Calculus Challenge 2004

Directions: Each of the following six problems will be scored equally. The problems are to be worked on individually, using only paper and writing implements. Clear and correct explanations of your solutions are necessary for full credit. Partial credit is possible. Good luck!

1) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with nonincreasing derivative such that $f(0) = 0$ and $f'(1) > 0$.

a) Show that $f(1) \geq f'(1)$.

b) Show that

$$\int_0^1 \frac{1}{1+f^2(x)} dx \leq \frac{f(1)}{f'(1)}.$$

When does equality hold, if ever?

2) Let $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$ with A_n and B_n rational numbers.

a) Express $(1-\sqrt{2})^n$ in terms of A_n and B_n.

b) Compute $\lim_{n\to\infty} \frac{A_n}{B_n}$.

3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with the property that:

$$\lim_{x\to\infty} f(x) = a$$

for some real number a. Show that $f(x)$ is the constant function $f(x) = a$.

4)Find the sum:

$$2 + 3 + \frac{12}{4} + \frac{20}{8} + \frac{30}{16} + \frac{42}{32} + \frac{56}{64} + \cdots$$

5) Compute the following limit or explain why it doesn't exist:

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$$

6) In a movie theater with level floor, the bottom of the screen is 1 unit above your eye level, and the top of the screen is 1 unit above that. How far back from the screen should you sit in order to maximize your viewing angle (α in the figure)?