via \( \phi(a) = \sum v_p(a) [A/p] \), where the sum runs over all primes of height one.

It turns out that \( \phi \) is a bijection. Moreover, the image of the subgroup of principal ideals under \( \phi \) is exactly the subgroup generated by cycles of the form \( \text{div}((0), x) \), which are exactly the cycles rationally equivalent to zero. This can be seen by the following:

If \( \frac{x}{y} \in K \), then \( \phi(\frac{x}{y}(A)) = \phi(xA) - \phi(yA) \). Also, \( \phi(xA) \) is the sum of \( v_p(xA) [A/p] \), and \( v_p(xA) \) is the length of \( A_p / xA_p \) in \( A/p \).

Therefore, we have shown that \( D(A)/P(A) \cong \text{CH}_{d-1}(A) \).

We can think of the Chow Group Problem as an attempt to generalize the fact that if \( A \) is a regular local ring, then \( \text{Cl}(A) = 0 \). It is true that for a regular local ring (which is a UFD) that \( \text{CH}_{d-1}(A) = 0 \), but what about the components below \( d-1 \)? Are they 0 as well?

Note that if we find any divisorial ideal of \( A \) that is not principal, then we have found a non-trivial element in \( \text{Cl}(A) \), and hence in \( \text{CH}_*(A) \).

Example Let \( A = k[[X, Y, Z]]/(XY - Z^2) \). Then the ideal \((x, z)\) is height 1 and prime, so divisorial. However, it’s not principal. Therefore, it defines a nonzero element of the divisor class group. In fact, the class of \((x, z)\) generates \( \text{Cl}(A) \), and it can be shown that \( \text{Cl}(A) \cong \mathbb{Z}/2 \).

REMARKS: (1) This ring is a complete intersection, but not a regular local ring, (2) \((x, z)^2 = (x^2, xz, z^2) = (x^2, xz, xy) = x, (x, y, z) \cong (x, y, z) \) and \((x, y, z) \) is not a divisorial ideal, which is the reason that \( \text{Cl}(A) \cong \mathbb{Z}/2 \).

Example Let \( A = k[[X, Y, Z, W]]/(XY - ZW) \). Then again the ideal \((x, z)\) is height 1, prime, and its class generates \( \text{Cl}(A) \). In fact, \( \text{Cl}(A) \cong \mathbb{Z} \).

Some results on Divisor Class Groups of a Noetherian normal domains \( A \) and \( B \):

1. If an \( A \)-algebra \( B \) is flat as an \( A \)-module, then there is a map \( \text{Cl}(A) \to \text{Cl}(B) \) \((p) \mapsto [B/pB] = \sum_{l=1} \ell(B_\Omega/pB_\Omega) (p)\).

2. \( \text{Cl}(A) \to \text{Cl}(A[X]) \) is an isomorphism. (Recall that Gauss’ showed that
A is a UFD if and only if \( A[X] \).

3. Let \( S \) be a multiplicatively closed subset of \( A \). Then (i) \( \text{Cl}(A) \to \text{Cl}(AS) \) is a surjection, (ii) the kernel is generated by the classes of the prime ideals which meet \( S \).

**REMARK:** Some good references for information on divisor class groups are (1) R. Fossum, The Divisor Class Group of a Krull Domain, and (2) N. Bourbaki, Commutative Algebra, Ch VII.

**PROPOSITION:** Let \( M \) be a finitely-generated \( A \)-module of dimension less than or equal to \( i + 1 \). Let \( x \in A \) such that \( x \) is contained in no minimal prime ideal of dimension \( i + 1 \) in the support of \( M \). Then

\[
[M/xM]_i - [xM]_i = \sum_{\{p|\text{dim}(A/p)=i+1\}} l_p(M_p)\text{div}(p, x),
\]

where \( xM \) is the set of elements of \( M \) annihilated by \( x \). In particular, if \( x \) is no a zero divisor on \( M \), then \( [M/xM]_i \) is rationally equivalent to zero.

**PROOF-sketch**

First of all, since \( x \) is not in any minimal prime in the support of \( M \) of dimension \( i + 1 \), both \( M/xM \) and \( xM \) have dimension at most \( i \). We want to reduce to the case where \( M = A/q \). To do this, we’ll show that both sides of the above equation are additive on short exact sequences.

Let \( 0 \to M' \to M \to M'' \to 0 \) be a short exact sequence of modules of dimension at most \( i + 1 \) such that \( x \) is contained in no minimal prime in their support of dimension \( i + 1 \). The RHS is clearly additive since length is additive. On the other hand, we obtain the long exact sequence:

\[
\begin{array}{ccccccccc}
0 & \to & xM' & \to & xM & \to & xM'' & \to & 0 \\
& | & & | & & | & & |
\end{array}
\]

If we localize at a prime of dimension \( i \), then the sequence we obtain is still exact. Again using the fact that length is additive, we see that the LHS is
additive. Therefore, by a standard filtration argument, we can assume that $M = A/q$.

- If the dimension of $q$ is $i+1$, then $x \notin q$ and the definition of $\text{div}(q, x) = [\frac{A/q}{x(A/q)}]_i = [M/xM]_i$ gives the result.
- If $\text{dim } M < i + 1$, then there are no primes of dimension $i + 1$ in the support of $M$, so the RHS is 0. If $x \notin q$ then $(x)$ annihilates $M$ so $xM = M/xM = M$, so LHS is 0. Finally, if $x \notin q$, then the dimensions of $M/xM$ and $xM$ are less than $i$, so LHS is again zero.

Definition: Let $k$ be a nonnegative integer. We say that a map $f : A \to B$ is flat of relative dimension $k$ if $f$ is a flat map of rings such that, for every prime ideal $p$ of $A$ of dimension $i$, every minimal prime ideal of $B/pB$ has dimension $i + k$.

PROPOSITION: If $f : A \to B$ is a flat map of relative dimension $k$, then the map from $Z_i(A)$ to $Z_{i+k}(B)$ that sends $[A/p]$ to $[B/pB]_{i+k}$ induces a map on Chow groups from $\text{CH}_i(A)$ to $\text{CH}_{i+k}(B)$.

PROOF

Let $p$ be a prime ideal of $A$ of dimension $i+1$ and let $x$ be an element of $A$ that is not in $p$. We must show that the cycle $\text{div}(p, x)$ is mapped to a cycle that is rationally equivalent to zero in $Z_{i+k}(B)$. We have the short exact sequence:

$$0 \to A/p \xrightarrow{x} A/p \to \frac{A/p}{x(A/p)} \to 0$$

and $\text{div}(p, x) = [\frac{A/p}{x(A/p)}]_i$. Then

$$0 \to B/pB \xrightarrow{x} B/pB \to \frac{B/pB}{x(B/pB)} \to 0$$

since $B$ is flat over $A$. Since dimension $B/pB \leq i + 1 + k$, by our lemma, $[\frac{B/pB}{x(B/pB)}]_{i+k}$ is rationally equivalent to zero (since $x$ is regular on $B/pB$). To complete the proof we must show that $[\frac{B/pB}{x(B/pB)}]_{i+k}$ is the image of $[\frac{A/p}{x(A/p)}]_i$.

Let

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = \frac{A/p}{x(A/p)}$$
be a filtration such that \(M_{j+1}/M_j \cong A/q_j\) for prime ideals \(q_j\) of \(A\). Then \([A/p]_x\), is the sum of \([A/q_j]\), over all \(q_j\) of dimension \(i\). Tensoring by \(B\) again, we have a filtration of \(A/q_j \otimes B\), with quotients of the form \(A/q_j \otimes B\), and the associated cycle \([A/q_j \otimes B]_{i+k}\) is the image of \([A/q_j]_i\). If \(q_j\) has dimension less than \(i\), then all components of \(A/q_j \otimes B\) have dimension less than \(i+k\), since \(f\) is flat of relative dimension \(k\). Thus \([A/q_j \otimes B]_{i+k}\) is the sum of \([A/q_j]_i\), so \([A/q_j \otimes B]_{i+k}\) is the image of \(\text{div}(p, x)\) as desired.

**REMARK:** The map on Chow groups whose existence is proven above is called a flat pull back by \(f\) and is denoted by \(f^*\).

This last proposition allows us to obtain some of the usual operations in Commutative Algebra with respect to the Chow group. In particular, we can obtain a map \(\text{CH}_*(A) \to \text{CH}_*(A_S)\), where \(S\) is a multiplicatively closed set. In addition, there is a map on Chow groups obtained by adjoining a polynomial variable. These are the next two results.

**PROPOSITION:** Let \(S\) be a multiplicatively closed subset of \(A\). For each prime ideal \(p_S\) of \(A_S\), define the dimension of \([A_S/p_S]\) to be the dimension of \([A/p]\). Let \(Z_*(S, A)\) denote the subgroup of \(Z_*(A)\) generated by those prime ideals \(q_j\) of \(A\) that meet \(S\). Then the inclusion \(Z_*(S, A)\) in \(Z_*(A)\) induces an exact sequence:

\[
Z_*(S, A) \to \text{CH}_*(A) \to \text{CH}_*(A_S) \to 0
\]

**PROOF**

Since the ideals of \(A_S\) are extended from prime ideals of \(A\), the map on the right is surjective. Moreover, the composition \(Z_*(S, A) \to \text{CH}_*(A) \to \text{CH}_*(A_S)\) is zero since every prime that meets \(S\) goes to zero in \(Z_*(A_S)\) and hence in \(\text{CH}_*(A_S)\). Therefore, we'll show exactness at \(\text{CH}_*(A)\). Let \(\sum_j n_j [A/p_j]\) be a cycle in \(Z_i(A)\) that goes to zero in \(\text{CH}_i(A_S)\). Then there exist prime ideals \(q_k\) of \(A_S\) of dimension \(i+1\) and elements \(x_k\) of \(A_S\) not in \(\{q_k\}\) such that

\[
\sum_j n_j [A_S/(p_j)_S] = \sum_j \text{div}(q_k, x_k)
\]
in $Z_i(AS)$. The prime ideals $q_k$ are extended from prime ideals of $A$, which we also denote $q_k$. Furthermore, by multiplying by units in $AS$, we may assume that the $x_k$ are in $A$. The difference

$$\sum_j n_j [A/p_j] - \sum_j \text{div}(q_k, x_k)$$

is a cycle in $Z_i(A)$. Since its image as a cycle in $Z_*(AS)$ is zero, all of its components with non-zero coefficients must be prime ideals that do not survive in $AS$, which means that they are in $Z_*(S, A)$. Thus $\sum_j n_j [A/p_j]$ is rationally equivalent to a cycle in $Z_*(S, A)$, and the above sequence is exact.

**THEOREM:** The map defined by flat pull-back of cycles from $CH_i(A) \to CH_{i+1}(A[T])$ is surjective for all $i$.

**PROOF**

**NOTE:** This map is in fact an isomorphism, but to define the inverse map requires a lot more machinery.

If there exists a $q \in \text{Spec}(A[T])$ such that $[A[T]/q]$ is not in the image of the Chow group of $A$ (i.e., no $p \in \text{Spec}(A)$ exists such that $[A/p] \mapsto [A[T]/q]$) then we choose one such that its intersection with $A$ is maximal among all prime ideals of $A[T]$ with this property. (This is possible since $A$ is Noetherian.) Let $\dim (A/q) = i + 1$ and $p = q \cap A$. Let $q' = p A[T]$ and consider the localization at the multiplicatively closed set $S = A - p$. The ring $(A[T]/q')_S = (AS/pAS)[T] = \kappa(p)[T]$. Thus, $(A[T]/q')_S$ is a PID, and hence $(q/q')_S$ is a principal ideal. By clearing denominators, we may assume that it is generated by an irreducible polynomial $f(T)$ with coefficients in $A$. Consider the cycle $[A[T]/q] - \text{div}(q', f(T))$. Since $f(T)$ generates $q'/q'$ in the localization of $A[T]/q'$ at $S = A - p$, the only prime ideal with non-zero coefficient in $\text{div}(q', f(T))$ that contracts to $p$ is $q$ and that coefficient is 1. Thus, every prime ideal with nonzero coefficient in $[A[T]/q] - \text{div}(q', f(T))$ must contract to a prime ideal of $A$ that properly contains $p$. By the maximality of $p$, these prime ideals are in the image of $p$, these ideals are in the image of $CH_*(A)$, so $[A[T]/q]$ is rationally equivalent to a cycle in the image of $CH_*(A)$, and thus is in the image of $CH_*(A)$ as well.