Intersection Multiplicity, Chow Groups, and the Canonical Element Conjecture

Abstract
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1 Serre’s Conjecture
All local rings are assumed to be Noetherian, \( M, N \) are finitely generated \( R \)-modules. If \( \text{proj dim}(M) \) finite or \( \text{proj dim}(N) \) finite and we have \( \ell(M \otimes_R N) < \infty \), then we may define

\[
\chi(M, N) := \sum_{i=0}^{d} (-1)^i \ell(\text{Tor}_i^R(M, N)),
\]

where \( d \) is \( \text{proj dim}(M) \) or \( \text{proj dim}(N) \) respectively.

1.1 Regular Rings

**Conjecture 1 (Nonnegativity)** If \( R \) is a regular local ring, then \( \chi(M, N) \geq 0 \).

This was proved by Gabber.

**Theorem 2 (Serre)** If \( R \) is a regular local ring and \( \ell(M \otimes_R N) < \infty \), then \( \dim(M) + \dim(N) \leq \dim(R) \)

**Conjecture 3 (Peskine-Szpiro)** If \( R \) is any local ring, \( M \) an \( R \)-module with \( \text{proj dim}(M) < \infty \), and \( \ell(M \otimes_R N) < \infty \), then \( \dim(M) + \dim(N) \leq \dim(R) \).

This is wide open except for hypersurface case.

**Conjecture 4 (Vanishing)** If \( R \) is a regular local ring and

\[
\dim(M) + \dim(N) < \dim(R),
\]

then \( \chi(M, N) = 0 \).
This was proved independently by Roberts and Gillet-Saulé.

**Conjecture 5 (Positivity)** If $R$ is a regular local ring and 
\[ \dim(M) + \dim(N) = \dim(R), \]
then $\chi(M, N) > 0$.

This conjecture is still open.

### 1.2 The General Case

**Theorem 6 (Serre)** If $R$ is a regular local ring, then 
\[ \max\{j : \text{Tor}_j^R(M, N) \neq 0\} = \dim(R) - \depth(M) - \depth(N). \]

**Lemma 1 (Hochster)** Let $R$ be Cohen-Macaulay and $M$ and $R$-module with 
\[ \proj \dim(M) < \infty. \] Vanishing holds if and only if it holds for every pair of 
Cohen-Macaulay $R$-modules $M, N$ such that, 
\[ \dim(M) + \dim(N) = \dim(R) - 1. \]

**Sketch of Proof** Write $\dim(M) + \dim(N) < \dim(R)$. So 
\[ \dim(R) - \height(\Ann(M)) + \dim(R) - \height(\Ann(N)) < \dim(R), \]
and so 
\[ \height(\Ann(M)) + \height(\Ann(N)) > \dim(R) \]
or 
\[ \height(\Ann(N)) > \dim(M). \]

If $r = \dim(M)$ and $s = \dim(N)$, we may choose $x_1, \ldots, x_{r+1} \in \Ann(N)$ such that 
$\ell(M/xM) < \infty$ and any $r$ elements of $x_1, \ldots, x_{r+1}$ is a system of parameters for $M$ with $x$ being $R$-regular.

Now we can construct $T$, a Cohen-Macaulay module, by taking a resolution of $N$ over $R/xR$
\[ 0 \to T \to \cdots \to (R/xR)^{i_1} \to (R/xR)^{i_0} \to N \to 0. \]
such that $\chi(M, T) = \chi(M, N)$. Note that $\dim(R/xR) = n - r - 1$ where $n = \dim(R)$. So $\dim(T) = n - r - 1$, but $\chi(M, R/xR) = 0$ as $\#(x) = r + 1 > \dim(M)$. Hence, $\chi(M, T) = 0$ if and only if $\chi(M, N) = 0$. In a similar manner we can show $M$ is a Cohen-Macaulay module.

**Proposition 7** Let $R$ be Gorenstein, $M, N$ are Cohen-Macaulay, where $M$ has finite projective dimension and $\ell(M \otimes_R N) < \infty$, $i = \dim(R) - \dim(M) - \dim(N)$, $r = \dim(M)$, $s = \dim(N)$, $\tilde{M} = \Ext^i_R(M, R)$, and $\tilde{N} = \Ext^{s-i}_R(N, R)$. Now 
\[ \chi(M, N) = (-1)^i \chi(\tilde{M}, \tilde{N}). \]
Sketch of Proof  The crucial step is a simple spectral sequence argument. First note
\[ \ell(T^R_j(M, N)) = \ell(T^R_j(M, N)). \]

Now write
\[ T^R_j(M, N) = \text{Ext}^R_n(T^R_j(M, N), R), \]
\[ = \text{Ext}^{n+j-(n-s)}_R(M, \text{Ext}^{n-s}_R(N, R)), \]
\[ = \text{Ext}^{r-+(i-j)}_R(M, \text{Ext}^{n-s}_R(N, R)), \]
\[ = T^R_i(M, \hat{N}). \]

\[ \text{Corollary 7.1} \quad \text{If } \dim(M) + \dim(N) = \dim(R) - 1, \text{ then } \chi(M, N) = -\chi(\hat{M}, \hat{N}). \]

\[ \text{Corollary 7.2} \quad \text{If } M \simeq \hat{M}, N \simeq \hat{N}, \text{ and } \dim(R) - \dim(M) - \dim(N) \text{ is odd, then } \chi(M, N) = 0. \]

\[ \text{Corollary 7.3} \quad \text{If } R, R/\mathfrak{p}, \text{ and } R/\mathfrak{q} \text{ are all Gorenstein, where } \mathfrak{p}, \mathfrak{q} \in \text{Spec}(R), \text{ and } \dim(R) - \dim(R/\mathfrak{p}) - \dim(R/\mathfrak{q}) \text{ is odd, then } \chi(R/\mathfrak{p}, R/\mathfrak{q}) = 0. \]

\[ \text{Corollary 7.4} \quad \text{If } i = 0, \text{ then } \ell(M \otimes_R N) = \ell(\hat{M} \otimes_R \hat{N}). \]

Theorem 8  If \( R \) is Gorenstein, then vanishing holds if and only if for every pair of Cohen-Macaulay modules \( M, \hat{N} \) where \( \text{proj dim}(M) < \infty \) and \( \dim(M) + \dim(N) = \dim(R) \), we have \( \ell(M \otimes_R N) = \ell(M \otimes_R \hat{N}). \)

Sketch of Proof  (⇒) Given \( M \) and \( N \) as above, we have by a result due to Serre that \( T^R_i(M, N) = 0 \) for \( i > 0 \). So,
\[ \chi(M, N) = \ell(M \otimes_R N). \]

Hence we have \( \chi(M, \hat{N}) = \ell(M \otimes_R \hat{N}). \) Now taking a prime filtration on \( N \) and using the additivity of \( \chi \) we have
\[ \chi(M, N) = \sum_{\dim(R/\mathfrak{p}) = \dim(N)} \ell(N/\mathfrak{p})\chi(M, R/\mathfrak{p}) + \sum_{\dim(Q_i) < \dim(N)} \chi(M, Q_i) \]

Similarly we have
\[ \chi(M, \hat{N}) = \sum_{\dim(R/\mathfrak{p}) = \dim(N)} \ell(\hat{N}/\mathfrak{p})\chi(M, R/\mathfrak{p}) + \sum_{\dim(Q_i) < \dim(N)} \chi(M, Q_i) \]

But by vanishing we have \( \sum \chi(M, Q_i) = 0. \) Since \( R \) is Gorenstein, we have \( \ell(N/\mathfrak{p}) = \ell(\hat{N}/\mathfrak{p}). \) Thus \( \chi(M, N) = \chi(M, \hat{N}). \)

Warning: One cannot use the same idea of additivity to prove an analogous statement when both \( M \) and \( N \) have finite projective dimension as \( R/\mathfrak{p} \) may no longer have finite projective dimension.
Recall if $M$ and $N$ are Cohen-Macaulay, then $\dim(M) + \dim(N) = \dim(R) - 1$. Write

$$0 \to T \to \left( \frac{R}{(x_1, \ldots, x_r)} \right)^t \to N \to 0$$

where the $x'_i s \in \text{Ann}(N)$ as earlier. So

$$\chi(M, N) = t\ell(M/xM) - \ell(M \otimes R T),$$

which leads us to:

$$0 \to \left( \frac{R}{(x_1, \ldots, x_r)} \right)^t \to \tilde{T} \to \tilde{N} \to 0$$

This shows that

$$\chi(M, \tilde{N}) = \ell(M \otimes R \tilde{T}) - t\ell(M/xM),$$

So

$$\chi(M, N) = -\chi(M, \tilde{N}).$$

Applying the above technique once more we see $\chi(\tilde{M}, \tilde{N})$. From a previous proposition we see that $\chi(M, N) = -\chi(\tilde{M}, \tilde{N})$. Hence, $\chi(M, N) = 0$. Note that the argument for this part of the proof would work if the projective dimension of both $M$ and $N$ are finite. 

**Remark** When $\dim(M) + \dim(N) = \dim(R)$ (as in the above theorem) we say we have a “proper intersection.”

**Sketch of Proof** This is implied by the fact that for every pair of Cohen-Macaulay modules $T$ and $Q$ with finite projective dimension such that $\ell(T \otimes R Q) < \infty$ and $\dim(T) + \dim(Q) = \dim(R)$, we have $\ell(T \otimes R Q) = \ell(T \otimes R \tilde{Q})$.

**Remark** If $R$ is regular and is a complete intersection ring, then $\ell(T \otimes R Q) = \ell(T \otimes R \tilde{Q})$ can be shown by local Chern characters.

**Theorem 9** If $R$ is Gorenstein and $\dim(R) \leq 5$, then vanishing holds for $R$-modules $M, N$ when both $M$ and $N$ have finite projective dimension.

**Open Problem 10** If $R$ is Gorenstein and $\dim(R) > 5$, does vanishing holds for pairs of $R$-modules $M, N$ when both $M$ and $N$ have finite projective dimension?

**Theorem 11** If $R$ is Gorenstein, then positivity (or nonnegativity) implies vanishing.

**Proof** We can assume $M$ to be Cohen-Macaulay with the projective dimension of $M$ finite. We know that if $\dim(M) < \dim(R)$ and $\ell(N) < \infty$, then $\chi(M, N) = 0$.

Suppose that $R/p$ has the least dimension such that we do not know about vanishing. Then
1. We have
\[
\chi(M, R/p^t) = \ell(R_p/p^tR_p)\chi(M, R/p) + \sum_{\dim(Q_i) < \dim(R/p)} \chi(M, Q_i).
\]
However, the last term in this sum goes to zero by our choice of \(R/p\).

2. If \(\dim(M) = r\) choose \(x_1, \ldots, x_r \in p\) such that \(\ell(M/xM) < \infty\). Set \(\overline{R} = R/x\) and \(\overline{M} = M/xM\), then \(\chi^\ell(M, R/p) = \chi^\ell(\overline{M}, R/p)\) as \(x\) is also an \(M\)-sequence and an \(R\)-sequence.

Thus we can assume that the projective dimension of \(M\) is finite, \(\ell(M) < \infty\), and \(\dim(R/p) = \dim(R) - 1\). So
\[
\chi(M, R/p) = \lim_{t \to \infty} \frac{\chi(M, R/p^t)}{\ell(R_p/p^tR_p)}
\]
Now look at
\[
0 \to p^t \to R \to R/p^t \to 0
\]
So, \(\chi(M, R/p^t) = \ell(M) - \chi(M, p^t)\). Now we have
\[
\chi(M, R/p) = -\lim_{t \to \infty} \frac{\chi(M, p^t)}{\ell(R_p/p^tR_p)}
\]
as the \(\ell(M)/\ell(R_p/p^tR_p)\) term goes to zero in the limit, note \(\dim(p^t) = \dim(R)\).
If positivity or nonnegativity holds, then \(\chi(M, p^t) \geq 0\) and thus \(\chi(M, R/p) \leq 0\).

So take \(y_1, \ldots, y_n\) a maximal \(R\)-sequence. Since \(\ell(M) < \infty\) we may assume that \(y_i \in \text{Ann}(M)\). Write
\[
0 \to N \to (R/y)^t \to M \to 0
\]
Then \(\chi((R/y)^t, R/p) = \chi(M, R/p) + \chi(N, R/p)\). But the left-hand side is zero by a result due to Serre and so each term of the right-hand side is less than or equal to zero. Thus both \(\chi(M, R/p) = 0\) and \(\chi(N, R/p) = 0\).

2 \(\chi_i\)-Conjecture

In this section we will assume that \(R\) is local, \(M, N\) are \(R\)-modules, the projective dimension of \(M\) is finite, \(\ell(M \otimes_R N) < \infty\), and we define
\[
\chi_i(M, N) := \sum_{j=0}^{\text{proj dim}(M) - i} (-1)^j \ell(\text{Tor}^R_{i+j}(M, N)).
\]

Conjecture 12 (Serre) If \(R\) is a regular local ring, then \(\chi_i(M, N) \geq 0\), or \(\chi_i(M, N) = 0\) if and only if \(\text{Tor}^R_j(M, N) = 0\) for \(j \geq i\).

Remark in the above conjecture, the conclusion \(\text{Tor}^R_j(M, N) = 0\) for \(j \geq i\) implies rigidity.
**Theorem 13 (Serre-Auslander)** The above conjecture is true when $R$ is of equal characteristic.

**Theorem 14 (Lichtenbaum)** The above conjecture is true when $R$ is unramified for all $\chi_i$ except possibly $i = 1$.

**Theorem 15 (Hochster)** The above conjecture is true when $R$ is unramified for $\chi_1$.

**Remark** Gabber also claims to have independently proven the above conjecture when $R$ is unramified for $\chi_1$.

**Open Problem 16** The above conjecture is open if $R$ is ramified. To clarify, it is still open when

$$R = \frac{V[[x_1, \ldots, x_n]]}{f}$$

where

$$f = x_n^t + a_1 x_n^{t-1} + \cdots + a_n,$$

$$a_i \in (p, x_1, \ldots, x_{n-1}),$$

$$a_t \in (p, x_1, \ldots, x_{n-1}) - (p, x_1, \ldots, x_{n-1})^2.$$  

In this case, $S = R \widehat{\otimes}_V R$ is no longer regular.

**Theorem 17** If $R$ is a regular local ring where the $\chi_2$-conjecture is valid, then $\chi(M, N) > 0$ when $M$ is Cohen-Macaulay and $\dim(M) + \dim(N) = \dim(R)$.

**Remark** The above conjectures make sense when $R$ is not regular and the projective dimension of $M$ or the projective dimension of $N$ is finite.

### 2.1 Counterexamples

**Example (Dutta-Hochster-Mclaughlin)** Let

$$R = \left( \frac{k[X, Y, U, V]}{(XY - UV)} \right)_{(X, Y, U, V)}.$$  

Now there exists an $R$-module $M$ such that $\ell(M) < \infty$, $\proj \dim(M) < \infty$, $\chi(M, R/p) = -1 \neq 0$, $\dim(M) = 0$, $\dim(R/p) = 1$ where $p = (X, U)$, and hence positivity is false, which implies $\chi_i$ is false.

**Example (Levine)** Let

$$R = \left( \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]}{\sum X_i Y_i} \right)_{(X_1, \ldots, X_n, Y_1, \ldots, Y_n)}.$$  

This was done using non-constructive $K$-theoretic techniques.

**Example (Roberts-Srinivas)**
1. \( R = k[X,Y,Z,W]/f \), where \( f \) has degree three and \( k \) is separable and algebraically closed - the coordinate ring of a cubic surface in \( \mathbb{P}^3 \).

2. \( R \) the coordinate ring \( \mathbb{P}^n \times \mathbb{P}^n \).

**Theorem 18 (Roberts, Gillet-Soulé)** Vanishing holds over complete intersection rings when both \( M \) and \( N \) have finite projective dimension.

**Theorem 19 (Dutta)** There exist complete intersection rings \( R \) along with \( R \)-modules \( M \) and \( N \) both with finite projective dimension such that \( \chi(M,N) = 0 \) but \( \chi_2(M,N) < 0 \). In fact, one can produce examples where all the \( \chi_i \)’s are negative for \( i > 2 \).

In light of the above theorem, we are not sure whether we should believe the positivity conjecture when both \( M \) and \( N \) have finite projective dimension over complete intersection rings.

To prove the above theorem, we need the following special case of a theorem by Auslander and Bridger.

**Theorem 20 (Auslander-Bridger)** Let \( R \) be Gorenstein and \( N \) any finitely generated \( R \)-module, then there exists an exact sequence

\[
0 \to T \to N \oplus R^t \to Q \to 0
\]

where \( \text{proj dim}(Q) < \infty \) and \( T \) is a maximal Cohen-Macaulay module.

**Theorem 21 (Auslander-Buchweitz)** Let \( R \) be Gorenstein and \( N \) any finitely generated \( R \)-module, then there exists an exact sequence

\[
0 \to N \to Q \to T \to 0
\]

where \( \text{proj dim}(Q) < \infty \) and \( T \) is a maximal Cohen-Macaulay module.

**Definition** Given a pair \( M, N \) such that \( \text{proj dim}(M) < \infty \), \( \ell(M \otimes_R N) < \infty \), and \( \text{dim}(M) + \text{dim}(N) = \text{dim}(R) \), we say a finitely generated \( R \)-module \( N' \) is a **companion module** of \( N \) with respect to \( M \) if the following hold:

1. \( \text{dim}(N') = \text{dim}(N) \).
2. \( \text{depth}(N') = \text{dim}(N') - 1 \).
3. \( \ell(M \otimes_R N') < \infty \) and \( \chi(M, N') = \chi(M, N) \).

**Proposition 22** With the above setup, if \( R \) is Gorenstein, \( N \) has a companion module.

**Proof** If \( \text{dim}(M) = r \) we can find \( x_1, \ldots, x_n \in \text{Ann}(N) \) a system of parameters that is an \( R \)-sequence. Set \( \overline{R} = R/xR \), so \( M \) is an \( \overline{R} \)-module. Applying Auslander-Bridger over \( \overline{R} \),

\[
0 \to T \to N \oplus \overline{R}^t \to Q \to 0,
\]
where $Q$ and $T$ are $R$-modules and $\text{proj dim}(Q) < \infty$ and $T$ is a maximal Cohen-Macaulay module. Now we have two cases. Case a: $\dim(Q) = \dim(R)$; and case b: $\dim(Q) < \dim(R)$. We want to reduce case a to case b. By the lectures of Paul Roberts in this mini-course, we have that $\dim(Q) = \dim(R)$ and $\text{proj dim}(Q) < \infty$ implies that $\text{Supp}(Q) = \text{Supp}(R)$. If $S$ is the set of non-zero-divisors of $R$, then $S^{-1}Q$ is $S^{-1}R$-free of rank $s$. Therefore we have the exact sequence

$$0 \rightarrow R^s \rightarrow Q \rightarrow Q' \rightarrow 0,$$

where $\dim(Q') < \dim(R)$ and the $\text{proj dim}(Q') < \infty$. So we have a diagram that looks like:

\[
\begin{array}{cccccc}
0 & \downarrow & 0 \\
R^s & \downarrow & T & \rightarrow & N \oplus R^t & \rightarrow & Q' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & Q' & & 0
\end{array}
\]

From this we obtain the exact sequence:

$$0 \rightarrow T \oplus R^t \rightarrow N \oplus R^t \rightarrow Q' \rightarrow 0$$

So $\dim(Q') < \dim(R)$.

Now we may assume case b. Write

$$0 \rightarrow T \rightarrow N \oplus R^t \rightarrow Q \rightarrow 0$$

with $\dim(Q) < \dim(R)$. So we have

$$\chi^R(M, N) + t\chi^R(M, R) = \chi^R(M, Y) + \chi^R(M, Q)$$

but

$$\chi^R(M, Q) = \sum (-1)^i \chi(\text{Tor}^R_i(M, R/\mathfrak{x}), Q).$$

Since $\text{Tor}^R_i(M, R/\mathfrak{x})$ has finite length, we are left with

$$\chi(M, N) = \chi(M, T) - t\chi(M, R).$$

Since $\dim(Q) < \dim(R)$,

$$0 \rightarrow R^t \rightarrow T \rightarrow N' \rightarrow 0$$

is an exact sequence. So

$$\chi(M, N') = \chi(M, T) - t\chi(M, R) = \chi(M, N).$$

So depth$(N') = \dim(N') - 1$ by depth counting, $\dim(N') = \dim(N)$. \sbox
2.1.1 Discussion of Proof

**Step 1**  
\( R \) is Gorenstein, so suppose vanishing does not hold. So we can find \( M \) Cohen-Macaulay with finite projective dimension, \( p \) a prime ideal such that \( \chi(M, R/p) > 0 \), \( \dim(M) + \dim(R/p) < \dim(R) \), and \( \chi(M, R/q) = 0 \) if \( q \supset p \).

**Step 2**  
From the previous section, we may assume that \( \ell(M) < \infty \) and so we have

\[
\chi(M, R/p) = \frac{\chi(M, R/p^t)}{\ell(R/p^t R_p)} = -\frac{\chi(M, p^t)}{\ell(R/p^t R_p)}.
\]

So \( \chi(M, R/p) > 0 \) which implies \( \chi(M, p^t) < 0 \), note that \( \dim(p^t) = \dim(R) \).

**Step 3**  
By an easy spectral sequence argument (which reduces to a long exact sequence) we find

\[
\chi(M, N) > \ell(\text{Tor}_R^1(M, \text{Ext}_R^1(N, R))) - \ell(\tilde{M} \otimes_R \text{Ext}_R^1(N, R))
\]

\[
\dim(\text{Ext}_R^1(N, R)) < \dim(R) \text{ since } R \text{ is Gorenstein. Suppose that }
\]

\[
\chi(\tilde{M}, \text{Ext}_R^1(N, R)) = 0.
\]

Then \( 0 > \chi(M, N) > \chi_2(\tilde{M}, \text{Ext}_R^1(N, R)) \). Letting \( x \in \text{Ann}(\text{Ext}_R^1(N, R)) \) a non-zero-divisor on \( R \), apply Auslander-Buchweitz to \( \text{Ext}_R^1(N, R) \). Write

\[
0 \rightarrow \text{Ext}_R^1(N, R) \rightarrow Q' \rightarrow T \rightarrow 0,
\]

with the projective dimension of \( Q' \) finite and \( T \) a maximal Cohen-Macaulay module over \( R \). We have

\[
\chi(\tilde{M}, T) = \ell(\text{Tor}_R^0(M, T)) - \ell(\text{Tor}_R^1(M, T)).
\]

So \( \chi_2(M, \text{Ext}_R^1(N, R)) = \chi_2(M, Q') < 0 \), but \( \chi(M, Q') = 0 \).

The condition \( \chi(M, \text{Ext}_R^1(N, R)) = 0 \) happens:

1. For all counterexamples to vanishing listed above,
2. When \( R \) is Gorenstein of dimension 3.

3 Some on Positivity

In this section we will assume that \( R \) is local and Noetherian, \( \dim(R) = d \), \( \text{char}(R) = p \) where \( p \) is a prime, and that \( R/m \) is perfect (Cohen-Macaulay with finite projective dimension) for convenience. \( M \) and \( N \) will be \( R \)-modules with \( \ell(M \otimes_R N) < \infty \) and \( \dim(M) + \dim(N) = \dim(R) \). Finally, \( f \) will denote the Frobenius endomorphism, specifically:

\[
f : R \rightarrow R \quad f^n : R \rightarrow R \quad r \mapsto r^p \quad r \mapsto r^{p^n}.
\]
The notation $f^nR$ represents the $R$-algebra structure defined by
\[ r \cdot x := r^{p^n}x \quad \text{and} \quad x \cdot r := x r, \]
where $x \in f^nR$. The notation $f^nN$ represents the left $R$-module structure defined by
\[ r \cdot x := r^{p^n}x, \]
where $x \in f^nN$. We define the Frobenius functor $F$ via
\[ F^n(-) := - \otimes R f^nR, \]
where the $R$-module structure is the normal one on the right.

**Theorem 23 (Peskine-Szpiro)** If $\text{proj dim}(M) < \infty$, then $\text{proj dim}(F^n(M)) < \infty$. Also $\text{Supp}(F^n(M)) = \text{Supp}(M)$, so $\ell(M \otimes_R N) = \ell(F^n(M) \otimes_R N) < \infty$.

### 3.1 Some Facts

Supposing $\text{proj dim}(M) < \infty$ and we have $M_R \xrightarrow{f^n} R N$, we have
\[ \text{Tor}_i^R(M, f^nN) = \text{Tor}_i^R(F^n(M), N). \]
This is because given a resolution $F_\bullet$ of $M$,
\[ F_\bullet \otimes_R f^nN \simeq F_\bullet \otimes_R f^nR \otimes_R N \simeq F^n(F_\bullet) \otimes_R N. \]

Now supposing $R$ is a complete local domain, where $k = R/\mathfrak{m}$, we have the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
R & \xrightarrow{f^n} & R \\
\downarrow \text{module finite} & & \uparrow \\
\mathbb{k}[X_1, \ldots, X_d] & \xrightarrow{f^n} & \mathbb{k}[X_1, \ldots, X_d]
\end{array}
\end{array}
\]

Note that because $k$ is perfect, the image of the bottom map is $\mathbb{k}[X_1^{p^n}, \ldots, X_d^{p^n}]$. So the torsion-free rank of $f^nR$ is $p^dn$.

Now we have a question: When $\text{proj dim}(M) < \infty$, how are $\chi(F^n(M), N)$ and $\chi(M, N)$ related?

When attacking this question we may assume that $N = R/\mathfrak{p} = \overline{R}$ as $\chi$ is additive. So we have
\[ 0 \rightarrow \bigoplus_{i=1}^{p} \overline{R} \rightarrow f\overline{R} \rightarrow Q \rightarrow 0 \]
where $\dim(Q) < \dim(\overline{R})$. So
\[ 0 \rightarrow \bigoplus_{i=1}^{p'} f^n\overline{R} \rightarrow f\overline{R} \rightarrow f^{n-1}Q \rightarrow 0, \]
where $\dim(Q) < \dim(\overline{R})$. So
and so \( \chi(M, \mathcal{F}^nR) = p^n \chi(M, \mathcal{F}^{n-1}R) + \chi(M, \mathcal{F}^{n-1}Q). \)

We obtain:

\[
\chi(F^n(M), R) = p^n \chi(M, R) + c_n \chi(M, R/p) + \cdots.
\]

By recalling: \( \chi(F^n(M), N) = \chi(M, f^nN). \)

**Definition** Define:

\[
\chi_{\infty} := \lim_{n \to \infty} \frac{\chi(F^n(M), N)}{p^n \cdot \text{codim}(M)}
\]

and

\[
\alpha_{\infty} := \lim_{n \to \infty} \frac{\chi(F^n(M), N)}{p^n \cdot \text{dim}(M)}
\]

Note that since \( \text{dim}(M) + \text{dim}(N) \leq \text{dim}(R) \), we have \( \text{dim}(M) \leq \text{codim}(N) \) and that we have equality in the positivity case.

**Theorem 24** We have that

\[
\alpha_{\infty}(M, R/p) = \chi(M, R/p) + \sum_{\text{dim}(R/p) < \text{dim}(R/p)} c_i \chi(M, R/p_i)
\]

where each \( c_i \in \mathbb{Q} \).

So when \( \text{dim}(M) + \text{dim}(N) < \text{dim}(R) \), \( \chi_{\infty}(M, N) = 0 \) and when \( \text{dim}(M) + \text{dim}(N) = \text{dim}(R) \), \( \chi_{\infty}(M, N) = \alpha_{\infty}(M, N). \)

**Theorem 25** If \( R \) is local, \( \text{proj dim}(M) < \infty \), \( M \) is Cohen-Macaulay, and \( \text{dim}(M) + \text{dim}(N) = \text{dim}(R) \), then \( \chi_{\infty}(M, N) > 0. \)

**Remark** If \( M \) is not assumed to be Cohen-Macaulay, then the theorem is still open!

Proof of the above statement can be made much simpler by the fact:

\[
\lim_{n \to \infty} \frac{\ell(\text{Tor}_i^R(F^n(M), N))}{p^n \cdot \text{codim}(M)} = \begin{cases} 0 & \text{for } i > 0, \\ \neq 0 & \text{for } i = 0. \end{cases}
\]

The first proof of this fact needed \( R \) to be Gorenstein. Now we know it for all \( R \). Also note that this is really a special case of the New Intersection Theorem.

**Theorem 26 (Seibert)**

1. If \( F_* \) is a finite complex of finitely generated free \( R \)-modules, \( N \) a finitely generated \( R \)-module of dimension \( r \) such that for each \( i \geq 0 \),

\[
\ell(H_i(F_* \otimes_R N)) < \infty,
\]

define

\[
\chi(F_*, N) = \sum (-1)^i \ell(H_i(F_* \otimes_R N)).
\]

Then \( \chi(F^n(F_*), N) = c_r p^{nr} + c_{r-1} p^{n(r-1)} + \cdots + c_0 \), where \( c_i \in \mathbb{Q} \).
2. Given an exact sequence

\[ 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0, \]

we have for some constant \( K \)

\[ \ell(H_i(F_\bullet \otimes_R N)) - \ell(H_i(F_\bullet \otimes_R N')) - \ell(H_i(F_\bullet \otimes_R N'')) \leq Kp^{n(r-1)}. \]

**Applications**

**Theorem 27**  
If \( R \) is a regular local ring, \( p \) a non-zero-divisor on \( M \), where \( M \) is a Cohen-Macaulay module, and \( p^tN = 0 \) for some \( t > 0 \), then \( \chi(M, N) > 0 \).

**Proof**  
Write \( N \supset pN \supset \cdots \supset p^{t-1}N \supset 0 \) \( \chi(M, N) = \sum \chi(M, p^iN/p^{i+1}N) \). So we can assume that \( pN = 0 \). Since \( p \) is a non-zero-divisor on \( R \) and on \( M \) we have \( \chi^R(M, N) = \chi^{R/pR}(M/pM, N) \)

but \( \overline{M} = M/pM \) is Cohen-Macaulay. So by vanishing,

\[ \overline{\chi^\infty(M, N)} \leq \chi^R(M, N) \]

by vanishing

So we see that \( \chi^R(M, N) > 0 \). \( \square \)

**Remark**  
This theorem was extended by Kurano and Roberts.

**Theorem 28 (Foxby)**  
If \( R \) is local and \( M \) is an \( R \)-module with finite projective dimension and the dimension of \( N \) is one, then \( \chi(M, N) > 0 \).

**Theorem 29 (Tennison)**  
If \( R \) is regular, \( M \) and \( N \) are \( R \)-modules, and suppose that \( \ell(G_m(M) \otimes G_m(N)) < \infty \).

Then \( \chi(M, N) = e_m(M)e_m(N) \).

More generally, if \( M = R/p, N = R/q, Y = \text{Spec}(M), Z = \text{Spec}(N), \) and \( \widetilde{Y}, \widetilde{Z} \) are the blow-ups of \( Y \) and \( Z \), then

\[ \ell(G_m(M) \otimes G_m(N)) < \infty \leftrightarrow \widetilde{Y} \cap \widetilde{Z} = \emptyset. \]

**Theorem 30 (Dutta)**  
If \( \widetilde{Y} \cap \widetilde{Z} \) is a finite set of points, then \( \chi(M, N) \geq e_m(M)e_m(N) \).

The proof of this last theorem uses nonnegativity results by Gabber and Intersection Theory as introduced in Fulton’s book.
3.2 Chow Groups

Let $\mathbb{A}_i(R)$ denote the $i$th Chow Group of $R$.

**Theorem 31 (Claborn-Fossum)**

1. For a field $k$, if $R = k[X_1, \ldots, X_n]$, then $\mathbb{A}_i(R) = 0$ for $i < n$ and $\mathbb{A}_n \simeq \mathbb{Z}$.

   For a DVR $V$, if $R = V[X_1, \ldots, X_n]$, then $\mathbb{A}_i(R) = 0$ for $i < n + 1$.

2. For a field $k$, if $R = k[[X_1, \ldots, X_n]]$, then $\mathbb{A}_i(R) = 0$ for $i < n$ and $\mathbb{A}_n \simeq \mathbb{Z}$.

   For a DVR $V$, if $R = V[[X_1, \ldots, X_n]]$, then $\mathbb{A}_i(R) = 0$ for $i < n + 1$.

**Conjecture 32 (Gersten)** If $R$ is any regular local ring, of dimension $n$, then $\mathbb{A}_i(R) = 0$ for $i < n$.

**Theorem 33 (Quillen)** If $R$ is a regular local ring smooth over $k$, then $\mathbb{A}_i(R) = 0$ for $i < n$.

His proof was geometric, looking at the tangent cone and tangent space.

**Theorem 34 (Gillet-Levine)** If $R$ is regular local and smooth over an excellent DVR $V$, then $\mathbb{A}_i(R) = 0$ for $i < n$.

This proof is an extension of Quillen’s arguments.

**Remark** Cannot assume $R$ is complete for the Chow group problem.

**Question** For $R \to \hat{R}$, can we say $\mathbb{A}_i(R) \hookrightarrow \mathbb{A}_i(\hat{R})$

While this is not true in general, (Hochster gave a counterexample in the non-normal case) we do have this:

**Theorem 35 (Kamoi-Kurano)** If $R$ is an excellent regular local ring, then $\mathbb{A}_i(R) \hookrightarrow \mathbb{A}_i(\hat{R})$.

Gersten’s Conjecture is still open when $R$ is ramified regular local. We have the following result:

**Theorem 36 (Dutta)** If $R$ is a ramified regular local ring, then $\mathbb{A}_1(R)$.

For $R = \frac{V[[X_1, \ldots, X_n]]}{p - \sum x_i^2}$, the result that $\mathbb{A}_i(R) = 0$ when $i < n$ was first proved by Levine using $K$-theoretic techniques. Dutta gives an algebraic proof which does not work for when the ring $R$ is not so nice.
Conjecture 37 (Bass-Quillen) If $R$ is a regular local ring and $P$ a finitely generated projective module over $R[X_1, \ldots, X_n]$, then $P = P_0 \otimes_R R[X]$ where $P_0$ is a finitely generated projective module over $R$.

The case where $R$ is a field, conjectured by Serre, was proved independently by Quillen and Suslin.

Theorem 38 (Lindel) Proved the above conjecture when $R$ is geometrically regular local ring. That is, when $R$ is a local ring which is smooth over $k$.

Lindel had a special proposition, which we will call a theorem:

Theorem 39 (Lindel) If $A$ is an affine domain over $k$ of dimension $d$ with maximal ideal $m$ such that $A_m$ is a regular local ring, and $A/m$ is a finite separable extension of $k$, then there exists $x_1, \ldots, x_t \in A$ such that

1. $A = k[x_1, \ldots, x_t]$ and $m = (f(x_1), x_2, \ldots, x_t)$ where $f$ is the monic irreducible polynomial of $\mathfrak{m}_1 \in A/m(= k(\mathfrak{m}_1))$ over $k$.

2. $B = k[x_1, \ldots, x_d]$, $n = B \cap m = (f(x_1), x_2, \ldots, x_d)$ and $B_n \to A_m$ is étale (flat with $\Omega_{A_m/B_n} = 0$).

Using Zariski’s Main Theorem we obtain an extension of this result:

Theorem 40 If $(R, m, k)$ is a regular local ring which is smooth over $k$, or an excellent DVR $V$, and $R/m$ is separably generated over $k$ or $V/m$, then there exists $(B, n, k)$ another regular local ring contained in $R$ such that

1. $B = W[X_1, \ldots, X_d](f(x_1), x_2, \ldots, x_d)$ where $W$ is a field or an excellent DVR contained in $R$ and $f(X_1)$ is a monic irreducible polynomial in $W[X_1]$.

2. If we take any $a \in m^2$ ($a \neq 0$), then we can choose $(B, n, k)$ such that $B \to R$ is étale, $B \cap aR = (h)$ and $B/ah \simeq R/aR$.

This theorem helps us to give an alternate proof of Serre’s Theorem on Intersection-Multiplicities without using “complete-Tor.” This also provides an alternate proof of Quillen’s Theorem on Chow groups. Take $a \in \text{Ann}(M) \cap \text{Ann}(N) \cap m^2$ and apply the above theorem. This pulls back our problem to the polynomial case. Thus, it brings the Intersection-Multiplicities and the Chow group problems back to the polynomial case. Hence, only the ramified case is left.

4 Canonical Element Conjecture

Let $(A, m, k)$ be a local ring of dimension $n$ and $x = x_1, \ldots, x_n$ a system of parameters for $A$. If we consider the Koszul complex $K(x, A)$ we can find a chain-map from the Koszul complex to a minimal free resolution $F_\bullet$ of $k$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow & \cdots & \longrightarrow & A^d & \longrightarrow & A & \longrightarrow & A/x & \longrightarrow & 0 \\
& | & & | & & | & & | & & | & & | & & | \\
& \phi_n & & \phi_{n-1} & & \phi_1 & & \text{id} & = \phi_0 & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
& A^{t_n+1} & \longrightarrow & A^{t_n} & \longrightarrow & A^{t_{n-1}} & \longrightarrow & \cdots & \longrightarrow & A^t & \longrightarrow & A & \longrightarrow & k & \longrightarrow & 0
\end{array}
$$

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Conjecture 41  
In the situation above, $\varphi_n \neq 0$ for any system of parameters $x$.

4.1 Supposing $\varphi_n = 0$

Suppose $\varphi_n = 0$. Applying $\text{Hom}_A(-, A)$, and denoting this with $(-)^*$, to the diagram above, we obtain:

\[
\begin{array}{c}
0 \rightarrow A \rightarrow (A^1)^* \rightarrow \cdots \rightarrow (A^{n-1})^* \rightarrow (A^n)^* \rightarrow \cdots \\
\downarrow \text{id} \quad \downarrow \varphi_1^* \quad \downarrow \varphi_{n-1}^* \quad \downarrow \varphi_n^* = 0 \\
0 \rightarrow A \rightarrow A^n \rightarrow \cdots \rightarrow A^n \rightarrow A \rightarrow 0
\end{array}
\]

Letting $G = \text{Coker}(A^2 \rightarrow A^n)$ and $\tilde{G} = \text{Coker}((A^{n-2})^* \rightarrow (A^{n-1})^*)$

\[
\begin{array}{c}
0 \rightarrow \text{Ext}_{A}^{n-1}(k, A) \rightarrow \tilde{G} \rightarrow \text{Im}(\tilde{G} \hookrightarrow (A^n)^*) \rightarrow \text{Ext}_{A}^{n}(k, A) \rightarrow 0 \\
\downarrow \kappa \quad \downarrow \eta \quad \downarrow 0 \\
0 \rightarrow H_1(x, A) \rightarrow G \rightarrow xA \rightarrow 0
\end{array}
\]

So, we have the complexes:

\[
\begin{array}{c}
0 \rightarrow A \rightarrow (A^1)^* \rightarrow \cdots \rightarrow (A^{n-1})^* \rightarrow \tilde{G} \rightarrow 0 \\
\downarrow \text{id} \quad \downarrow \varphi_1^* \quad \downarrow \varphi_{n-1}^* \quad \downarrow \eta \\
0 \rightarrow A \rightarrow A^n \rightarrow \cdots \rightarrow A^n \rightarrow \tilde{G} \rightarrow 0
\end{array}
\]

and

\[
\begin{array}{c}
0 \rightarrow A \rightarrow A^n \rightarrow \cdots \rightarrow A^n \rightarrow G \rightarrow 0 \\
\downarrow H_1(x, A) \\
0 \rightarrow A \rightarrow A^n \rightarrow \cdots \rightarrow A^n \rightarrow G \rightarrow 0
\end{array}
\]

Though $K_\bullet(x, A)$ is not necessarily exact, we still can prove the following:

Proposition 42  
There exists a free complex $L_\bullet$ of finitely generated free modules and maps $\psi_\bullet : L_\bullet \rightarrow K_\bullet(x, A)_{+1}$ such that

1. $L_\bullet$ is minimal and
2. $\psi_\bullet$ induces an isomorphism $H_i(L_\bullet) \simeq H_i(K_\bullet(x, A))_{+1}$ for $i > 0$.

Then the mapping cone of $\psi_\bullet$ gives a free resolution of $xA$.

This forces $\psi_{n-1} : A^{r_{n-1}} \rightarrow A$ to be onto. Actually, $\varphi_n \neq 0$ if and only if $\psi_{n-1}$ is not onto, which is the case if and only if $K_\bullet(x, A)$ embeds into the free minimal resolution of $A/xA$. This seems to be Robert’s way of looking at the Canonical Element Conjecture.
Consider the diagram

\[ \cdots \rightarrow A^n \xrightarrow{\alpha_n} A^{n-1} \xrightarrow{\alpha_{n-1}} A^{n-2} \xrightarrow{\alpha_{n-2}} \cdots \rightarrow A^0 \rightarrow H_1(x, A) \]

0 \rightarrow A \rightarrow A^n \rightarrow \cdots \rightarrow A^n \rightarrow G \rightarrow 0

and suppose that \( \psi_{n-1} \) is onto. Then we can break it up into:

1. \( A^{n-1} = Ae_1 \oplus (\bigoplus_{i=2}^{n-1} Ae_i) \).
2. \( \alpha_n(A) \subset \bigoplus_{i=2}^{n-1} Ae_i \)

\( \text{Coker}(\alpha_n) = A \oplus S'_{n-1} \) so the cokernel is a free summand. So if the Canonical Element Conjecture is true, this cannot happen.

From this with some work we get the following theorem:

**Theorem 43**  If \((A, m, k)\) is local, take a minimal resolution of \(k\) and let \(S_i = \text{Syz}^i(k)\). Then \(A\) is regular if and only if \(S_i\) has a free summand for some \(i > 0\).

Applying \(\text{Hom}_A(-, A)\) to the diagram above, we obtain

\[ P_\bullet : 0 \rightarrow (A^n)^* \rightarrow \cdots \rightarrow (A^{n-2})^* \rightarrow (A^{n-1})^* \rightarrow M \rightarrow 0 \]

where \(M = \text{Coker}(\alpha_{n-1}^*)\) and \(\theta(1) = \nu\), a minimal generator of \(M\). Such that \(x\nu = 0\). So we have that \(P_\bullet\) is a complex of finitely generated free \(A\)-modules such that \(\ell(H_i(P_\bullet)) < \infty\) for \(i > 0\) and \(H_0(P_\bullet)\) has a minimal generator killed by \(x\), and hence is killed by a power of \(m\). Thus the Canonical Element Conjecture is true if and only if the Improved New Intersection Theorem is true. It is enough to prove the Improved New Intersection Conjecture when \(M\) is locally free on \(\text{Spec}(A) - \{m\}\).

Suppose that \(\text{depth}(A) = \text{dim}(A) - 1\) and \(A\) is the homomorphic image of a Gorenstein ring \(R\) such that \(\text{dim}(R) = \text{dim}(A)\). Then the Canonical Element Conjecture holds in the following cases:

1. \(\text{Ext}_R^1(A, R)\) is decomposable.
2. \(\text{Ext}_R^1(A, R)\) is cyclic.

Now if \(\theta : \text{Ext}_R^1(k, \Omega) \rightarrow H_m^0(\Omega)\) where \(\Omega = \text{Hom}_R(A, R)\), the Canonical Element Conjecture says that \(\theta \neq 0\). Write

\[ I^\bullet : 0 \rightarrow \Omega \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^{n-1} \rightarrow E \rightarrow 0 \]
where $E$ is the injective hull of $A/\mathfrak{m}A$. By the same kind of argument as used before, but now using injective complexes we get a complex of injective modules $J^\bullet$ with $\varphi^\bullet: I^\bullet \to J^\bullet$ such that $\varphi^\bullet$ induces an isomorphism on cohomology,

$$
0 \to \Omega \to I^0 \to I^1 \to \cdots \to I^{n-1} \to E \to 0
$$

$$
0 \to J^0 \to \cdots \to J^{n-2} \to J^{n-1} \to J^n \to 0
$$

thus the mapping cone of $\varphi^\bullet$ gives an injective resolution of $\Omega$.

Following the same line of arguments, we can show that $\theta \neq 0$ if and only if $\varphi_{n-1}$ is not injective. Not injective means that the socle must get killed! See Shamash’s article.

Using these ideas we get that

1. If $x \in \mathfrak{m} \text{Ann}(\text{Ext}^1_R(A, R))$, then $A/xA$ satisfies the Canonical Element Conjecture.
2. If $\text{Ext}^1_R(A, R) = 0$, then $A$ satisfies the Canonical Element Conjecture. In particular
   
   (a) If $\Omega$ is $S_3$, $A$ satisfies the Canonical Element Conjecture.
   (b) $0 \to \Omega \to R \to R/\Omega \to 0$, $R/\Omega$ satisfies the Canonical Element Conjecture.
   (c) If $A$ is an almost complete intersection ring and $p$ is a non-zero-divisor on $A$, then $A$ satisfies the Canonical Element Conjecture.
   (d) If $A$ is almost a complete intersection ring, with $A = R/\lambda R$. Take $x_1, \ldots, x_n$ a system of parameters of $R$. Is $\ell(A/x) > \ell(\text{Tor}^R_1(x, R/\lambda R))$?

**Remark** For Canonical Element Conjecture, we may assume $A$ is almost a complete intersection ring and that $p$ is a parameter on $A$.

### 4.2 The Intersection Theorem in Characteristic $p$

Let us consider the Intersection Theorem in characteristic $p$ which is due independently to both Roberts and Peskine-Szpiro.

The statement is as follows: Consider a complex of finitely generated free modules of length $s$

$$
F_\bullet : 0 \to F_s \to \cdots \to F_1 \to F_0 \to 0
$$

where $\ell(H_i(F_\bullet)) < \infty$ and not all are zero for every $i$, then $s \geq d = \dim(A)$.

**Theorem 44** Let $A$ be local with dimension $d$ and of non-zero characteristic $p$. And consider the complex of free $A$-modules $F_\bullet$ with $\ell(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H^0_0(H_0(F_\bullet)) \neq 0$. Assume $M = H_0(F_\bullet)$ is locally free on $\text{Spec}(A) - \mathfrak{m}$ and take any finitely generated $A$-module $N$. Define

$$
\chi(F_\bullet, N) := \ell(H^0_m(M \otimes_A N)) + \sum_{i > 0} (-1)^i \ell(H_i(F_\bullet \otimes_A N)).
$$
Similarly define
\[ \chi_\infty(F_\bullet, N) := \lim_{n \to \infty} \frac{\chi(F^n(F_\bullet), N)}{p^{nd}}. \]

Then we have the following:

1. If \( \dim(N) < d \), then \( \chi_\infty(F_\bullet, N) = 0 \).

2. (a) If \( \dim(N) = d \) and \( s < d \), then \( \chi_\infty(F_\bullet, N) = 0 \).
   (b) If \( \dim(N) = d \) and \( s = d \), then \( \chi_\infty(F_\bullet, N) > 0 \).

**Corollary 44.1** The Improved New Intersection Theorem is true in characteristic \( p \).

**Proof** \( M \) has a minimal generator which is killed by \( m^t \). So,
\[ M \to A/I \]
where the minimal generator maps onto \( \mathbb{T} \) in \( A/I \). Hence we get an onto map \( F^n(M) \to A/I[p^n] \). This implies that
\[ \lim_{n \to \infty} \frac{\ell(A/I[p^n])}{p^{nd}} > 0. \]
But higher homologies go to zero in the limit, hence by the previous theorem, \( s \geq d \). \[ \square \]

**References and Further Reading**


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