Mathematical Models of Interactions between Species: Peaceful Co-existence of Vampires and Humans

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For Undergraduate Mathematical Colloquium

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Question:
Is coexistence of humans and vampires possible?

Building the model:
1. Humans

- Assume that the world's population is to follow the exponential growth rate *x(t)*
- Then the **dynamics of growth in human population is:** $\frac{dx}{dt}=kx$

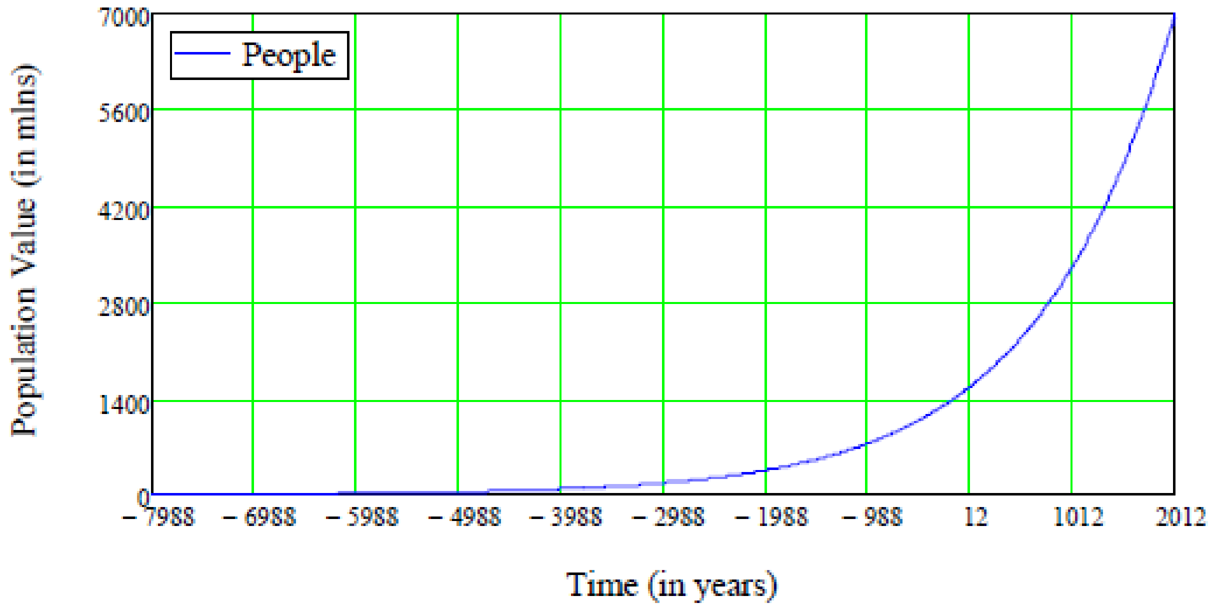
where *k* represents the coefficient of the population growth

- With the **solution:** $x(t) = x0e^{kt}$

where *x0* is the total volume of population at the initial time *t*.

Coefficient of the population growth: *K=0.000746*

Figure 1: Exponential growth model of world's population from the 8000 B.C.



2. Adding Vampires to the model

- The word "vampire" is considered to come from the Hungarian language where it is spelled "vampir"
- The fact that vampires constituted a threat to humans throughout the history of mankind (whether this threat was real or imaginary one) is real
- It can be illustrated by the examples of recent archaeological findings at ancient burial sites where some human remains showed signs of being killed in a typical way to slay the vampire: with a wooden stake put through the heart.

  (New Scientist, 2009).

Vampires in the model of human population growth

- Consider introducing vampires into the model of population growth
- Vampires are often described in legends and folklore as the man's natural **predators:**

Predator Prey Model

- Suppose the vampire population is denoted by the function *y(t), y0=1*.

- The human population dynamics can therefore be presented as:

$$\frac{dx}{dt} = kx - v(x)y$$

where *v(x)* is the rate at which humans are killed by vampires

Predator behaviors

- When vampire meets a human it either kills the human or turns it into a vampire
- Assume that the number of any vampire’s victims is growing proportionally.
- Thence, the function *v(x)* can be presented as the following:

$$v(x) = a \cdot x$$

where ***a>0*** is the coefficient of the human’s lethal interaction with a vampire

Differential Equations

- Then the differential equation describing the growth rate of human population can be formulated as the following:

$$\frac{dx}{dt} = x(k\text{-}ay)$$

- How does the population of **vampires** change?

- $$\frac{dy}{dt} = ?$$

How do vampires die?

- Let us also introduce vampire slayers into the model.
- The slayers regulate the population of vampires by periodically killing vampires.

- The equation for vampires will look like $\frac{dy}{dt} = ? - cy$

  where ***c≥0*** is the coefficient of lethal outcome of the interaction between a vampire and vampire slayer.

$dy/dt = ? - cy$

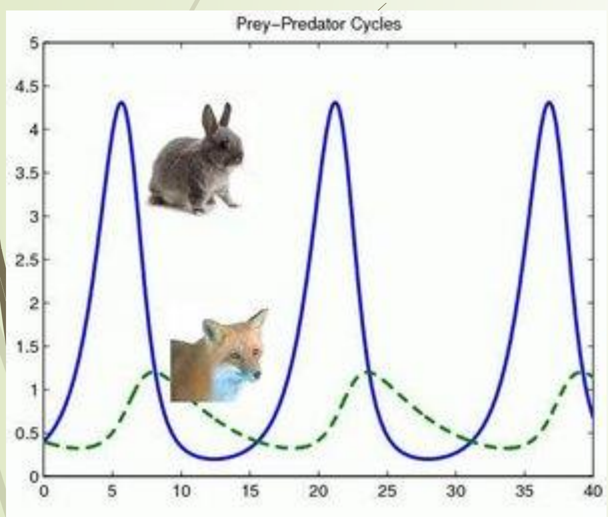
- How does the vampire population grow?
- *If* ***a>0*** is the coefficient of the human's lethal interaction with a vampire
- **b**: *0<b≤1* to be the coefficient reflecting the rate with which humans are turned into vampires
- The equation will then be modified to look like as the following:

$$\frac{dy}{dt} = baxy - cy$$

- Factor out y: $\frac{dy}{dt} = y(bax - c)$

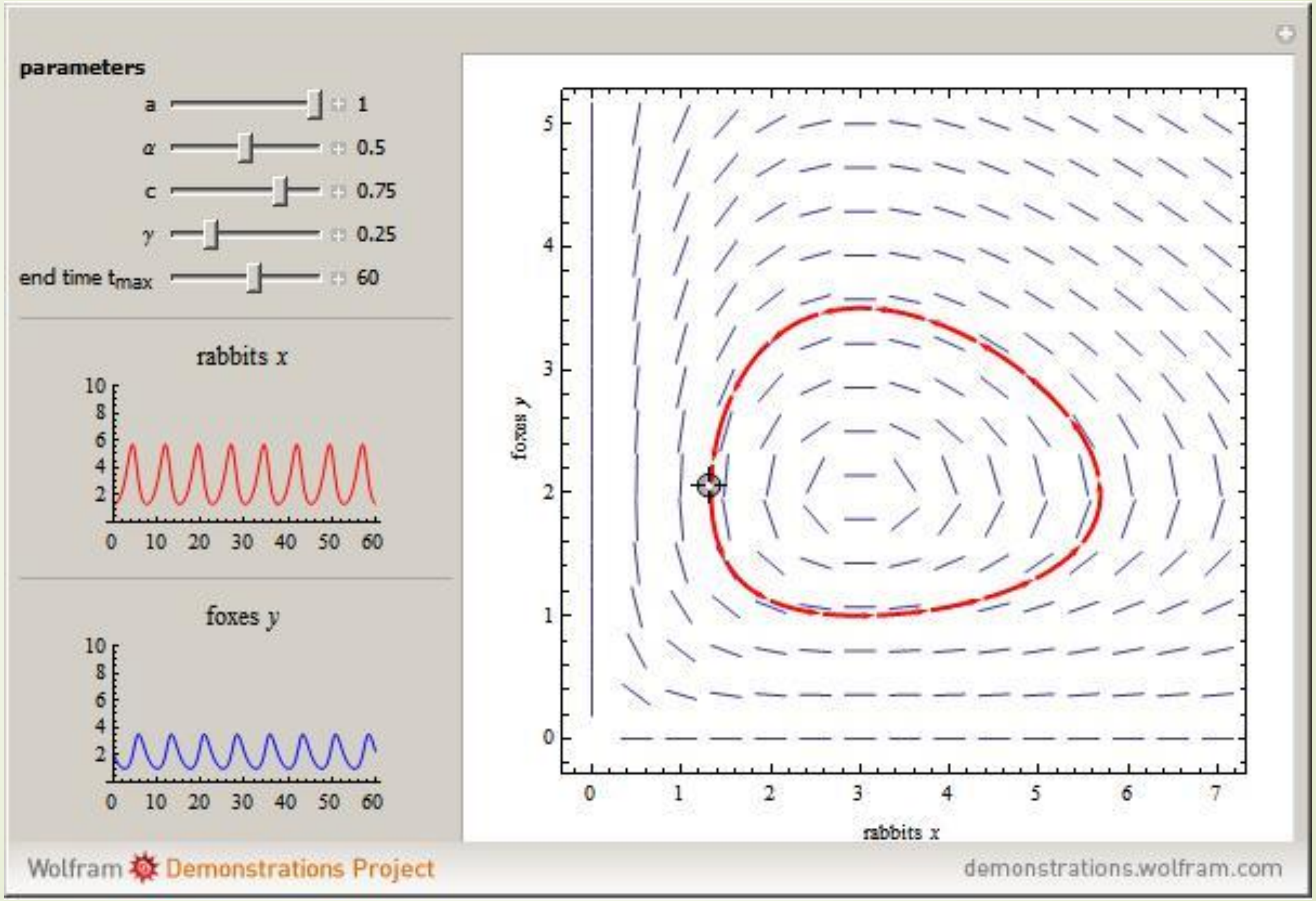
This system is classified as "predator-prey" type model

Lotka-Volterra predator-prey model



$$\begin{cases} \dfrac{dx}{dt} = x(k - ay) \\ \dfrac{dy}{dt} = y(bax - c) \end{cases}$$

Coexistence



Coexistence

- System of Ordinary Differential Equations (ODEs)
- The system allows for the stationary solution – a pair of solutions **(x_s, y_s)** for the system that creates a state when human and vampire populations can co-exist in time **without any change** in numbers

$$\begin{cases} \frac{dx}{dt} = x(k - ay) = 0 \\ \frac{dy}{dt} = y(bax - c) = 0 \end{cases}$$

Stationary case

$$\begin{cases} x(k - ay_s) = 0 \\ y(bax_s - c) = 0 \end{cases} \Rightarrow (x_s, y_s) = \left(\frac{c}{ba}, \frac{k}{a} \right)$$

- It is obvious from a stationary case that the size of human population is determined by the effectiveness **of slaying vampires by vampire hunters *c*** and the **number of cases when the humans are turned into vampires *ba***
- The size of vampire population depends on the **growth rate of human population *k*** and **vampires' thirst for human blood *a***.
- When vampires are capable of restraining their blood thirst, the size of both populations can be rather high in mutual co-existence.
- **Why the system is in balance?**

$$\begin{cases} \dfrac{dx}{dt} = xk - ayx \\ \dfrac{dy}{dt} = baxy - cy \end{cases}$$

$$(x_s, y_s) = \left(\frac{c}{ba}, \frac{k}{a} \right)$$

- The system is held in balance by the existence of vampire slayers

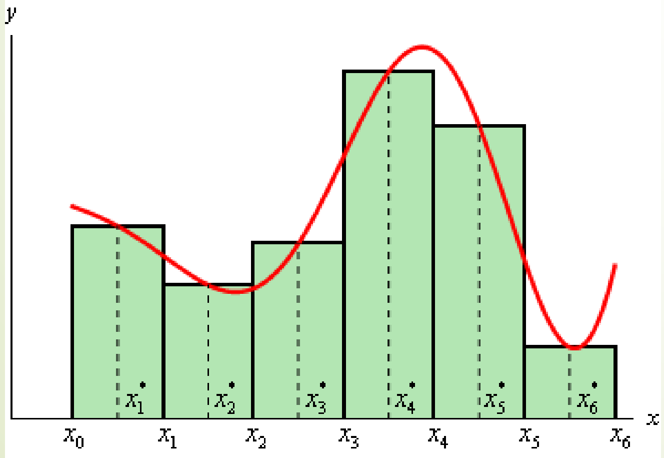
Solution of the System of ODEs

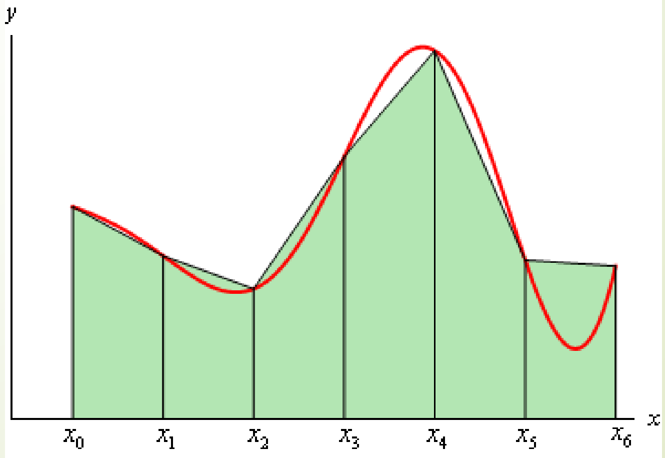
$$\begin{cases} \dfrac{dx}{dt} = x(k - ay) \\ \dfrac{dy}{dt} = y(bax - c) \end{cases}$$

- System of Ordinary Differential Equations (ODEs)
- Can be solved by using iterative numerical methods.
- Range-Kutta methods - that represent the modified and corrected Euler's method with a higher degree of precision

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} g(t, y)dt$$

- Integrals can be approximated using either *the rectangle method* or *Simpson's rule* of numerical approximation of definite integrals





Solution:

The formulae of calculations using the fourth-order Runge-Kutta method

$$x_{i+1} = x_i + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad (12)$$

$$y_{i+1} = y_i + \frac{1}{6}[m_1 + 2m_2 + 2m_3 + m_4]$$

$$k_1 = f(t_i, x_i, y_i)\Delta t$$

$$m_1 = g(t_i, x_i, y_i)\Delta t$$

$$k_2 = f(t_i + \frac{\Delta t}{2}, x_i + \frac{k_1}{2}, y_i + \frac{m_1}{2})\Delta t$$

$$m_2 = g(t_i + \frac{\Delta t}{2}, x_i + \frac{k_1}{2}, y_i + \frac{m_1}{2})\Delta t$$

$$k_3 = f(t_i + \frac{\Delta t}{2}, x_i + \frac{k_2}{2}, y_i + \frac{m_2}{2})\Delta t$$

$$m_3 = g(t_i + \frac{\Delta t}{2}, x_i + \frac{k_2}{2}, y_i + \frac{m_2}{2})\Delta t$$

$$k_4 = f(t_i + \Delta t, x_i + k_3, y_i + m_3)\Delta t$$

$$m_4 = g(t_i + \Delta t, x_i + k_3, y_i + m_3)\Delta t$$

$$f(t_i, x_i, y_i) = x_i(k - ay_i)$$

$$g(t_i, x_i, y_i) = y_i(bax_i - c)$$

Different scenarios with no slayers (c=0): *Scenario 1: The Stoker-King model*

- The vampire bites the victim and drinks the victim's blood, then returns to feed for 4-5 consecutive days
- Whereupon the victim dies, is buried and rises to become another vampire (unless a wooden stake is put through its heart).
- Vampires usually need to feed every day, so more and more human beings are constantly turned into vampires
- Bram Stoker's "Dracula" - 1897; Stephen King's "Salem's Lot" -1975

- Let us take 1897 as the starting point (i.e. the year Stoker's novel was first published).
- In 1897, the world population was about 1 650 million people (UN, 1999).
- The initial conditions:
  1 vampire, 1 650 million people, there are no organized groups of vampire slayers: **c=0**
- The calculation period is set at 1 year with a step of 5 days
- The coefficient of human population growth *k* for the given period is very small and can be neglected: ***k=0***.
- The probability of a human being turned into a vampire is very high: ***b=1***.

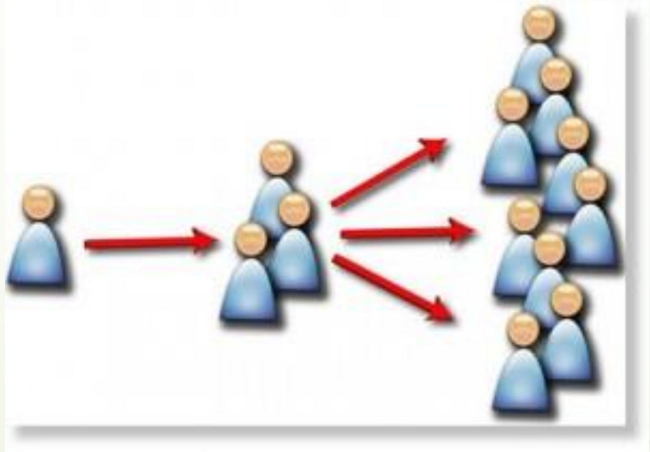
$$\begin{cases} \dfrac{dx}{dt} = x(\cancel{k} - ay) \\ \dfrac{dy}{dt} = y(bax - \cancel{c}) \end{cases}$$

Cauchy problem:

$$\begin{cases} \dfrac{dx}{dt} = -axy \\ \dfrac{dy}{dt} = axy \\ x(0) = 1.65 \cdot 10^9 \\ y(0) = 1 \end{cases}$$

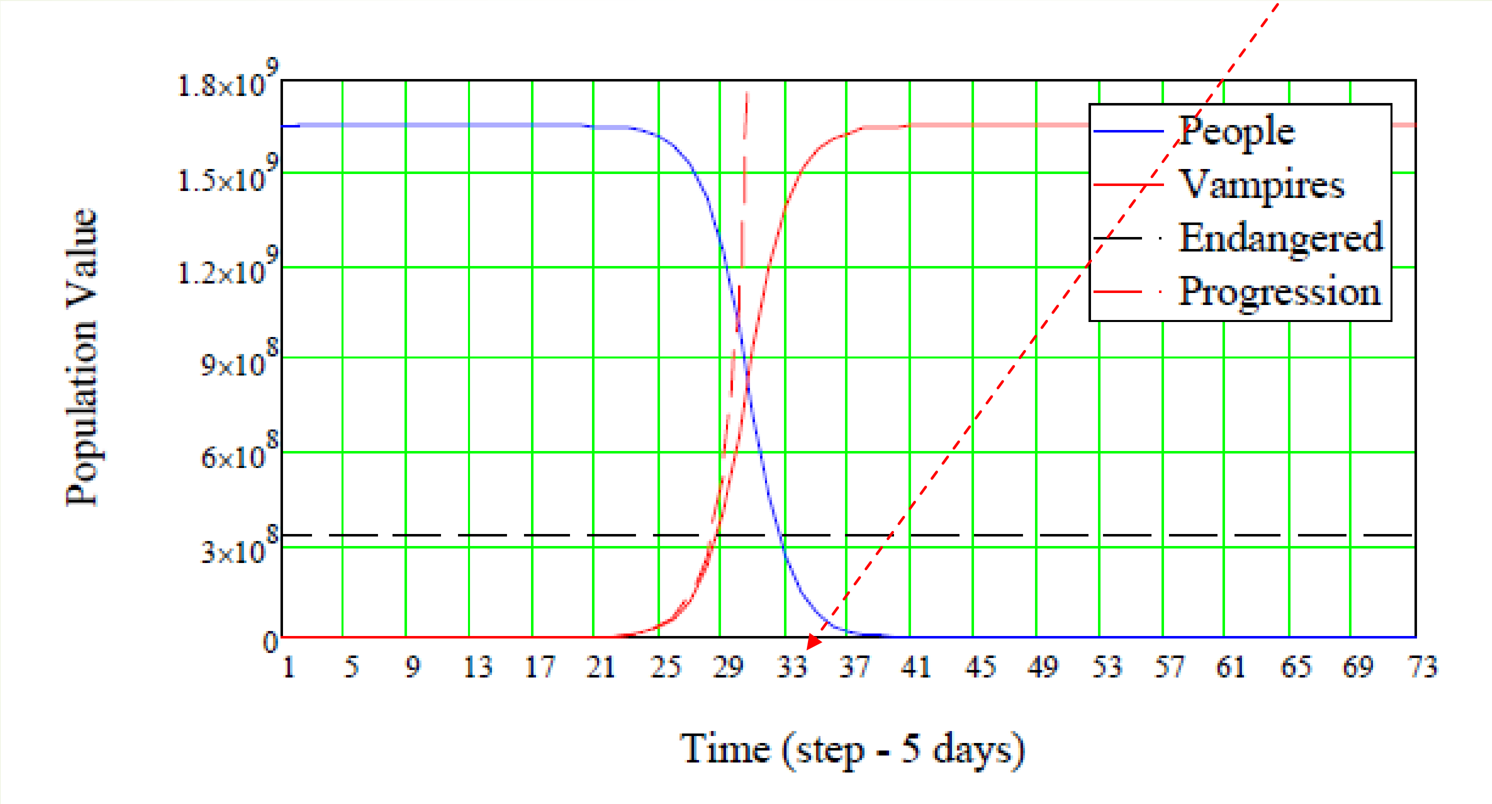
Epidemic?

- The total sum of humans and vampires does not change in time
- Human population does not grow and humans gradually become vampires
- The predator-prey model is diminished to a simple problem of an **epidemic outbreak**



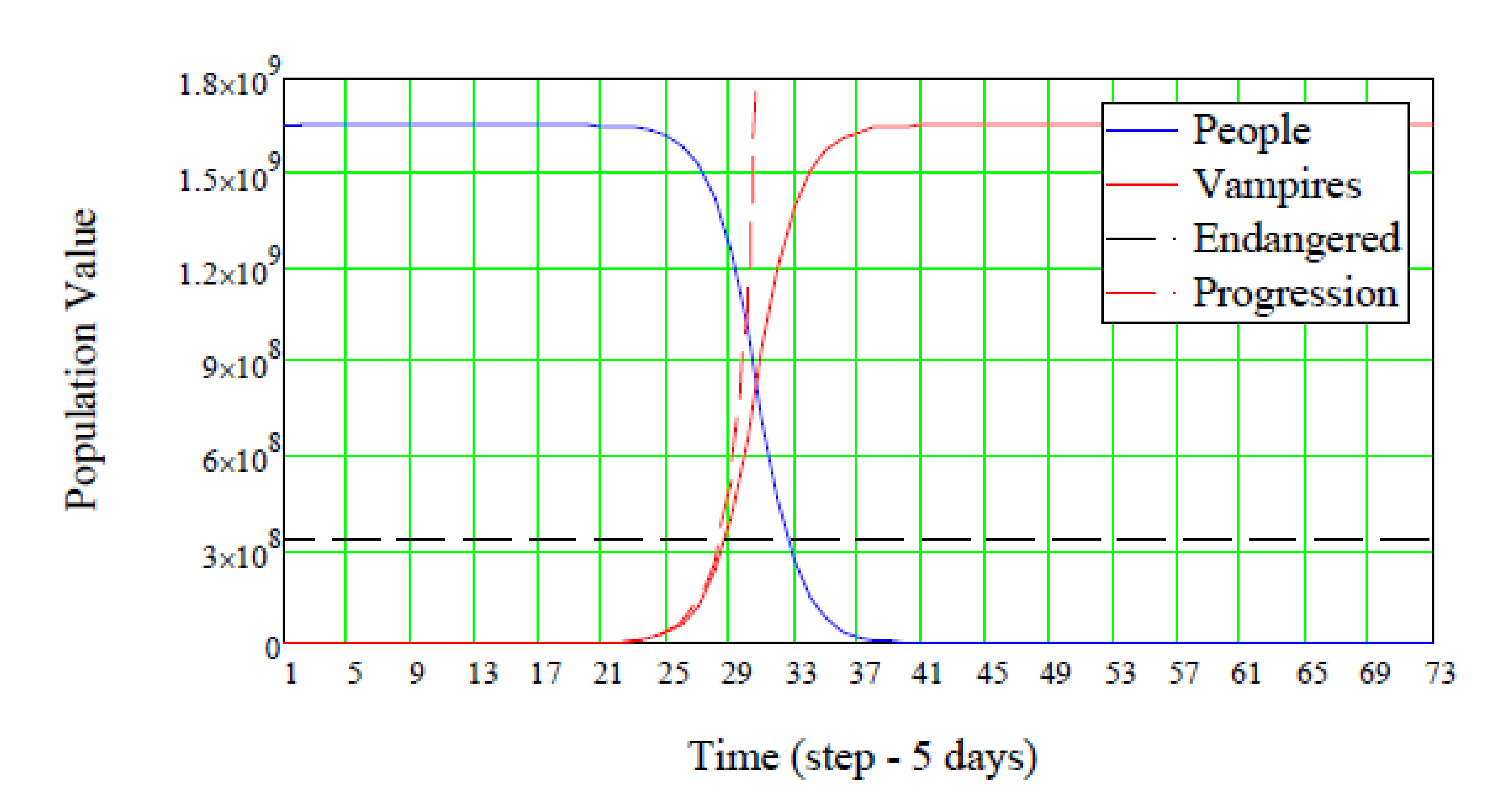
Epidemic

- The human population is drastically reduced by 80% by the 165th day from the moment when the first vampire arrives.
- This means that the human population reaches its critical value and practically becomes extinct
- At that precise moment, the world will be inhabited by 1 384 million vampires and 266 million people



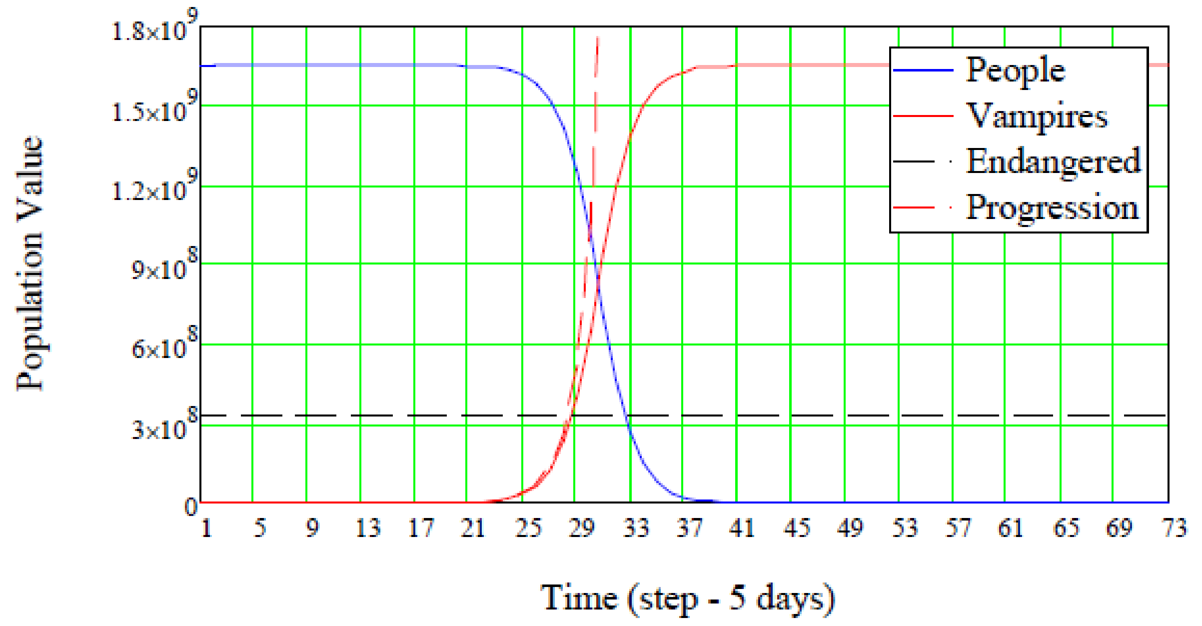
Epidemic

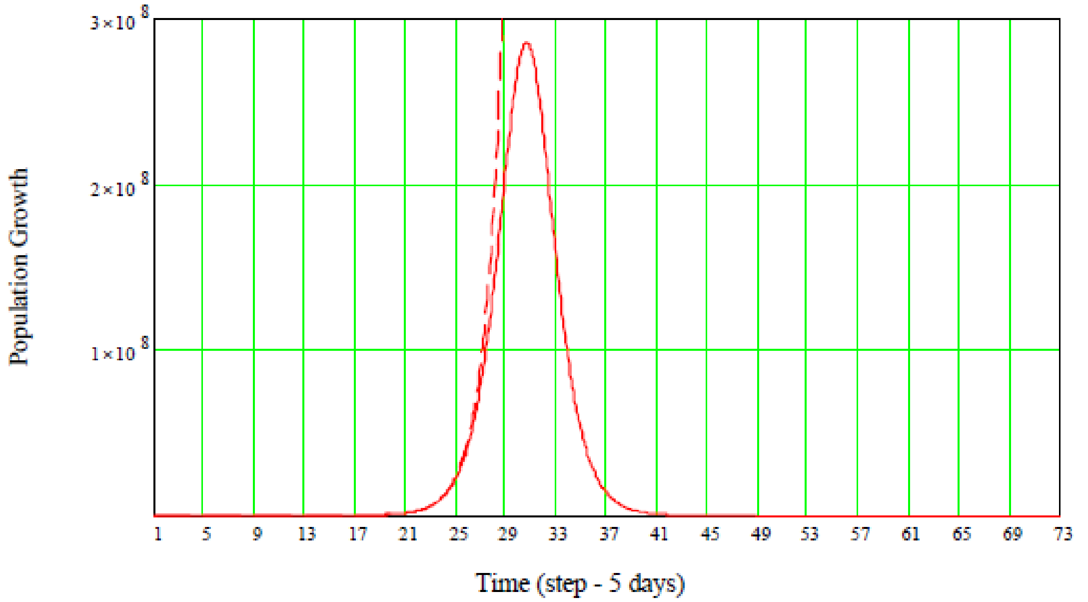
- With passing time the number of vampires grows and very soon there are no humans left
- **Severe epidemic outbreak**
- Leads first to the complete extinction of humans and then to the death of all vampires



The speed with which vampire population grows

$$\frac{dy}{dt}=\frac{a(y_0+x_0)^2\,y_0x_0e^{-a(y_0+x_0)(t-t_0)}}{(y_0+x_0e^{-a(y_0+x_0)(t-t_0)})^2}$$

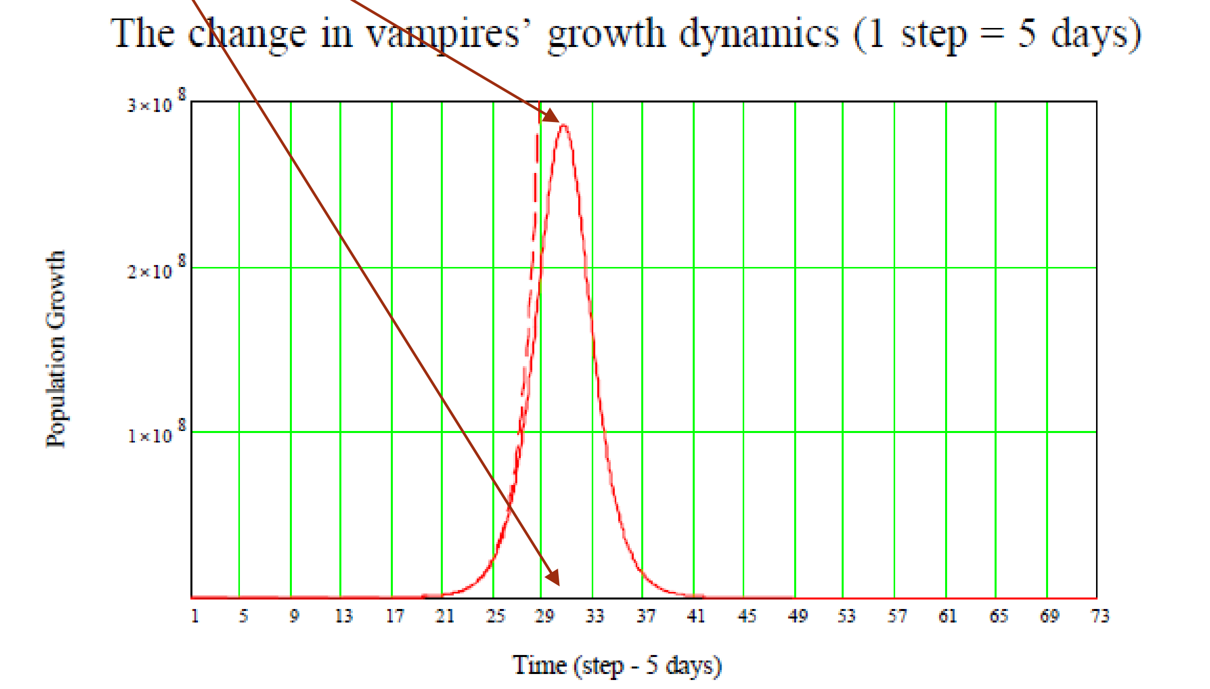




The maximal growth of the number of vampires (infected humans) will be observed in a moment of time t_{max}:

$$t_{\max} = \frac{\ln(x_0 / y_0)}{a(y_0 + x_0)} + t_0$$

where t_{max}= 153 is the day (153[rd] day) when the number of vampires is the highest, $x(t_{max})$=825 million is the number of vampires in a moment of time t_{max}, $x'(t_{max})$=286 million is the number of newly turned vampires in day t_{max}.



Scenario 2: The Rice model

- Anne Rice "Vampire Chronicles" describes the world with vampires, where vampires still need to feed on human but do so discretely
- The vampire can attack a human being, feed on it and leave it to live.
- In some cases (if they are too hungry), vampires kill their victims by draining their blood.
- BUT The vampire cannot easily turn the human into another vampire
- In order to do so, the victim's permission needs to be gained, it needs to drink some of vampire's blood, so it happens very rarely.
- They do not feed every day: some blood once a week or so is enough to survive

Scenario 2: The Rice model

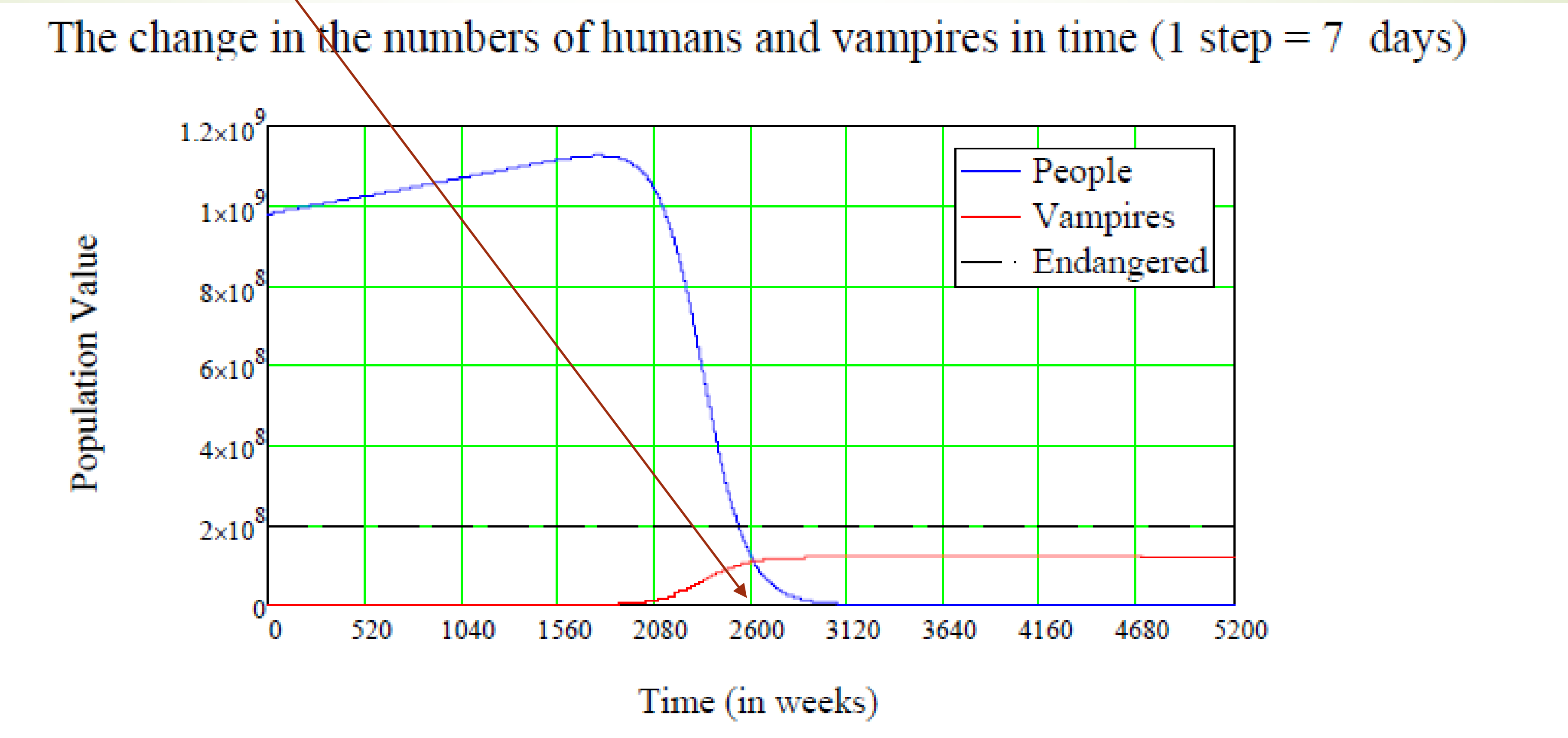
- The initial conditions of the Rice model are the following:
  2 vampires, 982 million people, there are no organized groups of vampire slayers: **c=0**
- The calculation period is set to 100 years with a step of 7 days
- Humans do not necessarily die or become vampires after their encounter with vampires, so the coefficient of lethal outcome ***a*** will be considerably lower than in the Stoker-King model: ***0.1a***.
- The probability of a human turned into a vampire is quite low: ***b=0.1***.

The coefficient of human population growth is calculated as $k = \dfrac{\ln(x_1 / x_0)}{t_1 - t_0}$

$$\begin{cases} \dfrac{dx}{dt} = x(k - ay) \\ \dfrac{dy}{dt} = baxy \\ x(0) = 982 \cdot 10^6 \\ y(0) = 2 \end{cases}$$

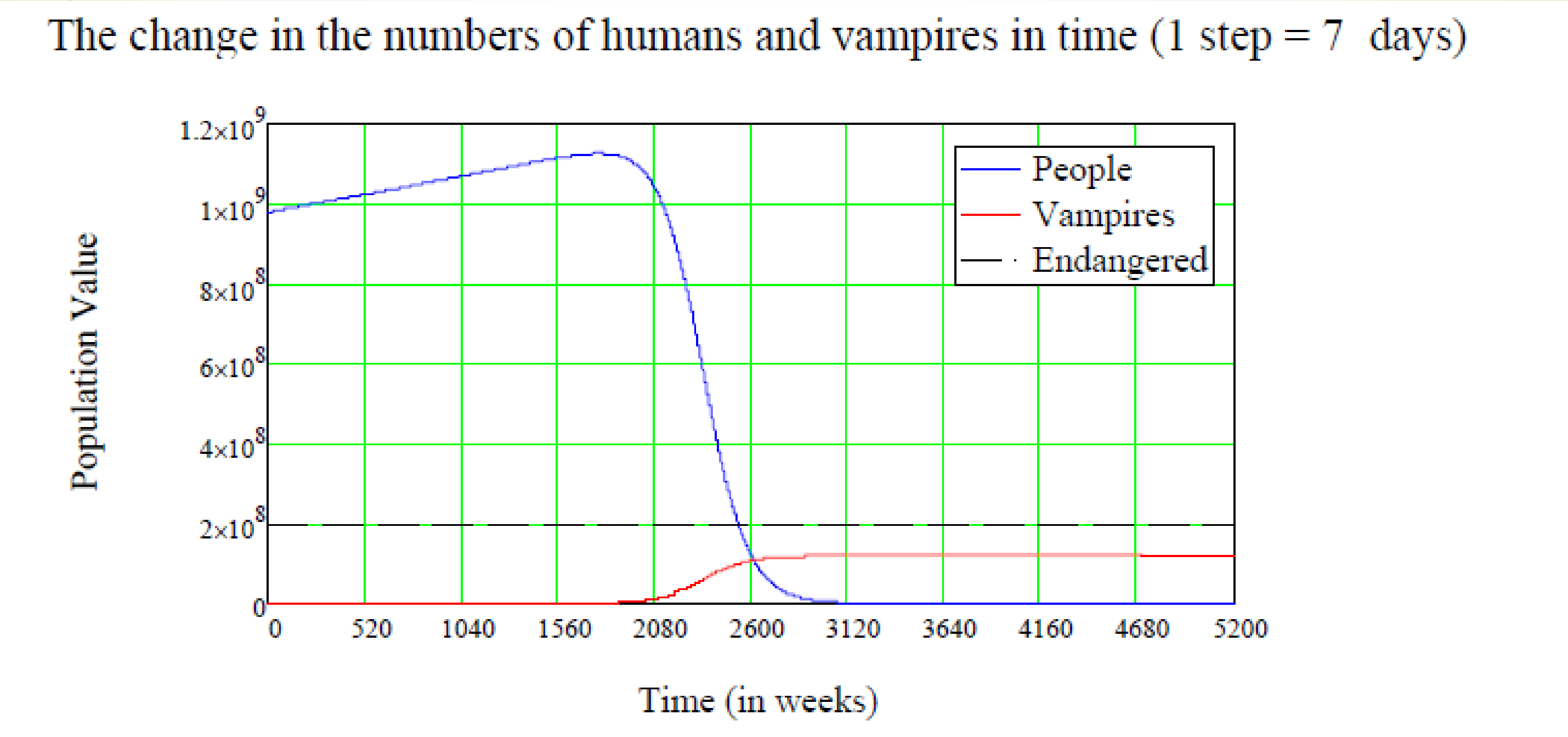
Epidemic?

- The human population grows in the beginning.
- When the number of vampires reaches its critical mass, the human population starts to shrink
- After 48.7 years human population is almost extinct.
  The number of vampires at this moment is equal to 100 million



Epidemic

- Delay of the total extinction of mankind by vampires by 48 years with respect to the first model



Scenario 3: The Harris-Meyer-Kostova model

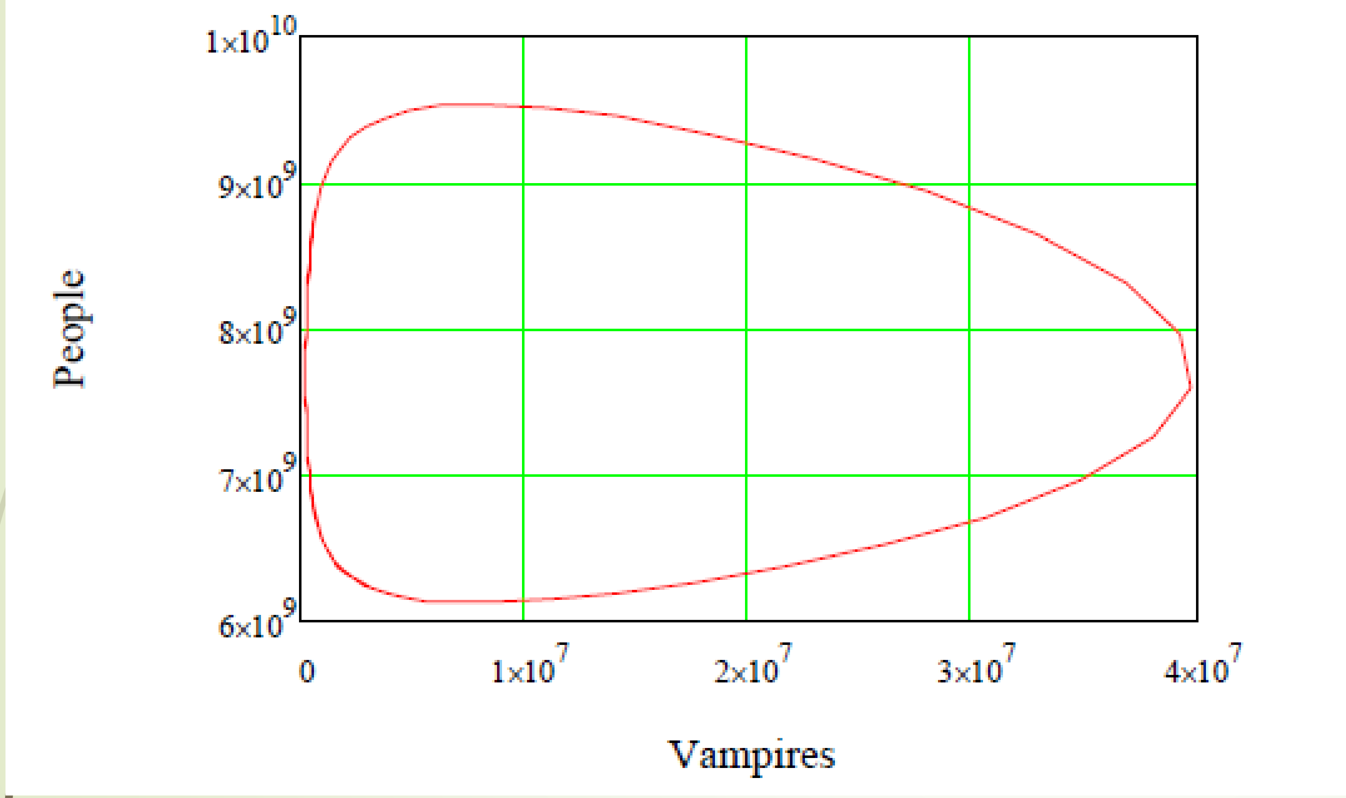
- In the books of Stephenie Meyer’s “Twilight series”, Charlaine Harris’s “Sookie Stockhouse (Southern Vampire) series”, “True Blood” (TV series) and Elizabeth Kostova’s “The Historian” there is a world drawn where **vampires peacefully coexist with humans.**
- Vampires interact with humans and drink animals’ or synthetic blood
- They either live in secrecy or side-by-side with humans.
- There is a possibility to turn a human being into a vampire, but it takes time and effort.

Scenario 3: coexistence?

- The calculation period is set at 100 years with a step of 1 year
- Humans almost always come out alive from their encounters with vampires, hence the coefficient of lethal outcome *a* will be low: ***0.01·a***
- The probability of a human being turned into a vampire is similar to the on in the Rice model: ***b=0.1***.
- Sometime vampires can be killed: ***c>0***
- The model allows for a stationary solution: there are system parameters (x_S, y_S) that would stabilize the populations of humans and vampires in time:
- **$(x_S, y_S) = (7704, 8)$ million individuals**

$$\begin{cases} \frac{dx}{dt} = x(k - ay) \\ \frac{dy}{dt} = y(bax - c) \\ x(0) = 6150 \cdot 10^6 \\ y(0) = 5 \cdot 10^6 \end{cases}$$

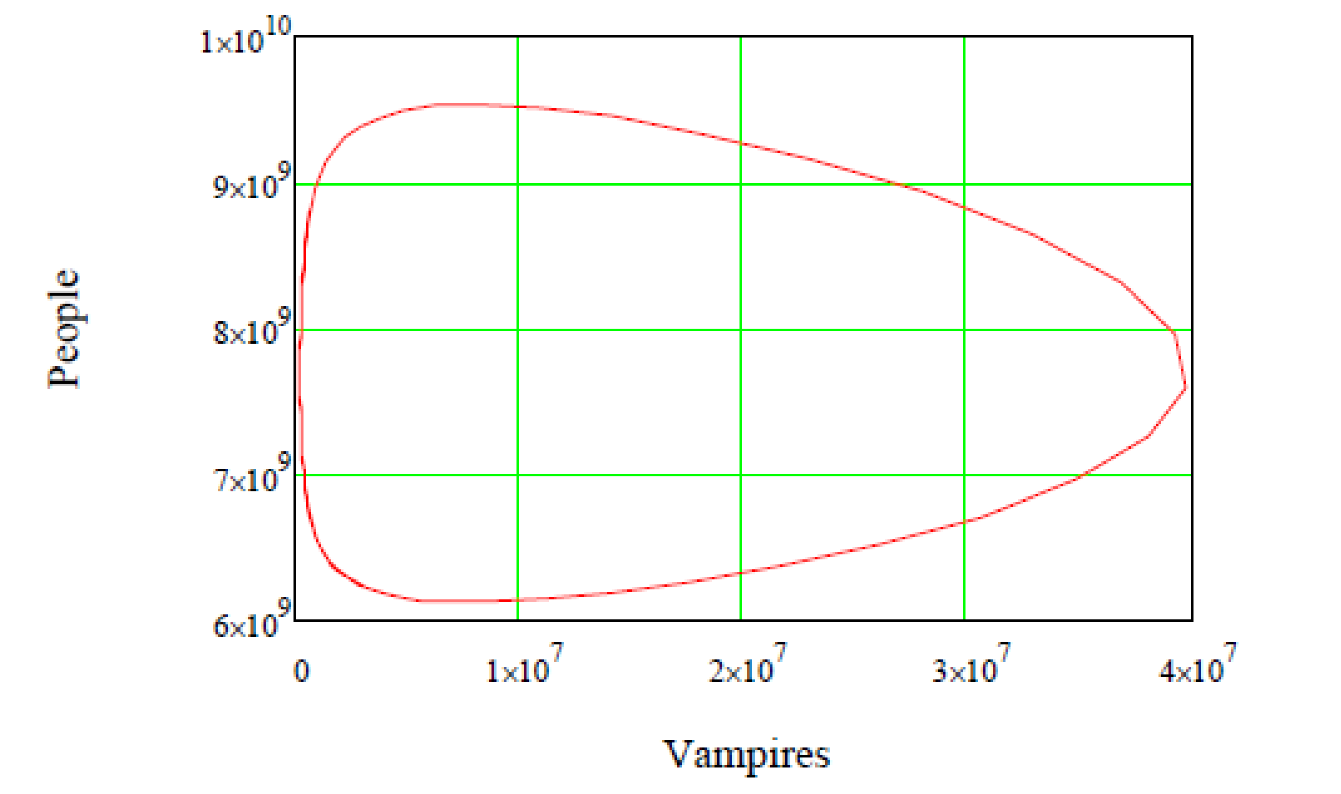
Scenario 3: coexistence



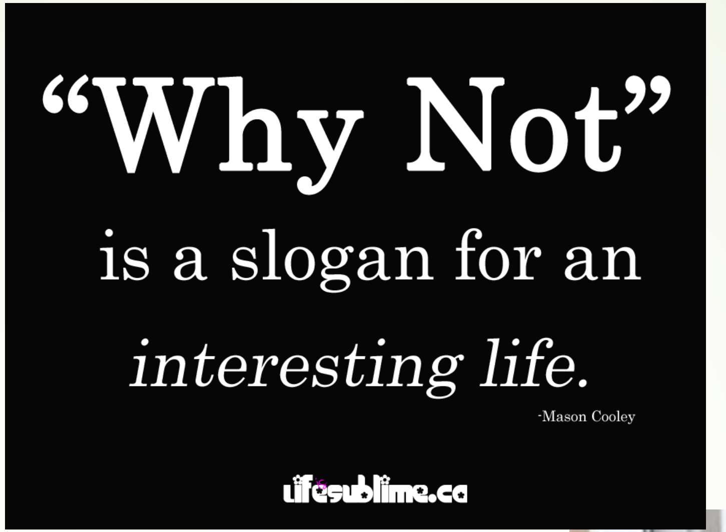
$$\begin{cases} \dfrac{dx}{dt} = x(k - ay) \\ \dfrac{dy}{dt} = y(bax - c) \\ x(0) = 6150 \cdot 10^6 \\ y(0) = 5 \cdot 10^6 \end{cases}$$

Peaceful co-existence of two spices

- This symbiosis, however, is very fragile
- Whenever the growth rate of human population slows down, the blood thirst of vampires accelerates, the whole system lies in ruins with just one population remaining.



Possible peaceful co-existence of two spices?



References:

- Applied Mathematical Sciences, Vol. 7, 2013, no. 10, 453 – 470

  “Mathematical Models of Interactions between Species: Peaceful Co-existence of Vampires and Humans Based on the Models Derived from Fiction Literature and Films”

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