Mathematical Models of Interactions between Species: Peaceful Co-existence of Vampires and Humans

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For Undergraduate Mathematical Colloquium

Source: Wadim Strielkowski, Evgeny Lisin and Emily Welkins, National University of Ireland, Galway and Charles University in Prague, Moscow Power Engineering Institute (Technical University), University of Strasbourg
Question: Is coexistence of humans and vampires possible?
Building the model:
1. Humans

- Assume that the world’s population is to follow the exponential growth rate \( x(t) \)
- Then the **dynamics of growth in human population** is:
  \[
  \frac{dx}{dt} = kx
  \]
  where \( k \) represents the coefficient of the population growth

- With the **solution**:
  \[
  x(t) = x_0 e^{kt}
  \]
  where \( x_0 \) is the total volume of population at the initial time \( t \).
Coefficient of the population growth: $K=0.000746$
2. Adding Vampires to the model

- The word “vampire” is considered to come from the Hungarian language where it is spelled “vampir”
- The fact that vampires constituted a threat to humans throughout the history of mankind (whether this threat was real or imaginary one) is real
- It can be illustrated by the examples of recent archaeological findings at ancient burial sites where some human remains showed signs of being killed in a typical way to slay the vampire: with a wooden stake put through the heart.

(New Scientist, 2009).
Vampires in the model of human population growth

- Consider introducing vampires into the model of population growth
- Vampires are often described in legends and folklore as the man’s natural predators:

  **Predator Prey Model**

- Suppose the vampire population is denoted by the function $y(t)$, $y_0=1$.
- The human population dynamics can therefore be presented as:

  \[
  \frac{dx}{dt} = kx - v(x)y
  \]

  where $v(x)$ is the rate at which humans are killed by vampires
Predator behaviors

- When a vampire meets a human, it either kills the human or turns it into a vampire.
- Assume that the number of any vampire’s victims is growing proportionally.
- Thence, the function $v(x)$ can be presented as the following:

$$v(x) = a \cdot x$$

where $a > 0$ is the coefficient of the human’s lethal interaction with a vampire.
Differential Equations

- Then the differential equation describing the growth rate of human population can be formulated as the following:
  \[ \frac{dx}{dt} = x(k-ay) \]
- How does the population of vampires change?
  \[ \frac{dy}{dt} = ? \]
How do vampires die?

- Let us also introduce vampire slayers into the model.
- The slayers regulate the population of vampires by periodically killing vampires.
- The equation for vampires will look like \( \frac{dy}{dt} = ? - cy \)

where \( c \geq 0 \) is the coefficient of lethal outcome of the interaction between a vampire and vampire slayer.
\[ \frac{dy}{dt} = ? - cy \]

- How does the vampire population grow?

- If \( a > 0 \) is the coefficient of the human’s lethal interaction with a vampire

- \( b : 0 < b \leq 1 \) to be the coefficient reflecting the rate with which humans are turned into vampires

- The equation will then be modified to look like as the following:

\[ \frac{dy}{dt} = baxy - cy \]

- Factor out \( y \): \[ \frac{dy}{dt} = y(bax - c) \]
This system is classified as “predator-prey” type model

Lotka-Volterra predator-prey model

\[
\begin{align*}
\frac{dx}{dt} &= x(k - ay) \\
\frac{dy}{dt} &= y(bax - c)
\end{align*}
\]
Coexistence
Coexistence

- System of Ordinary Differential Equations (ODEs)
- The system allows for the stationary solution – a pair of solutions \((x_s, y_s)\) for the system that creates a state when human and vampire populations can co-exist in time without any change in numbers

\[
\begin{align*}
\frac{dx}{dt} &= x(k - ay) = 0 \\
\frac{dy}{dt} &= y(bax - c) = 0
\end{align*}
\]
It is obvious from a stationary case that the size of human population is determined by the effectiveness of slaying vampires by vampire hunters \( c \) and the number of cases when the humans are turned into vampires \( ba \).

The size of vampire population depends on the growth rate of human population \( k \) and vampires’ thirst for human blood \( a \).

When vampires are capable of restraining their blood thirst, the size of both populations can be rather high in mutual co-existence.

Why the system is in balance?
The system is held in balance by the existence of vampire slayers.
Solution of the System of ODEs

- System of Ordinary Differential Equations (ODEs)
- Can be solved by using iterative numerical methods.
- Range-Kutta methods - that represent the modified and corrected Euler’s method with a higher degree of precision

\[ y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} g(t, y) dt \]

- Integrals can be approximated using either the *rectangle method* or *Simpson’s rule* of numerical approximation of definite integrals
Solution:

The formulae of calculations using the fourth-order Runge-Kutta method

\[
x_{i+1} = x_i + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad (12)
\]

\[
y_{i+1} = y_i + \frac{1}{6}[m_1 + 2m_2 + 2m_3 + m_4]
\]

\[
k_1 = f(t_i, x_i, y_i) \Delta t
\]

\[
m_1 = g(t_i, x_i, y_i) \Delta t
\]

\[
k_2 = f(t_i + \frac{\Delta t}{2}, x_i + \frac{k_1}{2}, y_i + \frac{m_1}{2}) \Delta t
\]

\[
m_2 = g(t_i + \frac{\Delta t}{2}, x_i + \frac{k_1}{2}, y_i + \frac{m_1}{2}) \Delta t
\]

\[
k_3 = f(t_i + \frac{\Delta t}{2}, x_i + \frac{k_2}{2}, y_i + \frac{m_2}{2}) \Delta t
\]

\[
m_3 = g(t_i + \frac{\Delta t}{2}, x_i + \frac{k_2}{2}, y_i + \frac{m_2}{2}) \Delta t
\]

\[
k_4 = f(t_i + \Delta t, x_i + k_3, y_i + m_3) \Delta t
\]

\[
m_4 = g(t_i + \Delta t, x_i + k_3, y_i + m_3) \Delta t
\]

\[
f(t_i, x_i, y_i) = x_i (k - ay_i)
\]

\[
g(t_i, x_i, y_i) = y_i (bax_i - c)
\]
Different scenarios with no slayers (c=0):

**Scenario 1: The Stoker-King model**

- The vampire bites the victim and drinks the victim’s blood, then returns to feed for 4-5 consecutive days.
- Whereupon the victim dies, is buried and rises to become another vampire (unless a wooden stake is put through its heart).
- Vampires usually need to feed every day, so more and more human beings are constantly turned into vampires.
- Bram Stoker’s “Dracula” - 1897; Stephen King’s “Salem’s Lot” - 1975
Let us take 1897 as the starting point (i.e. the year Stoker’s novel was first published).

In 1897, the world population was about 1 650 million people (UN, 1999).

The initial conditions:
1 vampire, 1 650 million people, there are no organized groups of vampire slayers: \( c=0 \)

The calculation period is set at 1 year with a step of 5 days

The coefficient of human population growth \( k \) for the given period is very small and can be neglected: \( k=0 \).

The probability of a human being turned into a vampire is very high: \( b=1 \).
Epidemic?

- The total sum of humans and vampires does not change in time
- Human population does not grow and humans gradually become vampires
- The predator-prey model is diminished to a simple problem of an epidemic outbreak
Epidemic

- The human population is drastically reduced by 80% by the 165th day from the moment when the first vampire arrives.
- This means that the human population reaches its critical value and practically becomes extinct.
- At that precise moment, the world will be inhabited by 1,384 million vampires and 266 million people.
Epidemic

- With passing time the number of vampires grows and very soon there are no humans left

- **Severe epidemic outbreak**

- Leads first to the complete extinction of humans and then to the death of all vampires
The speed with which vampire population grows

\[ \frac{dy}{dt} = \frac{a(y_0 + x_0)^2 y_0 x_0 e^{-a(y_0 + x_0)(t-t_0)}}{(y_0 + x_0 e^{-a(y_0 + x_0)(t-t_0)})^2} \]

The change in vampires’ growth dynamics (1 step = 5 days)
The maximal growth of the number of vampires (infected humans) will be observed in a moment of time \( t_{max} \):

\[
t_{max} = \frac{\ln(x_0 / y_0)}{a(y_0 + x_0)} + t_0
\]

where \( t_{max} = 153 \) is the day (153\textsuperscript{rd} day) when the number of vampires is the highest, \( x(t_{max}) = 825 \) million is the number of vampires in a moment of time \( t_{max} \), \( x'(t_{max}) = 286 \) million is the number of newly turned vampires in day \( t_{max} \).

The change in vampires’ growth dynamics (1 step = 5 days)
Scenario 2: The Rice model

- Anne Rice “Vampire Chronicles” describes the world with vampires, where vampires still need to feed on human but do so discretely.
- The vampire can attack a human being, feed on it and leave it to live.
- In some cases (if they are too hungry), vampires kill their victims by draining their blood.
- BUT The vampire cannot easily turn the human into another vampire.
- In order to do so, the victim’s permission needs to be gained, it needs to drink some of vampire’s blood, so it happens very rarely.
- They do not feed every day: some blood once a week or so is enough to survive.
Scenario 2: The Rice model

- The initial conditions of the Rice model are the following: 2 vampires, 982 million people, there are no organized groups of vampire slayers: \( c=0 \)
- The calculation period is set to 100 years with a step of 7 days
- Humans do not necessarily die or become vampires after their encounter with vampires, so the coefficient of lethal outcome \( a \) will be considerably lower than in the Stoker-King model: \( 0.1a \).
- The probability of a human turned into a vampire is quite low: \( b=0.1 \).

The coefficient of human population growth is calculated as

\[
k = \frac{\ln(x_1 / x_0)}{t_1 - t_0}
\]

\[
\begin{align*}
\frac{dx}{dt} &= x(k - ay) \\
\frac{dy}{dt} &= baxy \\
x(0) &= 982 \cdot 10^6 \\
y(0) &= 2
\end{align*}
\]
Epidemic?

- The human population grows in the beginning.
- When the number of vampires reaches its critical mass, the human population starts to shrink.
- After 48.7 years human population is almost extinct.
  The number of vampires at this moment is equal to 100 million.
Epidemic

- Delay of the total extinction of mankind by vampires by 48 years with respect to the first model
Scenario 3: The Harris-Meyer-Kostova model

- In the books of Stephenie Meyer's “Twilight series”, Charlaine Harris’s “Sookie Stockhouse (Southern Vampire) series”, “True Blood” (TV series) and Elizabeth Kostova's “The Historian” there is a world drawn where vampires peacefully coexist with humans.
  - Vampires interact with humans and drink animals’ or synthetic blood
  - They either live in secrecy or side-by-side with humans.
  - There is a possibility to turn a human being into a vampire, but it takes time and effort.
Scenario 3: coexistence?

- The calculation period is set at 100 years with a step of 1 year.
- Humans almost always come out alive from their encounters with vampires, hence the coefficient of lethal outcome \( a \) will be low: \( 0.01 \cdot a \).
- The probability of a human being turned into a vampire is similar to the one in the Rice model: \( b = 0.1 \).
- Sometimes vampires can be killed: \( c > 0 \).

The model allows for a stationary solution: there are system parameters \((x_s, y_s)\) that would stabilize the populations of humans and vampires in time:

\[(x_s, y_s) = (7704, 8) \text{ million individuals}\]
Scenario 3: coexistence

\[
\begin{align*}
\frac{dx}{dt} &= x(k - ay) \\
\frac{dy}{dt} &= y(bax - c)
\end{align*}
\]

\[\begin{align*}
x(0) &= 6150 \cdot 10^6 \\
y(0) &= 5 \cdot 10^6
\end{align*}\]
Peaceful co-existence of two spices

- This symbiosis, however, is very fragile
- Whenever the growth rate of human population slows down, the blood thirst of vampires accelerates, the whole system lies in ruins with just one population remaining.
Possible peaceful co-existence of two spices?

“Why Not” is a slogan for an interesting life.

-Misson Conley
References:


“Mathematical Models of Interactions between Species: Peaceful Co-existence of Vampires and Humans Based on the Models Derived from Fiction Literature and Films”

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