Calculus Challenge 2004 Solutions

1)Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with nonincreasing derivative such that $f(0) = 0$ and $f'(1) > 0$.

a) Show that $f(1) \geq f'(1)$.

b) Show that

$$\int_0^1 \frac{1}{1 + f^2(x)} dx \leq \frac{f(1)}{f'(1)}.$$

When does equality hold, if ever?

Answer:

a) By the Mean Value Theorem $f(1) - f(0) = f'(c)$ for some $c \in (0, 1)$. But f' is nonincreasing and hence $f'(c) \geq f'(1)$. So $f(1) = f(1) - f(0) \geq f'(1)$.

b) We see that $\frac{1}{1+f^2(x)}$ is less or equal to 1 and hence the integral is also less or equal to 1. Equality can only occur when $f^2(x) = 0$ for all x, but this is impossible since $f'(1) > 0$.

2) Let $(1 + \sqrt{2})^n = A_n + B_n\sqrt{2}$ with A_n and B_n rational numbers.

a) Express $(1 - \sqrt{2})^n$ in terms of A_n and B_n.

b) Compute $\lim_{n\to\infty} \frac{A_n}{B_n}$.

Answer:

a) Prove by induction that $(1 - \sqrt{2})^n = A_n - B_n\sqrt{2}$.

b) Since $(1 + \sqrt{2})^n = A_n + B_n\sqrt{2}$ and $(1 - \sqrt{2})^n = A_n - B_n\sqrt{2}$ one can solve for A_n and B_n.

$A_n = ((1 + \sqrt{2})^n + (1 - \sqrt{2})^n)/2$ and $B_n = ((1 + \sqrt{2})^n - (1 - \sqrt{2})^n)/2\sqrt{2}$. So,

$$\frac{A_n}{B_n} = \sqrt{2} \cdot \frac{1 + a^n}{1 - a^n},$$

where $a = \frac{1-\sqrt{2}}{1+\sqrt{2}}$ is less than 1 in absolute value. So the limit equals $\sqrt{2}$.

3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with the property that:

$$\lim_{x\to\infty} f(x) = a$$

for some real number a. Show that $f(x)$ is the constant function $f(x) = a$.

Answer: By definition of periodic, we know that there exists a real number $t > 0$ such that:

$$f(x) = f(x + nt)$$

for all natural numbers n and any real number x. It follows that:

$$f(x) = \lim_{n\to\infty} f(x + nt) = \lim_{x\to\infty} f(x) = a$$

4)Find the sum:

$$2 + 3 + \frac{12}{4} + \frac{20}{8} + \frac{30}{16} + \frac{42}{32} + \frac{56}{64} + \cdots$$

Answer: The sum can be written:

$$\frac{2\cdot 1}{2^0} + \frac{3\cdot 2}{2^1} + \frac{4\cdot 3}{2^2} + \frac{5\cdot 4}{2^3} + \frac{6\cdot 5}{2^4} + \cdots =$$

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^n} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \text{ where } x = 1/2.$$

This last series can be summed by differentiating the geometric series twice. For any $|x| < 1$:

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}$$

Evaluating this last expression at $x = 1/2$, and seeing that this is the same as the series we need to sum, we get that the sum is 16.

5) Compute the following limit or explain why it doesn't exist:

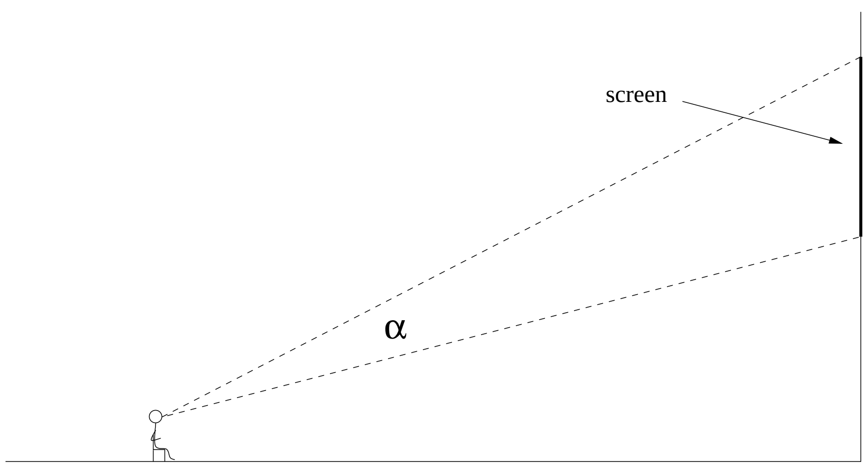
$$\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$$

Answer: For any $x \neq 0$ for which $\sin x \neq 0$,

$$\frac{x^2 \sin \frac{1}{x}}{\sin x} = \left(\frac{x}{\sin x}\right)\left(x \sin \frac{1}{x}\right)$$

The limit as $x \to 0$ of the first term in parentheses is 1. We also have that $0 \leq |x \sin \frac{1}{x}| \leq |x|$, so the second term goes to zero as $x \to 0$. Therefore, the limit is zero.

6) In a movie theater with level floor, the bottom of the screen is 1 unit above your eye level, and the top of the screen is 1 unit above that. How far back from the screen should you sit in order to maximize your viewing angle (α in the figure)?



Answer: As in the figure, the viewing angle is $\theta_1 - \theta_2$. Thus, we need to minimize the following function over the positive real numbers:

$$f(x) = \arctan(\frac{2}{x}) - \arctan(\frac{1}{x})$$

We have:

$$f'(x) = \frac{1}{1 + (\frac{2}{x})^2}\left(\frac{-2}{x^2}\right) - \frac{1}{1 + (\frac{1}{x})^2}\left(\frac{-1}{x^2}\right)$$

After doing a little algebra (flipping over fractions), we get:

$$f'(x) = -\frac{2}{x^2+4} + \frac{1}{x^2+1}$$

So we have $f'(x) = 0$ when:

$$x^2 + 4 = 2(x^2 + 1)$$

$$x^2 = 2$$

By examination, we see that $f(x)$ does have a maximum, and so it must occur at $x = \sqrt{2}$.