Calculus Challenge 2001

1. Evaluate the integral $\int_0^\infty \frac{dx}{1+x^3}$
   The method of partial fractions and completing the square yields $\frac{1}{x^3+1} = \frac{1}{3}(\frac{1}{x+1} - \frac{x-2}{x^2-x+1}) = \frac{1}{3}(\frac{1}{x+1} - \frac{u-\frac{3}{2}}{u^2+\frac{3}{4}})$ where $u = x - \frac{1}{2}$. This leads to
   $\int \frac{dx}{x^3+1} = \frac{1}{3}(ln(x+1) - \frac{1}{2}ln(u^2+\frac{3}{4}) + \frac{3}{2}\frac{2}{\sqrt{3}}tan^{-1}(\frac{u}{\frac{\sqrt{3}}{2}})) = \frac{1}{3}(ln(\frac{x+1}{\sqrt{x^2-x+1}}) + \sqrt{3}tan^{-1}(\frac{2x-1}{\sqrt{3}})$. From this we obtain
   $\int_0^\infty \frac{dx}{X^3+1} = \frac{1}{3}(\frac{\sqrt{3}\pi}{2} - \sqrt{3}tan^{-1}(\frac{-1}{\sqrt{3}})) = \frac{\sqrt{3}\pi}{3}(\frac{1}{2}+\frac{1}{6}) = \frac{2\pi\sqrt{3}}{9}$.

2. Show that $x^e \le e^x$ for all positive real numbers x and determine all number x for which equality holds.
   The inequality is equivalent to $x^{\frac{1}{x}} \le e^{\frac{1}{e}}$. Let $f(x) = x^{\frac{1}{x}}$. Then $f'(x) = x^{\frac{1}{x}}(\frac{1-ln(x)}{x^2})$. This is positive if $0 < x < e$ and is negative if $e < x$. Therefore, $f(x)$ has an absolute maximum at $x = e$ and so $f(x) \le f(e)$ for all $x > 0$ and equality holds if, and only if, $x = e$. Thus $x^e \le e^x$ for all $x > 0$ and equality holds if, and only if, $x = e$.

3. Let $\alpha > -1$ and $\beta > -1$. Calculate $\lim_{n\to\infty} n^{\beta-\alpha}\frac{1^\alpha+2^\alpha+\cdots+n^\alpha}{1^\beta+2^\beta+\cdots+n^\beta}$
   $\lim_{n\to\infty} n^{\beta-\alpha}\frac{1^\alpha+2^\alpha+\cdots+n^\alpha}{1^\beta+2^\beta+\cdots+n^\beta} = \lim_{n\to\infty}\frac{\sum_{k=1}^n(\frac{k}{n})^\alpha}{\sum_{k=1}^n(\frac{k}{n})^\beta} = \frac{\int_0^1 x^\alpha dx}{\int_0^1 x^\beta dx} = \frac{\beta+1}{\alpha+1}$

4. Find the sum of the following series: $\sum_{n=1}^\infty \frac{ln(2^n)}{e^n}$
   $\sum_{n=1}^\infty \frac{ln(2^n)}{e^n} = ln(2)\sum_{n=1}^\infty ne^{-n} = ln(2)f(\frac{1}{e})$ where $f(x) = \sum_{n=1}^\infty nx^n = x(\sum_{n=0}^\infty x^n)' = x(\frac{1}{1-x})' = \frac{x}{(1-x)^2}$ for $|x| < 1$. Thus the value of the series is $\frac{e^{-1}ln(2)}{(1-e^{-1})^2} = \frac{eln(2)}{(e-1)^2}$.

5. Find the antiderivative: $\int \frac{dx}{\sqrt{e^a - e^x}}$

The substitution $sin\theta = \frac{e^{\frac{x}{2}}}{e^{\frac{a}{2}}}$ reduces the integral to

$\int \frac{2cos\theta d\theta}{e^{\frac{a}{2}} sin\theta cos\theta} = 2e^{\frac{-a}{2}} \int csc\theta d\theta = -2e^{\frac{-a}{2}} ln|cot\theta + csc\theta| + C = -2e^{\frac{-a}{2}} ln(\frac{\sqrt{e^a - e^x} + e^{\frac{a}{2}}}{e^{\frac{x}{2}}}) + C = xe^{\frac{-a}{2}} - 2e^{\frac{-a}{2}} ln(e^{\frac{a}{2}} + \sqrt{e^a - e^x}) + C$

6. Does the series $\sum_{n=1}^{\infty} ne^{-\sqrt{n}}$ converge or diverge? Justify your answer.

This converges because of the integral test. Using the substitution $u^2 = x$ together with integration by parts,

$\int_1^{\infty} xe^{-\sqrt{x}} dx = \int_1^{\infty} 2u^3 e^{-u} du = 2(-u^3 e^{-u} - 3u^2 e^{-u} - 6ue^{-u} - 6e^{-u})|_1^{\infty} = \frac{32}{e}$