2006 Calculus Challenge

University of Utah Math Department

Instructions

1. Full credit will not be given without reasonable supporting work. Be sure to cite any theorems you use.
2. No books.
3. No notes.
4. No calculators.
5. No talking.
6. Write your name and UID on each sheet you submit and do not put multiple solutions on the same sheet. Please include your major on your top sheet.

Problems

1. Show that, for all real numbers a and b, $\dfrac{a^2}{2} + \dfrac{b^2}{2} \geq ab$.
2. We say that two polynomials $f(x)$ and $g(x)$ are *relatively prime* if there are no non-constant polynomials that divide both $f(x)$ and $g(x)$.

   Show that a polynomial $f(x)$ has no multiple roots if and only if $f(x)$ and $f'(x)$ are relatively prime.
3. Assuming that $a > -1$ and $b > -1$, use Riemann sums to calculate

$$\lim_{n\to\infty} n^{b-a} \frac{1^a + 2^a + \cdots + n^a}{1^b + 2^b + \cdots + n^b}.$$

4. Find the global maximum of the function

$$f(x) = \frac{1}{1 + |x - 2|} + \frac{1}{1 + |x + 6|}.$$

5. Compute $\displaystyle\int \frac{dx}{\sqrt{1 + \sqrt{1 + \sqrt{x}}}}$.
6. Use the Maclaurin series for e^1 to compute

$$\lim_{n\to\infty} n \sin(2\pi e n!).$$