Problem Set 3: Pebbles

Opens: 3 p.m. November 15th, 2021

Due: 3 p.m. November 29th, 2021

- You must work independently.
- Write your solutions clearly and show all of your work.
- Include your name, student ID number, and email address.
- Email a pdf file of your solution to `ugrad_services@math.utah.edu` by the deadline.
- A winner will be decided on the basis of the best solution submitted. If no best solution can be determined (i.e. there exist relatively identical solutions), the winner will be the student who submitted their solution first.
- Each submission will be given 3 points for a fully correct solution and 1-2 points for a partially correct solution. The winner of each problem set will get a bonus of ϵ points.
- Please don't just search online for a solution - that isn't the point of this contest.
- Enjoy the problems!

Problem 1 [1 point]: Alice and Bob play a game in which they take turns removing pebbles from a heap that initially has n pebbles. The number of pebbles removed at each turn must be one less than a prime number. The winner is the player who takes the last pebble. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. For example, if n = 17, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining pebbles to win.

Problem 2 [1 point]: There is a heap of 1050 pebbles on a table. One is allowed to perform the following operation: choose one of the heaps on the table containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Prove it is impossible to reach a situation where all the heaps on the table have exactly n stones for any $1 < n < 1050$ (and specifically none of the heaps are empty).

Problem 3 [1 point]: Consider the integer lattice in the plane with one pebble placed at the origin. We play a game in which at each step one pebble is removed from a node of the lattice and two new pebbles are placed at two neighboring nodes, provided that those nodes are unoccupied by another pebble. A neighboring node is a node one unit to the left, right, up, or down. Prove that at any time there will be at least one pebble at taxi-cab distance of at most 5 from the origin.