CHOW GROUP

Let A be a Noetherian ring and $X = \text{Spec}(A)$.

Notation	Definition
$Z_i(X)$ or $Z_i(A)$	The group of cycles of A of dimension i For each non-negative integer i this is the free Abelian group with basis consisting of all primes $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}) = i$
$[A/\mathfrak{p}]$	generator of $Z_i(A)$ corresponding to $\mathfrak{p}$, if $\dim(A/\mathfrak{p}) = i$
$Z_*(X)$ or $Z_*(A)$	The group of cycles of A direct sum of $Z_i(A)$ over all i

REMARK: Geometers index the other way (by codimension)

Example: Let $A = k[x, y]$, where k is a field. Since dim $A = 2$, $Z_i(A) = 0$ except for possibly $i = 0, 1, 2$.

- $Z_0(A)$ consists of the free Abelian group on the set of maximal ideals of A.

- $Z_1(A)$ consists of the free Abelian group on the set of primes of height one

- $Z_2(A)$ is free of rank one on the class $[A]$ since (0) is the only height zero prime of A

Definition: For an A-module M with dim $M \leq i$, let the **cycle of dim** i **associated to** M be $\sum_{\dim(A/\mathfrak{p})=i} \text{length}(M_\mathfrak{p})[A/\mathfrak{p}]$, where $\text{length}(M_\mathfrak{p})$ is the length of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Denote this sum by $[M]_i$.

(Recall, dim M = dim $A/\text{ann}(M)$, and $\mathfrak{p} \in \text{Supp}(M) \Leftrightarrow \text{ann}(M) \subset \mathfrak{p}$.)

REMARKS: Assume dim $M \leq i$.

(1.) If $\mathfrak{p} \in \text{Spec}(A)$ with $\dim(A/\mathfrak{p}) = i$, then $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module of finite length (possibly zero) .

(2.) If $M = A/\mathfrak{p}$ and $\dim(A/\mathfrak{p}) = i$, then $[A/\mathfrak{p}]_i = [A/\mathfrak{p}]$.

(3.) If dim $M < i$, then $[M]_i = 0$ (since no prime of dimension i contains ann$M$).

Definition: Let $\mathfrak{p}$ be a prime such that $\dim(A/\mathfrak{p}) = i+1$ and let x be an element of A that is not in $\mathfrak{p}$. Then $\mathfrak{p} \notin \text{Supp}(\frac{(A/\mathfrak{p})}{x(A/\mathfrak{p})})$, so $\dim(\frac{(A/\mathfrak{p})}{x(A/\mathfrak{p})}) \leq i$ (since x is not in the unique minimal prime of the domain $A/\mathfrak{p}$, which is $\mathfrak{p}$.)

Denote the cycle $[\frac{(A/\mathfrak{p})}{x(A/\mathfrak{p})}]$ by $\text{div}(\mathfrak{p}, x)$.

Definition: **Rational equivalence** is the equivalence relation on A generated by setting $\text{div}(\mathfrak{p}, x) = 0$ for all primes $\mathfrak{p}$ of dim $i+1$ and all the elements $x \notin \mathfrak{p}$. (In other words, two cycles are rationally equivalent if their difference lies in the subgroup generated by the cycles of the form $\text{div}(\mathfrak{p}, x)$.)

Definition: The **Chow group of** A is the direct sum of the groups $\text{CH}_i(A)$, where $\text{CH}_i(A)$ is the group of cycles $Z_i(A)$ modulo rational equivalence.

Denote the Chow group by $\text{CH}_*(A)$. (Also, sometimes the notations $A_i(A)$ and $A_*(A)$ are used instead of the CH.)

Example: Set $A = k[x, y]$, where k is algebraically closed.

- Since k is algebraically closed, every maximal ideal $\mathfrak{m}$ is generated by two elements $x - a$ and $y - b$, with $a, b \in k$, by the Nullstellensatz. We have $\text{div}((x-a), y-b) = [\frac{(A/(x-a))}{(y-b)(A/(x-a))}] = [A/\mathfrak{m}]$, so we have $\text{CH}_0(A) = 0$.
- Since A is a UFD, every height one prime is principal. Thus, every generator $[A/\mathfrak{p}]$ of $Z_1(A)$ is of the form $[A/fA] = \text{div}((0), f)$; so $\text{CH}_1(A) = 0$.
- Recall that $Z_2(A) \cong \mathbb{Z}$. Since there are no prime ideals of dim 3 (i.e., $i+1$, where $i = 2$), there are no relations in dimension 2. Therefore, $\text{CH}_2(A) \cong \mathbb{Z}$.

Thus, $\text{CH}_*(A) \cong \mathbb{Z}$.

REMARK 1: Any ring has non-zero Chow group because $[A] \neq 0$ since there are no relations in dimension $d+1$, where d= dim A.

REMARK 2: In general, the Chow group is very difficult to compute!

REMARK 3: To check that an element of the Chow group is zero is often do-able, but to show that an element is non-zero is more difficult. The fact that the divisor class group is isomorphic to the $d - 1^{textst}$ component of the Chow group allows us to obtain non-trivial examples of Chow groups.

Chow Group Problem: If A is a regular local ring, then $\mathrm{CH}_i(A) = 0$ for $i \neq d$, and $\mathrm{CH}_d(A) \cong \mathbb{Z}$ (where the generator is $[A]$).

Interlude on Divisor Class Groups

In fact, when A is an integrally closed domain of dimension d, $\mathrm{CH}_{d-1}(A)$ can be expressed in terms of the *divisor class group*. Recall that an integrally closed (or normal) domain has the properties S_2 and R_1. In particular, for any height 1 prime $\mathfrak{p}$ of A, $A_\mathfrak{p}$ is a regular local ring of dimension 1. In other words, $A_\mathfrak{p}$ is a **discrete valuation ring**. Consequently, any ideal of $A_\mathfrak{p}$ is a power of the maximal ideal $\mathfrak{p}A_\mathfrak{p}$. We use the notation $v_\mathfrak{p}(\mathfrak{a})$ to denote this power, where $v_\mathfrak{p}$ is the valuation on $A_\mathfrak{p}$. Note that $v_\mathfrak{p}(\mathfrak{a})$ = length of $(A/\mathfrak{a})_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

Let K be the quotient field of A.

Definition: A nonzero finitely-generated A-submodule of K is called a **fractional ideal**.

Note that a nonzero ideal of A is just a special case of this; if $\mathfrak{a}$ is a fractional ideal of A, but not necessarily an ideal of A, then there exists a nonzero element $x \in A$ such that $x\mathfrak{a}$ is an ideal of A. Therefore, one can think of fractional ideals as being ideals of A.

Definition: A fractional ideal is called **divisorial** if $\mathrm{Ass}(K/\mathfrak{a})$ consists only of height 1 primes.

Definition: Let $D(A)$ be the set of divisorial ideals and $P(A)$ the set of principal fractional ideals. (In fact, every principal fractional ideal is divisorial.) Then the **divisor class group of** A, denoted $\mathrm{Cl}(A)$, is the quotient $D(A)/P(A)$.

REMARK: The best way to think of the divisor class group of a normal domain A is that it is a measure of the extent to which A fails to be a Unique Factorization Domain. Recall that a Noetherian integral domain A is a UFD if and only if every height 1 prime ideal is principal. This result is the reason that A is a UFD if and only if $\mathrm{Cl}(A) = 0$.

Discussion: Recall that $Z_{d-1}(A)$ denotes the free Abelian group $\oplus_{\dim(A/\mathfrak{p})=d-1}\mathbb{Z}$; in other words, we're summing over the set of height one primes. Elements of $\mathrm{Z}_{d-1}(A)$ are formal sums $\sum_{\dim(A/\mathfrak{p})=d-1} n_{\mathfrak{p}} \cdot [A/\mathfrak{p}]$, where $n_{\mathfrak{p}} \in \mathbb{Z}$ and all but finitely many of the $n_{\mathfrak{p}}$ are 0. There is a bijection $\phi : D(A) \to Z_{d-1}(A)$ via $\phi(\mathfrak{a}) = \sum v_{\mathfrak{p}}(\mathfrak{a})[A/\mathfrak{p}]$, where the sum runs over all primes of height one.

It turns out that ϕ is a bijection. Moreover, the image of the subgroup of principal ideals under ϕ is exactly the subgroup generated by cycles of the form $\mathrm{div}((0), x)$, which are exactly the cycles rationally equivalent to zero. This can be seen by the following:

If $\frac{x}{y} \in K$, then $\phi(\frac{x}{y}(A)) = \phi(xA) - \phi(yA)$. Also, $\phi(xA)$ is the sum of $v_{\mathfrak{p}}(\mathfrak{x}A)[A/\mathfrak{p}]$, and $v_{\mathfrak{p}}(\mathfrak{x}A)$ is the length of $A_{\mathfrak{p}}/xA_{\mathfrak{p}}$ in $A/\mathfrak{p}$.

Therefore, we have shown that $D(A)/P(A) \cong \mathrm{CH}_{d-1}(A)$.

We can think of the Chow Group Problem as an attempt to generalize the fact that if A is a regular local ring, then $\mathrm{Cl}(A) = 0$. It is true that for a regular local ring (which is a UFD) that $\mathrm{CH}_{d-1}(A) = 0$, but what about the components below $d - 1$? Are they 0 as well?

Note that if we find any divisorial ideal of A that is not principal, then we have found a non-trivial element in $\mathrm{Cl}(A)$, and hence in $\mathrm{CH}_*(A)$.

<u>Example</u> Let $A = k[[X, Y, Z]]/(XY - Z^2)$. Then the ideal (x, z) is height 1 and prime, so divisorial. However, it's not principal. Therefore, it defines a nonzero element of the divisor class group. In fact, the class of (x, z) generates $\mathrm{Cl}(A)$, and it can be shown that $\mathrm{Cl}(A) \cong \mathbb{Z}_2$.

REMARKS: (1) This ring is a complete intersection, but not a regular local ring, (2) $(x, z)^2 = (x^2, xz, z^2) = (x^2, xz, xy) = x, (x, y, z) \cong (x, y, z)$ and (x, y, z) is not a divisorial ideal, which is the reason that $\mathrm{Cl}(A) \cong \mathbb{Z}_2$

<u>Example</u> Let $A = k[[X, Y, Z, W]]/(XY - ZW)$. Then again the ideal (x, z) is height 1, prime, and its class generates $\mathrm{Cl}(A)$. In fact, $\mathrm{Cl}(A) \cong \mathbb{Z}$.

Some results on Divisor Class Groups of a Noetherian normal domains A and B:

1. If an A-algebra B is flat as an A-module, then there is a map $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ ($< \mathfrak{p} > \mapsto [B/\mathfrak{p}B] = \sum_{\mathrm{ht}\mathfrak{Q}=1} l(B_{\mathfrak{Q}}/\mathfrak{p}B_{\mathfrak{Q}}) < \mathfrak{Q} >$.)

2. $\mathrm{Cl}(A) \to \mathrm{Cl}(A[X])$ is an isomorphism. (Recall that Gauss' showed that

A is a UFD if and only if $A[X]$.)

3. Let S be a multiplicatively closed subset of A. Then (i) $\mathrm{Cl}(A) \to \mathrm{Cl}(A_S)$ is a surjection, (ii) the kernel is generated by the classes of the prime ideals which meet S.

REMARK: Some good references for information on divisor class groups are (1) R. Fossum, The Divisor Class Group of a Krull Domain, and (2) N. Bourbaki, Commutative Algebra, Ch VII.

PROPOSITION: *Let M be a finitely-generated A-module of dimension less than or equal to $i+1$. Let $x \in A$ such that x is contained in no minimal prime ideal of dimension $i+1$ in the support of M. Then*

$$[M/xM]_i - [{}_xM]_i = \sum_{\{\mathfrak{p} | dim(A/\mathfrak{p}) = i+1\}} l_{\mathfrak{p}}(M_{\mathfrak{p}})\, div(\mathfrak{p}, x),$$

where ${}_xM$ is the set of elements of M annihilated by x. In particular, if x is no a zero divisor on M, then $[M/xM]_i$ is rationally equivalent to zero.

PROOF-sketch

First of all, since x is not in any minimal prime in the support of M of dimension $i+1$, both M/xM and ${}_xM$ have dimension at most i. We want to reduce to the case where $M = A/\mathfrak{q}$. To do this, we'll show that both sides of the above equation are additive on short exact sequences.

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules of dimension at most $i+1$ such that x is contained in no minimal prime in their support of dimension $i+1$. The RHS is clearly additive since length is additive. On the other hand, we obtain the long exact sequence:

$$0 \to_x M' \to_x M \to_x M'' \to M'/xM' \to M/xM \to M''/xM'' \to 0$$

This is just the Snake Lemma applied to:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot x} & & \downarrow{\scriptstyle \cdot x} & & \downarrow{\scriptstyle \cdot x} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

If we localize at a prime of dimension i, then the sequence we obtain is still exact. Again using the fact that length is additive, we see that the LHS is additive. Therefore, by a standard filtration argument, we can assume that $M = A/\mathfrak{q}$.

• If the dimension of $\mathfrak{q}$ is $i+1$, then $x \notin \mathfrak{q}$ and the definition of $\operatorname{div}(\mathfrak{q}, x) = [\frac{A/\mathfrak{q}}{x(A/\mathfrak{q})}]_i = [M/xM]_i$ gives the result.

• If dim $M < i+1$, then there are no primes of dimension $i+1$ in the support of M, so the RHS is 0. If $x \in \mathfrak{q}$ then (x annihilates M so) ${}_xM = M/xM = M$, so LHS is 0. Finally, if $x \notin \mathfrak{q}$, then the dimensions of M/xM and ${}_xM$ are less then i, so LHS is again zero.

Definition: Let k be a nonnegative integer. We say that a map $f : A \to B$ is **flat of relative dimension** k if f is a flat map of rings such that, for every prime ideal $\mathfrak{p}$ of A of dimension i, every minimal prime ideal of $B/\mathfrak{p}B$ has dimension $i + k$.

PROPOSITION: *If $f : A \to B$ is a flat map of relative dimension k, then the map from $Z_i(A)$ to $Z_{i+k}(B)$ that sends $[A/\mathfrak{p}]$ to $[B/\mathfrak{p}B]_{i+k}$ induces a map on Chow groups from $CH_i(A)$ to $CH_{i+k}(B)$.*

PROOF

Let $\mathfrak{p}$ be a prime ideal of A of dimension $i + 1$ and let x be an element of A that is not in $\mathfrak{p}$. We must show that the cycle $\operatorname{div}(\mathfrak{p}, x)$ is mapped to a cycle that is rationally equivalent to zero in $Z_{i+k}(B)$. We have the short exact sequence:

$$0 \to A/\mathfrak{p} \xrightarrow{\cdot x} A/\mathfrak{p} \to \frac{A/\mathfrak{p}}{x(A/\mathfrak{p})} \to 0$$

and $\operatorname{div}(\mathfrak{p}, x) = [\frac{A/\mathfrak{p}}{x(A/\mathfrak{p})}]_i$. Then

$$0 \to B/\mathfrak{p}B \xrightarrow{\cdot x} B/\mathfrak{p}B \to \frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)} \to 0$$

since B is flat over A. Since dimension $B/\mathfrak{p}B \leq i + 1 + k$, by our lemma, $[\frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)}]_{i+k}$ is rationally equivalent to zero (since x is regular on $B/\mathfrak{p}B$). To complete the proof we must show that $[\frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)}]_{i+k}$ is the image of $[\frac{A/\mathfrak{p}}{x(A/\mathfrak{p})}]_i$. Let

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = \frac{A/\mathfrak{p}}{x(A/\mathfrak{p})}$$

be a filtration such that $M_{j+1}/M_j \cong A/\mathfrak{q}_j$ for prime ideals $\mathfrak{q}_j$ of A. Then $[\frac{A/\mathfrak{p}}{x(A/\mathfrak{p})}]_i$ is the sum of $[A/\mathfrak{q}_j]_i$ over all $\mathfrak{q}_j$ of dimension i. Tensoring by B again, we have a filtration of $\frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)}$ with quotients of the form $A/\mathfrak{q}_j \otimes B$, and the associated cycle $[A/\mathfrak{q}_j \otimes B]_{i+k}$ is the image of $[A/\mathfrak{q}_j]_i$. If $\mathfrak{q}_j$ has dimension less than i, then all components of $A/\mathfrak{q}_j \otimes B$ have dimension less than $i+k$, since f is flat of relative dimension k. Thus $[\frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)}]_{i+k}$ is the sum of $[A/\mathfrak{q}_j \otimes B]_{i+k}$ for $\mathfrak{q}_j$ of dimension i, so $[\frac{B/\mathfrak{p}B}{x(B/\mathfrak{p}B)}]_{i+k}$ is the image of $\mathrm{div}(\mathfrak{p}, x)$ as desired.

REMARK: The map on Chow groups whose existence is proven above is called a **flat pull back** by f and is denoted by f^*.

This last proposition allows us to obtain some of the usual operations in Commutative Algebra with respect to the Chow group. In particular, we can obtain a map $\mathrm{CH}_*(A) \to \mathrm{CH}_*(A_S)$, where S is a multiplicatively closed set. In addition, there is a map on Chow groups obtained by adjoining a polynomial variable. These are the next two results.

PROPOSITION: *Let S be a multiplicatively closed subset of A. For each prime ideal $\mathfrak{p}_S$ of A_S, define the dimension of $[A_S/\mathfrak{p}_S]$ to be the dimension of $[A/\mathfrak{p}]$. Let $Z_*(S, A)$ denote the subgroup of $Z_*(A)$ generated by those prime ideals of A that meet S. Then the inclusion $Z_*(S, A)$ in $Z_*(A)$ induces an exact sequence:*

$$Z_*(S, A) \to \mathrm{CH}_*(A) \to \mathrm{CH}_*(A_S) \to 0$$

PROOF

Since the ideals of A_S are extended from prime ideals of A, the map on the right is surjective. Moreover, the composition $Z_*(S, A) \to \mathrm{CH}_*(A) \to \mathrm{CH}_*(A_S)$ is zero since every prime that meets S goes to zero in $Z_*(A_S)$ and hence in $\mathrm{CH}_*(A_S)$. Therefore, we'll show exactness at $\mathrm{CH}_*(A)$. Let $\sum_j n_j[A/\mathfrak{p}_j]$ be a cycle in $Z_i(A)$ that goes to zero in $\mathrm{CH}_i(A_S)$. Then there exist prime ideals $\mathfrak{q}_k$ of A_S of dimension $i+1$ and elements x_k of A_S not in $\{\mathfrak{q}_k\}$ such that

$$\sum_j n_j[A_S/(\mathfrak{p}_j)_S] = \sum_j \mathrm{div}(\mathfrak{q}_k, x_k)$$

in $Z_i(A_S)$. The prime ideals $\mathfrak{q}_k$ are extended from prime ideals of A, which we also denote $\mathfrak{q}_k$. Furthermore, by multiplying by units in A_S, we may assume that the x_k are in A. The difference

$$\sum_j n_j[A/\mathfrak{p}_j] - \sum_j \operatorname{div}(\mathfrak{q}_k, x_k)$$

is a cycle in $Z_i(A)$. Since its image as a cycle in $Z_*(A_S)$ is zero, all of its components with non-zero coefficients must be prime ideals that do not survive in A_S, which means that they are in $Z_*(S, A)$. Thus $\sum_j n_j[A/\mathfrak{p}_j]$ is rationally equivalent to a cycle in $Z_*(S, A)$, and the above sequence is exact.

THEOREM: *The map defined by flat pull-back of cycles from* $CH_i(A) \to CH_{i+1}(A[T])$ *is surjective for all* i*.*

PROOF

NOTE: This map is in fact an isomorphism, but to define the inverse map requires a lot more machinery.

If there exists a $\mathfrak{q} \in \operatorname{Spec}(A[T])$ such that $[A[T]/\mathfrak{q}]$ is not in the image of the Chow group of A (i.e., no $\mathfrak{p} \in \operatorname{Spec}(A)$ exists such that $[A/\mathfrak{p}] \mapsto [A[T]/\mathfrak{q}]$) then we choose one such that its intersection with A is maximal among all prime ideals of $A[T]$ with this property. (This is possible since A is Noetherian.) Let $\dim A/\mathfrak{q} = i+1$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Let $\mathfrak{q}' = \mathfrak{p}A[T]$ and consider the localization at the multiplicatively closed set $S = A - \mathfrak{p}$. The ring $(A[T]/\mathfrak{q}')_S = (A_S/\mathfrak{p}A_S)[T] = \kappa(\mathfrak{p})[T]$. Thus, $(A[T]/\mathfrak{q}')_S$ is a PID, and hence $(\mathfrak{q}/\mathfrak{q}')_S$ is a principal ideal. By clearing denominators, we may assume that it is generated by an irreducible polynomial $f(T)$ with coefficients in A. Consider the cycle $[A[T]/\mathfrak{q}] -$ $\operatorname{div}(\mathfrak{q}', f(T))$. Since $f(T)$ generates $\mathfrak{q}/\mathfrak{q}'$ in the localization of $A[T]/\mathfrak{q}'$ at $S = A - \mathfrak{p}$, the only prime ideal with non-zero coefficient in $\operatorname{div}(\mathfrak{q}', f(T))$ that contracts to $\mathfrak{p}$ is $\mathfrak{q}$ and that coefficient is 1. Thus, every prime ideal with nonzero coefficient in $[A[T]/\mathfrak{q}] -$ $\operatorname{div}(\mathfrak{q}', f(T))$ must contract to a prime ideal of A that properly contains $\mathfrak{p}$. By the maximality of $\mathfrak{p}$, these prime ideals are in the image of $\mathfrak{p}$, these ideals are in the image of $\mathrm{CH}_*(A)$, so $[A[T]/\mathfrak{q}]$ is rationally equivalent to a cycle in the image of $\mathrm{CH}_*(A)$, and thus is in the image of $\mathrm{CH}_*(A)$ as well.