Calculus Challenge, Spring 2005

Directions: Each of the following six problems will be scored equally. The problems are to be worked on individually, using only paper and writing implements. Clear and correct explainations of your solutions are necessary for full credit. Partial credit is possible. Good Luck!

1. Determine if the following infinite series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$$

Hint: $\ln x < \sqrt{x}$.

2. f is differentiable in $(0, 1)$ and continuous on $[0, 1]$.

(a) Show that

$$|f(b)| - |f(a)| \leq \int_0^1 |f'(x)|dx \quad \forall a, b \in (0, 1)$$

Hint: Start with the interval $[a, b]$, and apply the first fundamental theorem of calculus to it.

(b) By choosing a appropriately, show that

$$|f(b)| \leq \int_0^1 |f'(x)| + |f(x)|dx \quad \forall b \in (0, 1)$$

3. Let $f : [0, 1] \rightarrow [0, 1]$ be continuous and onto, i.e. for all $y \in [0, 1]$, there exists a $x \in [0, 1]$ such that $f(x) = y$. Show that there is a $c \in [0, 1]$ such that $f(c) = c$.

   Hint: Define an appropriate function g and apply the intermediate value theorem.

4. Evaluate the definite integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

5. Let $f(x) = 0$ if x is rational, and $f(x) = x$ if x is not rational. Show f is continuous only at 0.

6. Suppose f is an even or odd function. If f is continuous at a, show f is continuous at $-a$.