Vulgar Finance, Divine Mathematics

Sports Gambling, Financial Derivatives, and Arbitrage
Flip a penny

**Heads:** you owe me $1

**Tails:** I owe you $1

Wanna play?
Spin a penny

**Heads:** you owe me $1

**Tails:** I owe you $1

Wanna play?
If you spin a penny, it comes up tails roughly 80% of the time.

Why? The heads side is slightly heavier.

We have an edge!...but we can still lose a lot of money...

LISTEN  I WOULD

BUT THE WAY MY BANK ACCOUNT IS SET UP...
Let’s Get Ready to Rumble

VS

[Image of TV show characters]
Arbitrage = Free Lunch

It doesn’t matter who we think is going to win; we can lock in a riskless profit by betting the right amounts on BOTH outcomes.
Arbitrage = Free Lunch

Notice that you don’t need any money upfront to use this strategy; first, get paid by the person who owes you and with the proceeds, pay off the other person.
Stocks

Represent ownership shares in a company
Options: Calls

Give the owner the RIGHT to BUY a certain amount of STOCK (S) at a specified STRIKE PRICE (K) at a specified EXPIRATION TIME (T)

Example: The May 2016 call with strike price $50 gives the owner the right to buy stock for $50 in May 2016

The value of the call option at expiration is:

\[ C(T) = \max\{S(T) - K, 0\} \]
Options: Puts

Give the owner the RIGHT to SELL a certain amount of STOCK (S) at a specified STRIKE PRICE (K) at a specified EXPIRATION TIME (T)

Example: The May 2016 put with strike price $50 gives the owner the right to sell stock for $50 in May 2016

The value of the put option at expiration is:

$$P(T) = \max\{K - S(T), 0\}$$
Options: Payoff Diagrams

**LONG CALL**

- **Profit or Loss**
  - $0
  - -$200

- **Stock Price at Expiration**
  - 30
  - 40
  - 50

**LONG PUT**

- **Profit or Loss**
  - $0
  - -$200

- **Stock Price at Expiration**
  - 30
  - 40
  - 50
Compound Interest

Suppose the interest rate is $r$ and you want to borrow $P$ for $T$ years. At time $T$, how much will you owe?

Example:

$r = 0.1 \text{ (10\%)}$, $P = 100$, $T = 2$

Compounded annually: $P(1 + r)^T = 100(1.1)^2 = 121$

Compounded semi-annually: $P(1 + r/2)^{2T} = 100(1.05)^4 = 121.55$

Compounded quarterly: $P(1 + r/4)^{4T} = 100(1.025)^8 = 121.84$

Compounded continuously: $\lim_{n \to \infty} P(1 + r/n)^{nT} = Pe^{rT} = 122.14$
Two Portfolios

**Portfolio 1:** Long call, short put with strike $K \rightarrow C - P$

**Portfolio 2:** Long stock, borrow $Ke^{-rT} \rightarrow S - Ke^{-rT}$

What are the payoffs at time $T$?

**Portfolio 1:** $C(T) - P(T) = \max\{S(T) - K, 0\} - \max\{K - S(T), 0\} = S(T) - K$

**Portfolio 2:** $S(T) - Ke^{-rT}e^{rT} = S(T) - K$

They have the same payoff, so they should cost the same
Arbitrage Part II

**Portfolio 1:** Long call, short put with strike K

**Portfolio 2:** Long stock, borrow $K e^{-rT}$

What happens if portfolio 1 is cheaper than portfolio 2?

Short portfolio 2, and with the proceeds, go long portfolio 1. This will make you net positive and at time $T$, the two portfolios will offset each other.
An Example

$K = 50, \ r = 0.1, \ T = 0.25$ (3 months from now)

Stock is trading at $55$

Call costs $5$, put costs $1$

**Portfolio 1:** Long call, short put: $5 - $1 = $4$

**Portfolio 2:** Long stock, borrow $Ke^{-rT}$: $55 - 50e^{-(0.1)(0.25)} = 6.23$

Short portfolio 2 (+$6.23), long portfolio 1 (-$4) → net profit of $2.23$