# Calculus Challenge 2003

1) Which is larger (for $n > 8$): $(\sqrt{n+1})^{\sqrt{n}}$ or $(\sqrt{n})^{\sqrt{n+1}}$?
2) a) Show that $\lim_{n\to\infty} \sqrt{n^{200}+n^{100}+1} - n^{100} = 1/2$.

b) Compute $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^{200}+n^{100}+1})$.
3) Define a sequence by:

$$a_n = \int_0^1 (1-x^2)^n \, dx.$$

a) Show that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

b) Calculate $\sum_{n=1}^{\infty} a_n$.
4) a) Show that $\sqrt{\alpha} \leq \frac{1+\alpha}{2}$ for all positive numbers $\alpha$.

b) Show that the sequence $x_n$ converges, where

$$x_n = \sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{n}}}}.$$

5) Let $f$ be a continuous function such that $f$ is strictly increasing, $f(0) = 0$ and $f(1) = 1$. Let $g$ be the inverse of $f$ (so that for every $x$, $f(g(x)) = x$ and $g(f(x)) = x$). Show that

$$\int_0^1 f(x)\, dx + \int_0^1 g(y)\, dy = 1.$$

6) Suppose $\lim_{x\to\infty}(f(x)+f'(x)) = 0$, and that $\lim_{x\to\infty} f(x)$ exists. Show that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0.$$