Calculus Challenge 2002

1. Find the sum: \( \frac{1}{4} + \frac{3}{16} + \frac{5}{64} + \frac{7}{256} + \cdots \)

2. Find \( \lim_{x \to \infty} [(x^6 + x^5)^{\frac{1}{x}} - (x^6 - x^5)^{\frac{1}{x}}] \).

3. The horizontal line \( y = c, c > 0 \) intersects the curve \( y = 2x - 3x^3 \) in the first quadrant as shown. Find \( c \) so that the areas of the two shaded regions are equal.

4. Find \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} \)

5. Find \( \int \frac{dx}{x(x+1)(x+2)\cdots(x+m)} \), where \( m \) is a positive integer.

6. Let \( C \) be the curve \( x^{\frac{2}{3}} + y^{\frac{2}{3}} \) where \( x \geq 0 \) and \( y \geq 0 \). Find the length of the longest line segment that lies in the first quadrant and is tangent to \( C \).