Calculus Challenge 2002 Solutions

1. Find the sum: $\frac{1}{4}+\frac{3}{16}+\frac{5}{64}+\frac{7}{256}+\cdots$

   $\frac{1}{4}+\frac{3}{16}+\frac{5}{64}+\frac{7}{256}+\cdots=\frac{1}{4^1}+\frac{3}{4^2}+\frac{5}{4^3}+\frac{7}{4^4}+\cdots$
   $=\frac{1}{4}[1+\frac{3}{4}+\frac{5}{4^2}+\frac{7}{4^3}+\cdots]$
   $=\frac{1}{4}[1+\frac{(2)(1)+1}{4^1}+\frac{(2)(2)+1}{4^2}+\frac{(2)(3)+1}{4^3}+\cdots]$
   $=\frac{1}{4}\sum_{n=0}^{\infty}\frac{2n+1}{4^n}=\frac{1}{2}\sum_{n=0}^{\infty}n(\frac{1}{4})^n+\frac{1}{4}\sum_{n=0}^{\infty}(\frac{1}{4})^n$
   $=\frac{1}{8}\sum_{n=1}^{\infty}n(\frac{1}{4})^{n-1}+\frac{1}{4}\cdot\frac{1}{1-\frac{1}{4}}=\sum_{n=1}^{\infty}nx^{n-1}$
   $=\frac{d}{dx}\sum_{0}^{\infty}x^n=\frac{d}{dx}\frac{1}{1-x}=\frac{1}{(1-x)^2}$
   Therefore, given sum $=\frac{1}{8}\cdot\frac{1}{(1-\frac{1}{4})^2}+\frac{1}{4}\cdot\frac{1}{(1-\frac{1}{4})}$
   $=\frac{1}{8}\cdot\frac{1}{(\frac{3}{4})^2}+\frac{1}{4}\cdot\frac{1}{\frac{3}{4}}=\frac{1}{8}\cdot\frac{16}{9}+\frac{1}{4}\cdot\frac{4}{3}=\frac{2}{9}+\frac{1}{3}=\frac{5}{9}$

2. Find $\lim_{x\to\infty}[(x^6+x^5)^{\frac{1}{6}}-(x^6-x^5)^{\frac{1}{6}}]$.

   $(x^6\pm x^5)^{\frac{1}{6}}=x(1\pm\frac{1}{x})^{\frac{1}{6}}$
   Now, by the binomial theorem $(1+h)^{\frac{1}{6}}=1+\frac{1}{6}h+\circ(h)$ so
   $(x^6+x^5)^{\frac{1}{6}}-(x^6-x^5)^{\frac{1}{6}}=x(1+\frac{1}{x})^{\frac{1}{6}}-x(1-\frac{1}{x})^{\frac{1}{6}}$
   $=x[(1+\frac{1}{6}\cdot\frac{1}{x}+\circ(\frac{1}{x}))-(1-\frac{1}{6}\cdot\frac{1}{x}+\circ(\frac{1}{x}))]$
   $=x[\frac{1}{3x}+\circ(\frac{1}{x})]=\frac{1}{3}+\frac{\circ(\frac{1}{x})}{\frac{1}{x}}\to\frac{1}{3}$

3. The horizontal line $y=c, c>0$ intersects the curve $y=2x-3x^3$ in the first quadrant as shown. Find c so that the areas of the two shaded regions are equal.

   Let x_1, x_2 be the points where the curve $y=2x-3x^3$ intersects the line $y=c$. Then the left-hand shaded region has area
   $A_1=\int_0^{x_1}[c-(2x-3x^3)]dx$
   and the right-hand shaded region has area $A_2=\int_{x_1}^{x_2}[(2x-3x^3)-c]dx$
   $A_1=(cx-x^2+\frac{3}{4}x^4)|_0^{x_1}, A_2=(x^2-\frac{3}{4}x^4-cx)|_{x_1}^{x_2}$
   So for equal areas we have
   $cx_1-x_1^2+\frac{3}{4}x_1^4=(x_2^2-\frac{3}{4}x_2^4-cx_2)-(x_1^2-\frac{3}{4}x_1^4-cx_1)$
   So we must have $x_2^2-\frac{3}{4}x_2^4-cx_2=0$, or $4x_2^2-3x_2^4-4cx_2=0$, and for $x_2>0, 3x_2^3-4x_2+4c=0(*)$
   But (x_2,c) is on the curve, so $c=2x_2-3x_2^2$, i.e. $3x_2^3=2x_2-c(\#)$

Substitution in (*) gives $2x_2 - c - 4x_2 + 4c = 0$, so $x_2 = \frac{3c}{2}$
Putting this into (#) we get: $3(\frac{3c}{2})^3 = 2(\frac{3c}{2}) - c$
$\frac{81}{8}c^3 = 3c - c$
$81c^3 - 16c = 0$
$c(81c^2 - 16) = 0$
$c(9c-4)(9c+4) = 0$
$c > 0$, so $c = \frac{4}{9}$

4. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$

$\sum_{k=1}^{n} \frac{1}{k^2+n^2} = \frac{1}{n^2}\sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2} \cdot \frac{1}{n}$
$\sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2} \cdot \frac{1}{n}$ is the Riemann sum for $f(x) = \frac{1}{1+x^2}$ on $[0,1]$ with right regular partition $(0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$
as $n \to \infty$ these sums $\to \int_0^1 \frac{dx}{1+x^2} = tan^{-1}(x)|_0^1 = \frac{\pi}{4}$
so $\sum_1^n \frac{1}{k^2+n^2} = \frac{1}{n}\sum_1^n \frac{1}{1+(\frac{k}{n}^2)} \cdot \frac{1}{n} \to 0 \cdot \frac{\pi}{4} = 0$

5. Find $\int \frac{dx}{x(x+1)(x+2)+\cdots+(x+m)}$, where m is a positive integer.

$\frac{1}{x(x+1)\cdots(x+m)} = \sum_{k=0}^{m} \frac{A_k}{x+k}$
Get a common denominator on right-hand side:
$1 = \sum_{k=0}^{m} A_k(x+0)\cdots(x+k)\cdots(x+m)$
Evaluate A_k - let $x = -k$
$1 = A_k(-k+0)\cdots(-k+(k-1))\cdot(-k+(k+1)\cdots(-k+m)$
$= (-1)^k(k(k-1)\cdots 1)(1\cdot 2\cdots(m-k))A_k$
Therefore, $A_k = \frac{-1^k}{k!(m-k)!}, 0 \leq k \leq m$
so $\int \frac{dx}{\Pi_{i=0}^{m}(x+i)} = \sum_0^m \frac{-1^k}{k!(m-k)!} ln(x+k)(+C)$

6. Let C be the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ where $x \geq 0$ and $y \geq 0$. Find the length of the longest line segment that lies in the first quadrant and is tangent to C.

Let (a,b) be the point of tangency. We can find the slope of the tangent using implicit differentiation: $\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}y' = 0$.
$y' = -\frac{x^{\frac{-1}{3}}}{y^{\frac{-1}{3}}} = -\sqrt[3]{\frac{y}{x}}$

Once we have the slope, we can find the equation of the tangent line:
$\frac{y-b}{x-a} = -\sqrt[3]{\frac{b}{a}}$
$y - b = -\sqrt[3]{\frac{b}{a}}(x - a)$
and the intercepts:
$x = 0 \rightarrow y = b + a\sqrt[3]{\frac{b}{a}} = b + a^{\frac{2}{3}}b^{\frac{1}{3}}$
$y = 0 \rightarrow x = a + b\sqrt[3]{\frac{a}{b}} = a + a^{\frac{1}{3}}b^{\frac{2}{3}}$
So the (square of the) distance between the intercepts is $(b + a^{\frac{2}{3}}b^{\frac{1}{3}})^2 + (a + a^{\frac{1}{3}}b^{\frac{2}{3}})^2$
$= b^{\frac{2}{3}}(b^{\frac{2}{3}} + a^{\frac{2}{3}})^2 + a^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})^2 = b^{\frac{2}{3}} + a^{\frac{2}{3}} = 1$ (greatest value)