Some Sample Problems

1. Find the antiderivative: $\int \sqrt{tan(x)}dx$
   Let $u = \sqrt{tan(x)}$; $u^2 = tan(x), dx = \frac{2udu}{1+u^4}$. Then $\int u \frac{2udu}{u^4+1} = 2\int \frac{u^2}{u^4+1}du$.
   $u^4 + 1 = (u^2 + \sqrt{2}u + 1)(u^2 - \sqrt{2}u + 1)$, so by partial fractions:
   $2\int \frac{u^2}{u^4+1}du = -\frac{1}{\sqrt{2}}\int \frac{u}{u^2+\sqrt{2}u+1}du + \frac{1}{\sqrt{2}}\int \frac{u}{u^2-\sqrt{2}u+1}du$.
   Now, $\int \frac{u}{u^2+Ku+1}du = \frac{1}{2}ln(u^2 + Ku + 1) - \frac{K}{4-K^2}tan^{-1}(\frac{2u+K}{\sqrt{4-K^2}})$.
   Substituting $K = \sqrt{2}, -\sqrt{2}$ gives:
   $\int \sqrt{tan(x)}dx = \frac{-1}{2\sqrt{2}}[ln(u^2+\sqrt{2}u+1)-ln(u^2-\sqrt{2}u+1)]+\frac{1}{\sqrt{2}}[tan^{-1}(\sqrt{2}u+1) + tan^{-1}(\sqrt{2}u - 1)]|_{u=\sqrt{tan(x)}}$ $(+C)$

2. What is the sum of $1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \cdots$
   Let $A = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \cdots$
   $B = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$
   Then $B = ln(2)$ $(= \sum_1^\infty \frac{(-1)^n}{n} = ln(1 + x)|_{x=1})$.
   And $A + B = 2[1 + \frac{1}{3} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} - \frac{1}{10} - \frac{1}{12} + \frac{1}{13} + \cdots]$
   $= 2[1 + (\frac{1}{3} - \frac{1}{4}) - (\frac{1}{6} - \frac{1}{7}) + (\frac{1}{9} - \frac{1}{10}) - (\frac{1}{12} - \frac{1}{13}) + \cdots]$
   $= 2[1 + \frac{1}{3*4} - \frac{1}{6*7} + \frac{1}{9*10} - \frac{1}{12*13} + \cdots]$
   $= 2[1 + \sum_1^\infty \frac{(-1)^{n+1}}{3n(3n+1)}]$

   $\sum_1^\infty \frac{(-1)^{n+1}}{3n(3n+1)} = \int_0^1 \sum_1^\infty \frac{(-1)^n x^{3n}}{3n}dx = \int_0^1 \frac{1}{3}ln(1 + x^3)dx$

   $(ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots)$

   Therefore, $A + B = 2 + 2\int_0^1 \frac{1}{3}ln(1 + x^3)dx$
   $= 2 + \frac{2}{3}\int_0^1 ln(1 + x^3)dx$
   Integration by parts gives: $\int_0^1 ln(1 + x^3)dx = xln(1+x^3)|_0^1 - \int_0^1 \frac{3x^2}{1+x^3}xdx$
   $= ln(2) - 3\int_0^1 \frac{x^3}{x^3+1}dx$
   So $A + B = 2 + \frac{2}{3}ln(2) - 2\int_0^1 \frac{x^3}{x^3+1}dx$

   Now $\int_0^1 \frac{x^3}{x^3+1}dx = \int_0^1 \frac{x^3+1}{x^3+1}dx - \int_0^1 \frac{dx}{x^3+1} = 1 - \int_0^1 \frac{dx}{x^3+1}$
   So $A + B = 2 + \frac{2}{3}ln(2) - 2[1 - \int_0^1 \frac{dx}{x^3+1}] = \frac{2}{3}ln(2) + 2\int_0^1 \frac{dx}{x^3+1}$

   Partial fractions gives:
   $\int_0^1 \frac{dx}{x^3+1} = \frac{1}{3}\int_0^1 \frac{dx}{x+1} - \frac{1}{3}\int_0^1 \frac{x-2}{x^2-x+1}dx = \frac{1}{3}ln(2) - \frac{1}{6}\int_0^1 \frac{2x-4}{x^2-x+1}dx$

$= \frac{1}{3}ln(2) - \frac{1}{6}\int_0^1 \frac{2x-1}{x^2-x+1}dx - \frac{1}{6}\int_0^1 \frac{-3}{x^2-x+1}dx$
$= \frac{1}{3}ln(2) - \frac{1}{6}ln(x^2-x+1)|_0^1 + \frac{1}{2}\int_0^1 \frac{dx}{x^2-x+1}$
$= \frac{1}{3}ln(2) + \frac{1}{2}\int_0^1 \frac{dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$
$= \frac{1}{3}ln(2) + \frac{1}{2} * \frac{2}{\sqrt{3}}tan^{-1}(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}})|_0^1$
$= \frac{1}{3}ln(2) + \frac{1}{\sqrt{3}}[tan^{-1}(\frac{1}{\sqrt{3}} - tan^{-1}(\frac{-1}{\sqrt{3}}]$
$= \frac{1}{3}ln(2) + \frac{1}{\sqrt{3}} * \frac{\pi}{3}$
Finally, $A + B = \frac{2}{3}ln(2) + 2[\frac{1}{3}ln(2) + \frac{\pi}{3\sqrt{3}}]$
$= \frac{4}{3}ln(2) + \frac{2\pi}{3\sqrt{3}}$
So $A = \frac{4}{3}ln(2) - ln(2) + \frac{2\pi}{3\sqrt{3}}$
$= \frac{1}{3}ln(2) + \frac{2\pi}{3\sqrt{3}}$

3. Find the total perimeter of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where a is a fixed positive number.
$y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$ (1st quadrant piece, take a look at the graph)
$1 + (y')^2 = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$
Therefore the total length $= 4\int_0^a a^{\frac{1}{3}} x^{\frac{-1}{3}} dx$
$= 4a^{\frac{1}{3}} \cdot \frac{3}{2}x^{\frac{2}{3}}|_0^a = 6a$

4. Let $f(x) = \int_1^x \frac{ln(t)}{1+t}dt$ for $x > 0$. Find $f(x) + f(\frac{1}{x})$.
(As a check: $f(2) + f(\frac{1}{2}) = \frac{1}{2}(ln2)^2$.)

Let $g(x) = f(x) + f(\frac{1}{x}), x > 0, g' = f'(x) - \frac{1}{x^2}f'(\frac{1}{x})$
$= \frac{ln(x)}{1+x} - \frac{1}{x^2}\frac{ln(\frac{1}{x})}{1+\frac{1}{x}}$
$= \frac{ln(x)}{1+x} + \frac{ln(x)}{x(1+x)}$
$= \frac{ln(x)}{x}$

$g'(x) = \frac{ln(x)}{x}$ implies $g(x) = \frac{1}{2}(ln(x))^2 + C$.
$g(\frac{1}{2}) = \frac{1}{2}(ln(2))^2$ implies $C = 0$
Therefore, $f(x) + f(\frac{1}{x}) = \frac{1}{2}(ln(x))^2$