Calculus Challenge, Spring 2005

Solutions

1. Observe
$$(\ln n)^{\ln(\ln n)} = e^{\ln(\ln n)^{\ln(\ln n)}} = e^{[\ln(\ln n)]^2}$$
and by the hint, $\ln(\ln n) < \sqrt{\ln n}$ implies
$$\frac{1}{e^{[\ln(\ln n)]^2}} > \frac{1}{e^{\ln n}} = \frac{1}{n}$$
Since $\sum_2^\infty \frac{1}{n}$ diverges, so does $\sum_2^\infty \frac{1}{(\ln n)^{\ln(\ln n)}}$.

2. (a)
$$|f(b)| - |f(a)| \leq |f(b) - f(a)| = |\int_a^b f'(x)dx| \leq \int_a^b |f'(x)|dx$$
$$\leq \int_0^1 |f'(x)|dx$$

   (b) Choose $a \in (0,1)$ such that $f(a) \leq f(x)$ for all $x \in (0,1)$ then
$$|f(a)| \leq |\int_0^1 f(x)dx| \leq \int_0^1 |f(x)|dx \Rightarrow$$
$$|f(b)| \leq \int_0^1 |f'(x)|dx + |f(a)| \leq \int_0^1 |f'(x)|dx + \int_0^1 |f(x)|dx$$
$$= \int_0^1 |f'(x)|dx + |f(x)|dx$$

3. Let $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. By intermediate value theotem there exists $c \in [0,1]$ such that $g(c) = 0$, whence the conclusion.

4. Substitute $x = \pi - z$. Then, $dx = -dz$, $x = 0$ implies $z = \pi$, and $x = \pi$ implies $z = 0$, so
$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x} = -\int_\pi^0 \frac{(\pi - z)\sin(\pi - z)}{1+\cos^2(\pi - z)}dz = \int_0^\pi \frac{\pi \sin z}{1+\cos^2 z}dz - \int_0^\pi \frac{z \sin z}{1+\cos^2 z}dz$$
Therefore,
$$\int_0^\pi \frac{x \sin x}{1+cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+cos^2 x}dx = \int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}(\frac{\pi}{4}+\frac{\pi}{4}) = (\frac{\pi}{2})^2$$

5. Let $\varepsilon > 0$, and let $\delta = \varepsilon$. Then, for any $|x| < \varepsilon$, $|f(x)| = 0$ if x is rational and $|f(x)| = |x| < \varepsilon$ if x is irrational. Thus, f is continuous at 0. For $a \neq 0$ and rational, $|f(x)| = 0$ or $|x|$. For $\varepsilon < |a|$, any $\delta > 0$, there is an $|x-a| < \delta$ such that $|x| > |a|$, so f is not continuous at a. For a irrational, $|f(x)-f(a)| = |a|$ or $|x-a|$, so again we fail for $\varepsilon < |a|$.

6. Let $\lim_{x\to a} f(x) = L$. We note f is continuous at $-a$ if the limit $\lim_{x\to -a} f(x)$ exists and equals $f(-a) = L$ if f is even, $-L$ if f is odd. This limit is equivalent to $\lim_{x\to a} f(-x) =: (*)$. If f is even, $f(-x) = f(x)$, so $(*) = L$. If f is odd, $f(-x) = -f(x)$, so $(*) = -L$. We conclude f is continuous at $-a$.