Calculus Challenge 2003 Solutions

1) Which is larger (for $n > 8$): $(\sqrt{n+1})^{\sqrt{n}}$ or $(\sqrt{n})^{\sqrt{n+1}}$?

Solution:
Answer: $(\sqrt{n})^{\sqrt{n+1}}$.

Since ln is an increasing function:

$$(\sqrt{n})^{\sqrt{n+1}} > (\sqrt{n+1})^{\sqrt{n}} \iff \ln((\sqrt{n})^{\sqrt{n+1}}) > \ln((\sqrt{n+1})^{\sqrt{n}}).$$

Using properties of ln, the last line is equivalent to

$$\sqrt{n+1}\ln n > \sqrt{n}\ln(n+1) \iff \frac{\ln n}{\sqrt{n}} > \frac{\ln(n+1)}{\sqrt{n+1}}.$$

Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = x^{-3/2}(1 - \frac{1}{2}\ln x)$. When $x > 8$, $\ln x > 2$, so $f'(x) < 0$ and f is decreasing. Therefore, $(\sqrt{n})^{\sqrt{n+1}} > (\sqrt{n+1})^{\sqrt{n}}$.

2) a) Show that $\lim_{n\to\infty} \sqrt{n^{200} + n^{100} + 1} - n^{100} = 1/2$.

Solution:

$$\sqrt{n^{200} + n^{100} + 1} - n^{100} = (\sqrt{n^{200} + n^{100} + 1} - n^{100}) \cdot \frac{\sqrt{n^{200} + n^{100} + 1} + n^{100}}{\sqrt{n^{200} + n^{100} + 1} + n^{100}} =$$

$$= \frac{n^{100} + 1}{\sqrt{n^{200} + n^{100} + 1} + n^{100}}.$$

Now, divide the denominator and the numerator by the same quantity, n^{100}. By letting n approach infinity, one can now establish the desired conclusion.

b) Compute $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^{200} + n^{100} + 1})$.

Solution:
First note that $\sin^2$ is a periodic function of period π. So,

$$\sin^2(\pi\sqrt{n^{200} + n^{100} + 1}) = \sin^2(\pi\sqrt{n^{200} + n^{100} + 1} - \pi n^{100})$$

Use part a) and note that hence $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^{200} + n^{100} + 1} - \pi n^{100}) = \sin^2(\pi/2) = 1$.

3) Define a sequence by:

$$a_n = \int_0^1 (1-x^2)^n \, dx.$$

a) Show that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

Solution:

First, note that $1-x^2 \leq 1$ for all $0 \leq x \leq 1$. So, by integrating this inequality, we see that $a_n \leq 1$, for every n.

Now, for $0 \leq x \leq 1$, we have that $1 - x^2 \geq 1 - x$. By integrating this inequality we get that $a_n \geq 1/(n+1)$.

So, $1 \geq \sqrt[n]{a_n} \geq \sqrt[n]{1/(n+1)}$. Since $\lim_{n\to\infty} \sqrt[n]{1/(n+1)} = 1$, we are done.

b) Calculate $\sum_{n=1}^{\infty} a_n$.

Solution:

Note that we have shown, in part a), that $a_n \geq 1/(n+1)$. So, $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} 1/(n+1) = \infty$.

4) a) Show that $\sqrt{\alpha} \leq \frac{1+\alpha}{2}$ for all positive numbers α.

Solution:

The inequality $\sqrt{\alpha} \leq \frac{1+\alpha}{2}$ is equivalent to $(1-\sqrt{\alpha})^2 \geq 0$.

b) Show that the sequence x_n converges, where

$$x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}.$$

Solution:

We will use the inequality proven in part a).

So,

$$x_n \leq \frac{1}{2}(1 + 1 + \sqrt{2 + \cdots + \sqrt{n}}) = \frac{2}{2} + \frac{1}{2}(\sqrt{2 + \cdots + \sqrt{n}}).$$

Continue the procedure with $\sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}$.

Since $\sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}} \leq \frac{1}{2}(3 + \sqrt{3 + \cdots + \sqrt{n}})$ we get that

$$x_n \leq \frac{2}{2} + \frac{3}{2^2} + \frac{1}{2^2}\sqrt{3 + \cdots + \sqrt{n})}$$

Repeat the procedure and see that

$$x_n \leq \sum_{k=1}^{n} (k+1)/2^k$$

Now, $\sum_{k=0}^{\infty} x^k = 1/1 - x$ for every $0 < x < 1$.
By taking the derivative in both sides one gets that
$\sum_{k=0}^{\infty}(k+1)x^k = 1/(1-x)^2$ for every $0 < x < 1$.
So, by taking $x = 1/2$ we get that
$x_n \leq 4 - 1 = 3$.
On the other hand, x_n is obviously increasing. So, x_n does converge to a number less than or equal to 3.

5) Let f be a continuous function such that f is strictly increasing , $f(0) = 0$ and $f(1) = 1$. Let g be the inverse of f (so that for every x, $f(g(x)) = x$ and $g(f(x)) = x$. Show that

$$\int_0^1 f(x)\, dx + \int_0^1 g(y)\, dy = 1.$$

Solution:
The curve $y = f(x)$ cuts the unit square into two pieces. The curve starts at the origin and ends at $(1, 1)$. The area of the part of the square below $y = f(x)$ is $\int_0^1 f(x)\, dx$. By integrating $x = g(y)$ from $y = 0$ to $y = 1$, we get the area of the part of the square that is above the curve. Hence, the sum of the integrals gives us the area of the square, which is 1.

6) Suppose $\lim_{x\to\infty}(f(x) + f'(x)) = 0$, and that $\lim_{x\to\infty} f(x)$ exists. Show that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0.$$

Solution:
First solution:

Let us suppose that the conclusion is wrong, i.e. assume that

$$\lim_{x\to\infty} f(x) \neq 0.$$

Write

$$f(x) = \frac{e^x f(x)}{e^x}$$

and apply l'Hospital rule. Since

$$\frac{e^x(f(x) + f'(x))}{e^x}$$

approaches zero as $x \to \infty$, we see that $\lim_{x\to\infty} f(x) = 0$. This contradicts the assumption we made.

So, $\lim_{x\to\infty} f(x)$ is indeed 0.

This conclusion together with the hypothesis immediately gives that, also,

$$\lim_{x\to\infty} f'(x) = 0.$$

Second solution:

Assume that $\lim_{x\to\infty} f(x)$ is not zero. This means that $\lim_{x\to\infty} f'(x)$ is not zero either. Let us suppose that $\lim_{x\to\infty} f'(x)$ equals L, with L positive or ∞ (the case where L is negative or $-\infty$ can be treated in similar fashion). This implies that $\lim_{x\to\infty} f(x) = -L$.

The assumption on f' implies now that there is $x_o > 0$ such that $f'(x) > l$ for all $x \geq x_o$ with l a fixed positive number (one can let l be anything less than L). Integrate from x_o to x, with $x > x_o$. It follows that

$$f(x) \geq l(x - x_o)$$

and hence $\lim_{x\to\infty} f(x)$ equals ∞. False, as $\lim_{x\to\infty} f(x) = -L$.

In conclusion, the only possibility is that $\lim_{x\to\infty} f(x)$ is in fact zero.