Solutions

1. The method of partial fractions produces

$$\frac{x^2+x-5}{(x-1)^2} = 1+\frac{3x-6}{(x-1)^2} = 1+\frac{3}{x-1}-\frac{3}{(x-1)^2}.$$

Integration by parts gives

$$\int \frac{e^x}{x-1}\,dx = \frac{e^x}{x-1} + \int \frac{e^x}{(x-1)^2}\,dx$$

Thus

$$\int \frac{x^2+x-5}{(x-1)^2}e^x dx = \int e^x dx + \frac{3e^x}{x-1} + \int \frac{3e^x}{x-1}dx - \int \frac{3e^x}{x-1}dx = e^x + \frac{3e^x}{x-1} + C.$$

2. If $|x| < 1$, then $\sum_{n=1}^{\infty} nx^n = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\frac{d}{dx}(1-x)^{-1} = x(1-x)^{-2}$. Hence

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = 2^{-1}(1-2^{-1})^{-2} = 2.$$

3. $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \left\{x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right\}dx =$

$\left\{\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x-4\tan^{-1}x\right\}_0^1 = \frac{22}{7}-\pi.$

4. Let $A = 1+\frac{1}{3^p}+\frac{1}{5^p}+\ldots$ and $B = \frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\ldots$ Then $2^pB = A + B$ from which it follows that $A = (2^p - 1)B$. Therefore

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots} = \frac{A+B}{A-B} = \frac{2^pB}{(2^p-2)B} = \frac{2^p}{2^p-2}.$$

5. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n^3} > \int_1^{n^3}\frac{dx}{x} > \frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n^3}$. It follows from this that

$$\ln n^3 - 1 > 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n^3} > \ln n^3.$$

Since $\ln n^3 = 3\ln n$, this implies that

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n^3}}{\ln n} = 3.$$

6. Let $f(x) = mx - 1 + x^{-1}$ for $x > 0$. If $m \le 0$, then $f(x) < 0$ for any $x > 1$. Thus we may assume that $m > 0$. Then $f'(x) = m - x^{-2}$ which is negative for $x \in (0, \sqrt{1/m})$ and positive for $x \in (\sqrt{1/m}, \infty)$. Thus f(x) has a minimum when $x = \sqrt{1/m}$. Therefore $f(x) \ge 0$ for $x \in (0, \infty)$ if, and only if, $f(\sqrt{1/m}) \ge 0$. Since $f(\sqrt{1/m}) = 2\sqrt{m} - 1$ and this is ≥ 0 if, and only if, $m \ge 1/4$, it follows that the smallest value of m such that $f(x) \ge 0$ for all positive x is $m = 1/4$.