ASYMPTOTIC CONES OF SYMMETRIC SPACES

by

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ABSTRACT

We study the asymptotic cone of arbitrary symmetric spaces. The main case to consider is when the symmetric space is irreducible and of noncompact type. It is known that if P is a symmetric space of noncompact type, then P is a homogeneous space and can be written as $P = G/K$ where G is a semisimple Lie group and K is a maximal compact subgroup. The asymptotic cone of P, denoted $\mathrm{Cone}_\omega P$, is naturally a homogeneous space with respect to the group $\mathrm{Cone}_\omega G$. We show that $\mathrm{Cone}_\omega G$ is an algebraic group that can be obtained from the group G by extending the field of real numbers. Using this description of the asymptotic cone as a homogeneous space, along with the study of the field extension, we identify the stabilizer of a point in $\mathrm{Cone}_\omega P$ and show that the asymptotic cone of a symmetric space is independent of the base point, scale factors and the ultrafilter.

To my family, especially Sierra.

CONTENTS

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Bruce Kleiner was a wonderful advisor throughout my studies. He always helped me with wonderful ideas and insights. He was always available to discuss mathematics with me, even after he moved from Utah. Without his help and encouragement, this dissertation would not have been possible.

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CHAPTER 1

INTRODUCTION

One of the main techniques in topology and geometry is to look closely. Given a topological space, we generally concentrate on small neighborhoods of points. One defines limits as sequences approaching a point. Similarly, derivatives are defined at points. Thinking infinitesimally has led to many deep and powerful results such as differential and integral calculus. Recently, many geometers and group theorists have begun to use large scale or asymptotic techniques. Instead of looking at shrinking neighborhoods, expanding neighborhoods are investigated. Rather than looking at sequences converging to a given point, one looks at diverging sequences.

This new approach has been particularly fruitful in the field of geometric group theory. Here, the geometric spaces are discrete metric spaces. The classical infinitesimal techniques are of little use when looking at discrete spaces. Rather than looking at the local level, geometers and group theorists look at large scale properties of these discrete spaces. This approach naturally led to the notion of *quasi-isometry* in contrast to the usual notion of isometry. A quasi-isometry is a map between spaces that may not even be continuous, but it does not distort distances "too much."

Definition 1.1. Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is a *quasi-isometry* if there exists some $L > 1$ and some $C > 0$ satisfying the following properties:

1. $\frac{1}{L}d(x, \tilde{x}) - C \leq d(f(x), f(\tilde{x})) \leq Ld(x, \tilde{x}) + C$, for all $x, \tilde{x} \in X$;

2. $d'(y, f(X)) < C$, for all $y \in Y$

Using quasi-isometries, one can talk about spaces that are quasi-isometric and properties that are invariant under quasi-isometry. This approach leads to many techniques that are quite different from classical infinitesimal techniques.

One of the new tools obtained as a result of looking at the large scale is the *asymptotic cone* of a metric space. Intuitively, if (X, d) is a metric space, then the asymptotic cone of X is the space that appears in the limit as one moves further and further from X. This is accomplished by rescaling the metric on X and looking for a limit space while the metric is scaled down to zero. Ultrafilters are required to ensure that there is always a limit. We will describe ultrafilters and the actual construction of the asymptotic cone briefly below.

The asymptotic cone of X, $\mathrm{Cone}_\omega X$, hides the local structure of the metric space X. Surprisingly, this allows one to use infinitesimal techniques to prove large scale results. For example, the asymptotic cone of $\mathbb{Z}^n$ is Euclidean space $\mathbb{R}^n$, and one can use infinitesimal techniques in $\mathbb{R}^n$ to recover large scale properties of $\mathbb{Z}^n$.

Using the asymptotic cone to study quasi-isometries between spaces is based on the fact that if $f : X \to Y$ is a quasi-isometry, then f induces a bi-Lipschitz map between $\mathrm{Cone}_\omega X$ and $\mathrm{Cone}_\omega Y$. This idea has been used by Kleiner and Leeb to prove that if two symmetric spaces are quasi-isometric, then they are actually homothetic [17]. This idea was also used by Kapovich and Leeb to classify many 3-manifolds groups according to quasi-isometry type [15].

An essential ingredient to actually construct the asymptotic cone of a metric space is the use of ultrafilters. To define an ultrafilter, let $\mathcal{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$, the set of all subsets of $\mathbb{N}$.

Definition 1.2. A family of subsets $\omega \subseteq \mathcal{P}(\mathbb{N})$ is said to be a *nonprincipal ultrafilter* if ω satisfies the following conditions:

1. $\emptyset \notin \omega$;
2. if $A \subseteq B \subseteq \mathbb{N}$ and $A \in \omega$, then $B \in \omega$;
3. if $A, B \in \omega$ then $A \cap B \in \omega$;

4. for any $A \subseteq \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} - A \in \omega$;

5. if $A \subseteq \mathbb{N}$ and A is finite, then $A \notin \omega$.

Definition 1.3. Given an ultrafilter ω, if $x_i \in \mathbb{R}$ is a sequence, then $x \in [-\infty, \infty]$ is an *ultralimit* of the sequence (written $\lim_\omega x_i = x$), if for every neighborhood $U \subseteq [-\infty, \infty]$ of x $\{i \mid x_i \in U\} \in \omega$.

It is well known that every sequence has a unique ultralimit (see Lemma 2.13).

To actually show that nonprincipal ultrafilters exist requires the Axiom of Choice. This is a potential problem because the use of the Axiom of Choice means there is no canonical choice for a nonprincipal ultrafilter. Instead there are many different nonprincipal ultrafilters, as will be described in Section 2.3.1.

After fixing a nonprincipal ultrafilter ω, we may fix scale factors $\lambda_i \in \mathbb{R}$, such that $\lim_\omega \lambda_i = \infty$. We also fix a base point $\star \in X$. The asymptotic cone is then the set of sequences $x_i \in X$ such that there is some constant $C \in \mathbb{R}$ with $d(x_i, \star) < C\lambda_i$ for all $i \in \mathbb{N}$. We define the distance between two sequences to be

$$d((x_i), (y_i)) = \lim_\omega \left(\frac{d(x_i, y_i)}{\lambda_i} \right)$$

and we identify two sequences if their distance is zero.

The construction outlined above is actually equivalent, in some cases, to looking at the Gromov-Hausdorff limit of the metric space X rescaled by the scale factors λ_i. In particular, given (X, d), one can consider the sequence of metric spaces

$$\left(X, \frac{1}{\lambda_i} d \right)$$

where $\frac{1}{\lambda_i} d$ is the distance function on X multiplied by $\frac{1}{\lambda_i}$. If this sequence of metric spaces converges in the sense of Gromov-Hausdorff, then it converges to the asymptotic cone. If the sequence does not converge, one can still construct the asymptotic cone using ultrafilters.

In some sense, asymptotic cones were first used by Gromov [12]. Gromov did not use ultrafilters which required extra work to extract a convergent subsequence from the sequence of metric spaces $(X, \frac{1}{\lambda_i} d)$. Later, van den Dries and Wilkie reworked Gromov's results using ultrafilters and defined the asymptotic cone using a nonprincipal ultrafilter as above [32].

As mentioned, the existence of a nonprincipal ultrafilter requires the Axiom of Choice. Thus, the construction of the asymptotic cone is not canonical. In fact, one can exhibit a metric space X and two different ultrafilters that result in asymptotic cones that are not homeomorphic. The example of Thomas and Velickovic is of a finitely generated but not finitely presented group [29]. It is still an open question if the asymptotic cone is unique for finitely presented groups. Given a metric space, an important question is if the asymptotic cone is independent of the base point, scale factors and ultrafilter. We answer this question affirmatively for symmetric spaces:

Theorem 1.4. Let X be a symmetric space. Then the asymptotic cone of X is independent of the base point, scale factors and the ultrafilter.

Asymptotic cones of symmetric spaces were first studied by Kleiner and Leeb [17]. Among their results, they show that the asymptotic cone of a symmetric space is a Euclidean building. Euclidean buildings are higher dimensional analogs of trees. Bruhat and Tits studied Euclidean buildings in [4] and [31]. They showed that Euclidean buildings arise from algebraic groups and valuation fields. In the case of the symmetric space for the group $SL(n, \mathbb{R})$, Leeb identified the asymptotic cone as a homogeneous space over an algebraic group [18]. Parreau showed that the asymptotic cone of the symmetric space for $SL(n, \mathbb{R})$ fits a certain model for Euclidean buildings over the group SL_n [22].

In Chapter 5, we study the asymptotic cone of an arbitrary symmetric space as a homogeneous space. We show that the transitive group acting on the asymptotic cone is an algebraic group over a valuation field obtained by a field extension of the field of real numbers. It is this description of the asymptotic cone as a homogeneous space together with an analysis of the valuation field that helps us establish Theorem 1.4.

CHAPTER 2

FIELDS

One goal of this chapter is to construct both the nonstandard real numbers and another similar nonarchimedean field. Before we actually do this, we need some background on ordered fields and valuation fields.

2.1 Ordered fields

A field k is ordered if there is a total ordering on k compatible with the field operations. This means that for all $u, x, y, z \in K$, $u > 0$, we have

$$\begin{aligned} x < y \quad &\Longrightarrow \quad x + z < y + z \\ x < y \quad &\Longrightarrow \quad xu < yu \end{aligned}$$

These properties immediately imply that k has characteristic zero. All ordered fields contain the rationals $\mathbb{Q}$, as a subfield.

Given an ordered field k, there exists a unique maximal algebraic extension of k which is still ordered [5, p. 269]. This is called the *real closure* of k. A field is said to be a *real closed field* if k is equal to its real closure. The standard example of a real closed field is the real numbers, $\mathbb{R}$. A basic result on real closures is the following theorem [5, Section 7.4].

Theorem 2.1. k is real closed if and only if it is ordered, every positive element of k is a square and every polynomial of odd degree has a root in k.

2.1.1 Order properties for fields

Definition 2.2. Let k be an ordered field and let $x \in k$. We say that x is

1. *infinite* if $|x| > n$ for all $n \in \mathbb{N}$;

2. *finite* if $|x| < n$ for some $n \in \mathbb{N}$;

3. *infinitesimal* if $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

k is *archimedian* if k contains no infinite elements. Otherwise, k is *nonarchimedean*.

Note that $\mathbb{R}$ is a real closed archimedian field. It is well known that any archimedian field is isomorphic to a subfield of $\mathbb{R}$. In fact, any complete, archimedian field is isomorphic to $\mathbb{R}$ [5, p. 259]. This is an example of order properties determining the ordered field. We will want a similar property for nonarchimedean fields.

Definition 2.3. Let S be an ordered set.

1. A subset $B \subseteq S$ is said to be *cofinal* if for any $x \in S$, there is some $b \in B$ with $x \leq b$. The *cofinality* of S, $\mathrm{cof}(S)$, is the minimum cardinality of a cofinal subset of S.

2. A subset $B \subseteq S$ is said to be *coinitial* if for any $x \in S$, there is some $b \in B$ with $b \leq x$. The *coinitiality* of S, $\mathrm{coi}(S)$, is the minimum cardinality of a coinitial subset of S.

Notice that if $B \subseteq k$ is cofinal in the field k, then $-B$ is coinitial. Therefore, for a field, the concepts of coinitiality and cofinality are equivalent. It is easy to see that $\mathrm{cof}(\mathbb{R}) = \mathrm{cof}(\mathbb{Q}) = \aleph_0$. A field with $\mathrm{cof}(k) > \aleph_0$ must necessarily be nonarchimedean.

The order properties that interest us are contained in the following definition. First, if $A, B \subseteq k$, we say $A \ll B$ if every element of A is less than every element of B.

Definition 2.4. Let k be an ordered field. We say that k is

1. an η_1-field if for each pair of (possibly empty) subsets $A, B \subseteq k$ with $A \ll B$ and $|A \cup B| < \aleph_1$, there is some $x \in k$ with $A \ll \{x\} \ll B$;

2. a semi-η_1-field if for each strictly increasing sequence $a_i \in k$ and strictly decreasing sequence $b_i \in k$, such that $a_i < b_i$ for all i, there exists some $x \in k$ such that $a_i < x < b_i$ for all i.

Note that $\mathbb{R}$ is a semi-η_1-field but not an η_1-field. As already mentioned, $\mathbb{R}$ is the unique complete archimedean field. For nonarchimedean fields, we have the following result of Erdös, Gillman and Henriksen [10, Theorem 2.1].

Theorem 2.5. Any two real closed η_1-fields of cardinality $\aleph_1$ are isomorphic.

It is clear that this isomorphism must preserve the orderings on the fields. This is because in a real closed field the set of all squares is the set of all nonnegative numbers. The set of all squares must be preserved in an isomorphism.

Without assuming the Continuum Hypothesis ($\aleph_1 = \mathfrak{c}$), Theorem 2.5 says nothing about fields of cardinality $\mathfrak{c}$. It is interesting to note that if we assume the negation of the Continuum Hypothesis, then as was shown by Roitman, there are infinitely many nonisomorphic η_1-fields of cardinality $\mathfrak{c}$ [26]. We will be assuming the Continuum Hypothesis throughout, so we can apply Theorem 2.5 to real closed η_1-fields of cardinality $\mathfrak{c}$.

2.2 Valuation fields

Definition 2.6. Let k be a field and let Γ be an ordered abelian group. A *valuation* on k is a surjective map $v : k \to \Gamma \cup \{\infty\}$ satisfying the following properties:

1. $v(x) = \infty \iff x = 0$;
2. $v(xy) = v(x) + v(y)$;
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Γ is called the *value group* of the valuation v.

In practice, the value group will generally be an additive subgroup of $\mathbb{R}$. In this case we say that v is a *real valuation.* If the value group is a discrete subgroup of $\mathbb{R}$, then we say that v is a *discrete valuation.*

Definition 2.7. Let (k, v_k) and (K, v_K) be fields with valuations, with the same value group. Let $\phi : k \to K$ be an isomorphism. Then ϕ is said to be *valuation preserving* if $v_K(\phi(x)) = v(x)$.

We will be interested in ordered fields that have a valuation. How the valuation and ordering interact is important and we make the following definition.

Definition 2.8. Let (k, v) be an ordered valuation field. The valuation is said to be *compatible with the ordering* if $0 \leq x < y \implies v(x) \geq v(y)$.

An immediate consequence of this definition is that if $v(x) > v(y)$ then $|x| < |y|$. It is important to note that $v(x) \geq v(y)$ does not imply that $|x| \leq |y|$.

Definition 2.9. Let (k, v) be a valuation field. The *valuation ring* of v is

$$O = \{x \in k \mid v(x) \geq 0\} \tag{2.1}$$

and the *valuation ideal* of v is

$$J = \{x \in k \mid v(x) > 0\} \tag{2.2}$$

One can show that J is a maximal ideal in O and therefore the quotient is a field. This quotient field is called the *residue field*: $\mathfrak{R} = O/J$.

2.2.1 Fields of power series

Let k be a field and Γ an ordered abelian group. We let k^Γ be the set of all functions from Γ to k. If $f \in k^\Gamma$, we define the support of f to be

$$\operatorname{supp} f = \{x \in \Gamma \mid f(x) \neq 0\}$$

We now define

$$k((\Gamma)) = \{f \in k^\Gamma \mid \operatorname{supp} f \text{ is well-ordered }\}$$

$k((\Gamma))$ is called a *Hahn field.* We think of elements of $k((\Gamma))$ as formal power series:

$$f = \sum_{\gamma \in I} k_\gamma t^\gamma \tag{2.3}$$

where $I \subseteq \Gamma$ is a well-ordered set. This formal power series makes the following definitions of addition and multiplication a bit more transparent:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (fg)(x) &= \sum_{u+v=x} f(u)g(v)\end{aligned}$$

It is not too difficult to show that $k((\Gamma))$ is closed under addition and multiplication, making $k((\Gamma))$ a ring with identity. It is much more difficult to show that $k((\Gamma))$ is actually a field. This was done by Hahn in 1907 [1, Sections 6.20,7.20].

Hahn fields can be given a valuation $v : k((\Gamma)) \to \Gamma$, defined by

$$v(f) = \inf(\operatorname{supp} f)$$

It is not difficult to show that this satisfies the definition of a valuation. If, in addition, k is an ordered field, we can order $k((\Gamma))$ by declaring f positive if and only if $f(v(f))$ is positive in k.

Proposition 2.10. Let k, K be fields. Let Γ be an ordered group. Suppose that $\phi : k \to K$ is an isomorphism. Then ϕ induces a valuation preserving isomorphism of Hahn fields $\Phi : k((\Gamma)) \to K((\Gamma))$ given by $\Phi(f)(x) = \phi(f(x))$. If k and K are ordered fields and ϕ is order preserving, then Φ is also ordered preserving.

In the power series notation (2.3), $\Phi : k((\Gamma)) \to K((\Gamma))$ is given by

$$\Phi\left(\sum k_\gamma t^\gamma\right) = \sum \phi(k_\gamma) t^\gamma$$

Proof: Given $f, g \in k((\Gamma))$ and $x \in \Gamma$:

$$\begin{aligned}\Phi(f+g)(x) &= \phi((f+g)(x)) = \phi(f(x)+g(x))\\ &= \phi(f(x)) + \phi(g(x))\\ &= \Phi(f)(x) + \Phi(g)(x)\end{aligned}$$

and therefore $\Phi(f+g) = \Phi(f) + \Phi(g)$. Similarly, $\Phi(fg) = \Phi(f)\Phi(g)$.

If $\Phi(f)(x) = \Phi(g)(x)$ for all $x \in \Gamma$, then $\phi(f(x)) = \phi(g(x))$ for all $x \in \Gamma$. Since ϕ is an isomorphism, $f(x) = g(x)$ for all x. Therefore $f = g$ and Φ is injective. Similarly, Φ^{-1} is injective and Φ must be an isomorphism.

Notice that because ϕ is an isomorphism, $f(x) = 0$ if and only if $\Phi(f)(x) = 0$. This shows that

$$\operatorname{supp} \Phi(f) = \operatorname{supp} f$$

and therefore $v(\Phi(f)) = v(f)$ and Φ is valuation preserving.

If ϕ is an isomorphism of ordered fields, then using the above results,

$$\Phi(f)(v(\Phi(f))) = \Phi(f)(v(f)) = \phi(f(v(f))) \tag{2.4}$$

Recall that $f > 0$ if and only if $f(v(f)) > 0$. The equation (2.4) now shows that $f > 0$ if and only if $\Phi(f) > 0$. Therefore Φ is order preserving because ϕ is order preserving. □

2.3 The nonstandard real numbers

2.3.1 Ultrafilters

We will let $\mathcal{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$, the set of all subsets of $\mathbb{N}$.

Definition 2.11. A family of subsets $\omega \subseteq \mathcal{P}(\mathbb{N})$ is said to be a *filter* if it satisfies the following conditions:

1. $\emptyset \notin \omega$;
2. if $A \subseteq B \subseteq \mathbb{N}$ and $A \in \omega$, then $B \in \omega$;
3. if $A, B \in \omega$ then $A \cap B \in \omega$.

ω is an *ultrafilter* if ω also satisfies

4. for any $A \subseteq \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} - A \in \omega$.

ω is a *nonprincipal* filter if ω also satisfies

5. if $A \subseteq \mathbb{N}$ and A is finite, then $A \notin \omega$.

Example 2.12. Fix $n \in \mathbb{N}$ and define

$$\omega_n = \{A \subseteq \mathbb{N} \mid n \in A\}$$

It is straightforward to check that ω_n satisfies all the properties in Definition 2.11, except for property 5. ω_n is the principal ultrafilter determined by $n \in \mathbb{N}$.

A small note about language and ultrafilters. If $A \in \omega$, then we will say that A is a set of full ω-measure. Suppose we have an ultrafilter and a sequence of statements P_i. If

$$\{i \mid P_i\} \in \omega$$

then we will say that P_i is true for ω-almost all i, or that P_i is true on a set of full ω-measure. This terminology comes from viewing ω as a finitely additive measure on $\mathbb{N}$.

Our main purpose for ultrafilters is to generalize convergence of sequences as in the following lemma. For a proof, see [15, Section 3.1].

Lemma 2.13. Let ω be an ultrafilter. Let $x_i \in \mathbb{R}$ be a real valued sequence. Then there exists a unique point $x \in [-\infty, \infty]$ such that for every neighborhood $U \subseteq [-\infty, \infty]$ of x, $\{i \mid x_i \in U\} \in \omega$. We define this point x, to be the ultralimit of the sequence, and we write

$$\lim_{\omega} x_i = x$$

Consider the principal ultrafilter ω_n of Example 2.12 for some $n \in \mathbb{N}$. Let $x_i \in \mathbb{R}$ be a sequence. It is fairly easy to see that

$$\lim_{\omega_n} x_i = x_n$$

The ultralimit of a principal ultrafilter "picks out" the respective term in the sequence. This makes principal ultrafilters uninteresting. We will therefore be interested in nonprincipal ultrafilters. To construct a nonprincipal ultrafilter, one begins with the cofinite filter η defined as:

$$\eta = \{A \subseteq \mathbb{N} \mid \mathbb{N} - A \text{ is finite}\}$$

A quick check shows that η satisfies all the properties in Definition 2.11, except property 4. We now consider

$$\mathcal{P}_\eta = \{\omega \subseteq \mathcal{P}(\mathbb{N}) \mid \eta \subseteq \omega, \ \omega \text{ a filter}\}$$

We can apply Zorn's Lemma to the set $\mathcal{P}_\eta$ to get an ultrafilter ω. In fact, it is easy to see that any nonprincipal ultrafilter must contain the cofinite filter η. The

Axiom of Choice is required to construct a nonprincipal ultrafilter, and therefore there is no canonical nonprincipal ultrafilter. To illustrate this problem, consider the following example.

Example 2.14. Let $E \subseteq \mathbb{N}$ denote the set of even integers. We can apply Zorn's Lemma to the following sets:

$$\begin{aligned} \mathcal{P} &= \{\omega \subseteq \mathcal{P}(\mathbb{N}) \mid \eta \subseteq \omega, E \in \omega,\ \omega \text{ a filter}\} \\ \mathcal{P}' &= \{\omega \subseteq \mathcal{P}(\mathbb{N}) \mid \eta \subseteq \omega, E \notin \omega,\ \omega \text{ a filter}\} \end{aligned}$$

This will give ultrafilters ω and ω'. These ultrafilters are different since $E \in \omega$ but $E \notin \omega'$. Consider the sequence (x_i) given by

$$x_i = \begin{cases} 0 & \text{if } i \text{ even} \\ 1 & \text{if } i \text{ odd} \end{cases}$$

Taking the ultralimit with respect to both these ultrafilters gives

$$\begin{aligned} \lim_{\omega} x_i &= 0 \\ \lim_{\omega'} x_i &= 1 \end{aligned}$$

For us, this is the inherent problem with ultrafilters, the ultralimits might be different.

2.3.2 Ultrapowers

Let X be some set and let ω be an ultrafilter. We consider the set of sequences in X:

$$X^{\mathbb{N}} = \{(x_i) \mid x_i \in X, i \in \mathbb{N}\}$$

We say two sequences are equivalent if they agree on a set of full ω-measure:

$$(x_i) \sim (y_i) \iff \{i \mid x_i = y_i\} \in \omega$$

We define ${}^*X$ as the set of equivalence classes under this equivalence relation. Notice that we can embed X into $X^{\mathbb{N}}$ as the set of constant sequences. If X is an infinite set then it is easy to see that ${}^*X \neq X$. We say that ${}^*X$ is an *ultrapower* of X. Notice that ${}^*X$ depends on the ultrafilter ω.

Remark 2.15. If $(x_i) \in X^{\mathbb{N}}$ is a sequence, then we will denote by $[x_i]$ the corresponding element in ${}^*X$. If $x \in X$, then we can consider x as a constant sequence, $(x) \in X^{\mathbb{N}}$ and the corresponding element in ${}^*X$ will be denoted by $[x]$.

2.3.3 The nonstandard real numbers

We construct an ultrapower of $\mathbb{R}$. This construction is due to Abraham Robinson and a good reference is his book [24]. One of the main points is that ${}^*\mathbb{R}$ is a field with field structure inherited from $\mathbb{R}$. There are many references for this construction as well as for the proofs of the properties stated below. In addition to [24], other good references include [11], [19] and [20].

Notice first, $\mathbb{R}^{\mathbb{N}}$ is a partially ordered ring with operations defined point wise:

$$\begin{aligned} (x_i) + (y_i) &= (x_i + y_i) \\ (x_i)(y_i) &= (x_i y_i) \\ (x_i) < (y_i) &\iff x_i < y_i \text{ for all } i \end{aligned}$$

Let ω be a nonprincipal ultrafilter and let ${}^*\mathbb{R}$ be the corresponding ultrapower of $\mathbb{R}$. We embed $\mathbb{R} \subseteq {}^*\mathbb{R}$ via the embedding $x \mapsto [x]$, where $[x]$ represents the constant sequence.

Proposition 2.16. ${}^*\mathbb{R}$ is a nonarchimedean ordered field with operations and ordering induced from $\mathbb{R}^{\mathbb{N}}$. The ultralimit $\lim_\omega$, is a homomorphism from the ring of finite elements in ${}^*\mathbb{R}$ to $\mathbb{R}$.

For a proof, see [11, Theorem 5.6.2]. To emphasize the field operations in ${}^*\mathbb{R}$, we have the following:

$$\begin{aligned} [x_i] + [y_i] &= [x_i + y_i] \\ [x_i][y_i] &= [x_i y_i] \\ [x_i] < [y_i] &\iff x_i < y_i \text{ for } \omega\text{-almost all } i \end{aligned}$$

Proposition 2.17. ${}^*\mathbb{R}$ is a real closed field.

The proof is straightforward, see [10, Lemma 3.3].

Proposition 2.18. ${}^*\mathbb{R}$ is an η_1-field.

The proof is basically a diagonal argument and can be found in [10, Theorem 3.4].

Notice that the set $\mathbb{R}^{\mathbb{N}}$ has cardinality $\mathfrak{c}$. Therefore the field ${}^*\mathbb{R}$ has cardinality $\mathfrak{c}$. Applying Theorem 2.5 gives the following isomorphism theorem [10, Theorem 3.4]. Note that the Continuum Hypothesis is required.

Corollary 2.19. For any ultrafilters, the fields ${}^*\mathbb{R}$ are all isomorphic.

Example 2.20.

1. Consider the sequence $(i) = (1, 2, 3, \ldots)$. This defines an element in $\mathbb{R}^{\mathbb{N}}$ and hence an element in ${}^*\mathbb{R}$. We will denote this element by $[i] \in {}^*\mathbb{R}$. If $x \in \mathbb{R}$, then $[x] \in {}^*\mathbb{R}$, represented as a constant sequence. Then, except for a finite number of i, $x < i$, and therefore $[x] < [i]$ in ${}^*\mathbb{R}$. Therefore $[i]$ is infinite and ${}^*\mathbb{R}$ is nonarchimedean. The element $[1/i]$ is a nonzero infinitesimal.

2. Let $p(t) \in {}^*\mathbb{R}[t]$ be a polynomial of one variable. We can consider each of the coefficients of $p(t)$ as a sequence of real numbers. As such, this will give a sequence of polynomials $p_i(t) \in \mathbb{R}[t]$. We can write $p = [p_i]$. To find roots of p, we can just find roots of the polynomials $p_i(t)$. If $p_i(t_i) = 0$ for ω almost all i, then $[t_i]$ is a root of p.

3. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then we can extend this function to a function ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ as follows. If $[x_i] \in {}^*\mathbb{R}$, we define

$$ {}^*f([x_i]) = [f(x_i)] $$

We will often write f instead of ${}^*f$ for this extension. See Section 2.5 for more details.

2.4 The field ${}^\rho\mathbb{R}$

We are going to use ${}^*\mathbb{R}$ to construct another field. This field, like ${}^*\mathbb{R}$, was introduced by Abraham Robinson. This new field is not as prevalent in the literature as the field ${}^*\mathbb{R}$. The references are [19], [25], [21] and [23]. Most of the elementary results can be found in [19].

2.4.1 The definitions

We fix a positive infinitesimal $\rho \in {}^*\mathbb{R}$. We consider the following sets

$$M_0 = \{x \in {}^*\mathbb{R} \mid |x| < \rho^{-n} \text{ for some } n \in \mathbb{N}\} \tag{2.5}$$

$$M_1 = \{x \in {}^*\mathbb{R} \mid |x| < \rho^{n} \text{ for all } n \in \mathbb{N}\} \tag{2.6}$$

M_0 is a ring of ${}^*\mathbb{R}$ and M_1 is a maximal ideal of M_0 [19, pp 77-78]. Therefore, M_0/M_1 is a field, which we define to be the field ${}^\rho\mathbb{R}$.

$$ {}^\rho\mathbb{R} = \frac{M_0}{M_1} \tag{2.7}$$

We denote the projection from M_0 to ${}^\rho\mathbb{R}$ by $\Pi : M_0 \to {}^\rho\mathbb{R}$. If $x \in M_0 \subseteq {}^*\mathbb{R}$, we will write $\overline{x}$ for the element $\Pi(x)$.

Notice that if $x \in M_1$ and $0 < y < x$ then $y \in M_1$ (which means that M_1 is a convex ideal). This implies that the ordering on ${}^*\mathbb{R}$ induces an ordering on ${}^\rho\mathbb{R}$. This ordering can be described as

$$\overline{x} < \overline{y} \text{ (in } {}^\rho\mathbb{R}) \iff x < y \text{ (in } M_0)$$

Proposition 2.21. ${}^\rho\mathbb{R}$ is a real closed field.

Proof: Recall Theorem 2.1, which states that an ordered field is real closed if and only if every positive element is a square and every polynomial of odd degree has a root. We first verify that every positive element in ${}^\rho\mathbb{R}$ is a square. Let $\overline{x} \in {}^\rho\mathbb{R}$ be given with $\overline{x} > 0$ ($x \in M_0$). Then $x > 0$ and because ${}^*\mathbb{R}$ is real closed, there is some $z \in {}^\rho\mathbb{R}$ such that $z^2 = x$. Since $x \in M_0$, we must have $z \in M_0$ and therefore $\overline{z}^2 = \overline{x}$ and $\overline{x}$ is a square.

If we have a polynomial of odd degree $\overline{P}(X) \in {}^\rho\mathbb{R}[X]$, we can lift the coefficients of $\overline{P}$ (which are elements of ${}^\rho\mathbb{R}$) to $M_0 \subseteq {}^*\mathbb{R}$. Thus we obtain a polynomial $P(X) \in M_0[X] \subseteq {}^*\mathbb{R}[X]$. Since ${}^*\mathbb{R}$ is real closed, there is some $x \in {}^*\mathbb{R}$ such that $P(x) = 0$.

If $x \in M_0$, then $\overline{P(x)} = \overline{P}(\overline{x}) = 0$ and we have a root of $\overline{P}$. So, we have to show that $x \in M_0$. Suppose to the contrary that $x \notin M_0$.

$$\begin{aligned} x \notin M_0 \quad &\Longleftrightarrow \quad \forall n \in \mathbb{N}, |x| > \rho^{-n} \\ &\Longleftrightarrow \quad \forall n \in \mathbb{N}, |x^{-1}| < \rho^{n} \\ &\Longleftrightarrow \quad x^{-1} \in M_1 - \{0\} \end{aligned} \tag{2.8}$$

Now suppose that $P(X)$ is given by

$$P(X) = a_n X^n + \cdots + a_1 X + a_0$$

where $a_i \in M_0$ for all i and $a_n \notin M_1$. We rearrange the equation $P(x) = 0$:

$$\begin{aligned} a_n x^n &= -a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0 \\ x &= -\left(\frac{a_{n-1}}{a_n}\right) - \left(\frac{a_{n-2}}{a_n}\right)\frac{1}{x} - \cdots - \left(\frac{a_0}{a_n}\right)\frac{1}{x^{n-1}} \\ |x| &\leq \left|\frac{a_{n-1}}{a_n}\right| + \left|\frac{a_{n-2}}{a_n}\right|\left|\frac{1}{x}\right| + \cdots + \left|\frac{a_0}{a_n}\right|\left|\frac{1}{x^{n-1}}\right| \end{aligned} \tag{2.9}$$

Notice that since $a_n \in M_0 - M_1$ and $a_i \in M_0$, we have $a_i/a_n \in M_0$ for all i. We are also assuming that $x \notin M_0$ which, by (2.8), means that $x^{-i} \in M_1 \subseteq M_0$ for all i. Because M_0 is a ring, the sum in (2.9) is in M_0. Because M_0 is convex, the inequality (2.9) now implies that $x \in M_0$, a contradiction. Therefore $x \in M_0$ and $\overline{x}$ will be a root of $\overline{P}(X)$. $\square$

2.4.2 The valuation

If $x \neq 0 \in {}^*\mathbb{R}$, we define $\log_\rho |x| = (\ln |x|)/(\ln \rho)$. We can compose $\log_\rho$ and the ultralimit:

$$\lim_\omega \circ \log_\rho |\cdot| : {}^*\mathbb{R} - \{0\} \to [-\infty, \infty]$$

If $x \in M_0 - M_1$, then $\rho^n < |x| < \rho^{-n}$ for some $n \in \mathbb{N}$. This implies $n \ln \rho < \ln |x| < -n \ln \rho$ and therefore $-n < \log_\rho |x| < n$. Thus, if $x \in M_0 - M_1$, $\log_\rho |x|$ is finite and

$$\lim_\omega \circ \log_\rho : M_0 - M_1 \to \mathbb{R}$$

Lemma 2.22. If $x \in M_0$ and $h \in M_1$ then $\lim_\omega \circ \log_\rho |x| = \lim_\omega \circ \log_\rho |x + h|$.

For a proof, see [21, p. 194].

Lemma 2.22 tells us that $\lim_\omega \circ \log_\rho |\cdot|$ descends to a well-defined function on ${}^\rho\mathbb{R} - \{0\}$. We now define

$$v : {}^\rho\mathbb{R} \to \mathbb{R} \cup \{\infty\}$$
$$v(\overline{x}) = \begin{cases} \lim_\omega \circ \log_\rho |x| & \text{if } \overline{x} \neq 0 \\ \infty & \text{if } \overline{x} = 0 \end{cases}$$

Proposition 2.23. v is a valuation on the field ${}^\rho\mathbb{R}$. The valuation is compatible with the ordering on ${}^\rho\mathbb{R}$.

For a proof, see [19, pp. 79-84].

Proposition 2.24. $\mathbb{R}$ embeds into ${}^\rho\mathbb{R}$, and for all $x \in \mathbb{R} - \{0\} \subseteq {}^\rho\mathbb{R}$, $v(x) = 0$.

See [19, p. 82] for a proof.

Example 2.25. Proposition 2.24 shows that the valuation of any finite element is zero. However, there are also infinite elements of ${}^\rho\mathbb{R}$ that have zero valuation. To see this, consider the element in ${}^\rho\mathbb{R}$ represented by $\log \rho$. It is clear that $\log \rho \in M_0$ and $\log \rho$ is infinite. A small calculation shows that $\log \rho$ has zero valuation as well.

2.4.3 Order properties

Proposition 2.26. $\mathrm{cof}({}^\rho\mathbb{R}) = \aleph_0$.

Proof: Consider the element $\overline{\rho} \in {}^\rho\mathbb{R}$ (the projection of ρ from M_0 to ${}^\rho\mathbb{R}$). The set $\{\overline{\rho}^{-n}\}$ is countable and cofinal in ${}^\rho\mathbb{R}$. For if $x \in {}^\rho\mathbb{R} - \{0\}$, then $v(x) = r$ for some $r \in \mathbb{R}$. Let $n \in \mathbb{N}$ be such that $-n < r$. Because the valuation is compatible with the ordering, we have

$$v(\overline{\rho}^{-n}) = -n < r = v(x) \implies \overline{\rho}^{-n} > x$$

Therefore the set $\{\overline{\rho}^{-n}\}$ must be cofinal. □

Proposition 2.27. ${}^\rho\mathbb{R}$ is not an η_1 field.

Proof: We need to exhibit sets $A \ll B$ such that $|A \cup B| < \aleph_1$ and there does not exist $x \in {}^\rho\mathbb{R}$ such that $A \ll \{x\} \ll B$. To do this let $A = \{0\}$ and let $B = \{\overline{\rho}^n\}$.

Suppose that x is such that $A \ll \{x\} \ll B$. Then, by taking valuations and using the fact that the valuation is compatible with the ordering, we see that

$$n = v(\overline{\rho}^n) \leq v(x) \leq v(0) = \infty$$

Since this must be true for all n, $v(x) = \infty$. However, because v is a valuation, this means that $x = 0$, a contradiction (since we had assumed $\{0\} = A \ll \{x\}$). $\square$

Notice in the proof that A is not an increasing sequence, even though B is a decreasing sequence. In fact, this must be the case in any example showing that ${}^{\rho}\mathbb{R}$ is not η_1.

Proposition 2.28. ${}^{\rho}\mathbb{R}$ is a semi-η_1 field.

Proof: Suppose $\overline{a}_n \in {}^{\rho}\mathbb{R}$ is a strictly increasing sequence and $\overline{b}_n \in {}^{\rho}\mathbb{R}$ is a strictly decreasing sequence such that $\overline{a}_n < \overline{b}_n$ for all n. We need to show that there is some $\overline{x} \in {}^{\rho}\mathbb{R}$ such that $\overline{a}_n < \overline{x} < \overline{b}_n$ for all n.

We lift $\overline{a}_n$ and $\overline{b}_n$ to strictly monotonic sequences $a_n, b_n \in {}^{*}\mathbb{R}$. Because $\overline{a}_n < \overline{b}_n$, we have $a_n < b_n$ for all n. We now apply the semi-η_1 property for ${}^{*}\mathbb{R}$. This implies that there is an element $x \in {}^{*}\mathbb{R}$ such that $a_n < x < b_n$. Now consider $\overline{x} \in {}^{\rho}\mathbb{R}$. Because $a_n < x < b_n$, we must have $\overline{a}_n \leq \overline{x} \leq \overline{b}_n$. Notice that since $\overline{a}_n$ is strictly increasing and $\overline{b}_n$ is strictly decreasing we cannot have $\overline{x} = \overline{a}_n$ or $\overline{x} = \overline{b}_n$ for any n. Therefore $\overline{a}_n < \overline{x} < \overline{b}_n$ and ${}^{\rho}\mathbb{R}$ is semi-η_1. $\square$

2.4.4 The residue field

Since ${}^{\rho}\mathbb{R}$ is a valuation field, there is a valuation ring O, a valuation ideal J and a residue field $\mathfrak{R}$. Recall that O, J and $\mathfrak{R}$ are defined as:

$$\begin{aligned} O &= \{x \in {}^{\rho}\mathbb{R} \mid v(x) \geq 0\} \\ J &= \{x \in {}^{\rho}\mathbb{R} \mid v(x) > 0\} \\ \mathfrak{R} &= O/J \end{aligned}$$

We will use the notation that if $x \in O$, then the corresponding element in $\mathfrak{R}$ will be denoted $\overline{x}$. Notice that J is a convex set, therefore the ordering on ${}^{\rho}\mathbb{R}$ induces an ordering on $\mathfrak{R}$. This ordering can be described as

$$\overline{x} < \overline{y} \text{ (in } \mathfrak{R}) \iff x < y \text{ (in } {}^\rho\mathbb{R})$$

Lemma 2.29. $\mathfrak{R}$ is real closed.

Proof: This is basically the same proof as when we proved that ${}^\rho\mathbb{R}$ is real closed, Proposition 2.21. It is clear that $\mathfrak{R}$ is ordered and that every positive element has a square. The key point here is that if $x \in {}^\rho\mathbb{R}$, $x \geq 0$ and $v(x) \geq 0$ then $v(\sqrt{x}) \geq 0$.

Now, suppose we have a polynomial $\overline{P}(X) \in \mathfrak{R}[X]$ of odd degree. We lift this to a polynomial $P(X) \in O[X]$ Since, ${}^\rho\mathbb{R}$ is real closed, there is some $x \in {}^\rho\mathbb{R}$ such that $p(x) = 0$. If $x \in O$, then x will project to a root $\overline{x} \in \mathfrak{R}$. So, we have to show that $x \in O$. Assume that $P(X)$ is given by

$$P(X) \quad = \quad a_n X^n + \cdots + a_1 X + a_0$$

where n is odd and $a_n \notin J$. In particular this means that $v(a_n) = 0$. Suppose that $x \notin O$. Then $v(x) < 0$, $v(x^{-1}) > 0$ and therefore $x^{-1} \in J$. We now manipulate the expression $P(x) = 0$.

$$\begin{aligned} a_n x^n &= -a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0 \\ x &= -\left(\frac{a_{n-1}}{a_n}\right) - \left(\frac{a_{n-2}}{a_n}\right)\frac{1}{x} - \cdots - \left(\frac{a_0}{a_n}\right)\frac{1}{x^{n-1}} \\ |x| &\leq \left|\frac{a_{n-1}}{a_n}\right| + \left|\frac{a_{n-2}}{a_n}\right|\left|\frac{1}{x}\right| + \cdots + \left|\frac{a_0}{a_n}\right|\left|\frac{1}{x^{n-1}}\right| \end{aligned} \tag{2.10}$$

Notice that $v(a_i/a_n) = v(a_i) - v(a_n) = v(a_i) \geq 0$ because $v(a_n) = 0$. This shows that $a_i/a_n \in O$. Also, by assumption, $x^{-1} \in J$. Because J is an ideal and O a ring, every term on the right in (2.10) is in O. O is a convex ring, so we must have $x \in O$, a contradiction. Thus we have $x \in O$ and $\overline{x}$ is a root of $\overline{P}(X)$. $\square$

Proposition 2.30. $\mathfrak{R}$ is a semi-η_1-field.

Proof: This follows from the corresponding fact in ${}^\rho\mathbb{R}$. We suppose that we have a strictly increasing sequence $\overline{a}_n \in \mathfrak{R}$ and a strictly decreasing sequence $\overline{b}_n \in \mathfrak{R}$ such that $\overline{a}_n < \overline{b}_n$ for all n where $a_n, b_n \in {}^\rho\mathbb{R}$. Because the ordering on $\mathfrak{R}$ is induced from the ordering on ${}^\rho\mathbb{R}$, we also have $a_n < b_n$ for all n. We can now use that fact that ${}^\rho\mathbb{R}$ is semi-η_1 and find $x \in {}^\rho\mathbb{R}$ such that $a_n < x < b_n$ for all n. Then $\overline{a}_n < \overline{x} < \overline{b}_n$ and therefore $\mathfrak{R}$ is semi-η_1. □

Proposition 2.31. $\mathfrak{R}$ is an η_1 field.

Proof: We need to show that given any two subsets $A, B \subseteq \mathfrak{R}$ such that $|A \cup B| < \aleph_1$ and $A \ll B$, there exists $x \in \mathfrak{R}$ such that $A \ll \{x\} \ll B$. Notice first that if $\operatorname{cof}(A) = 1$, then we can replace A by a single element set. Similarly if $\operatorname{coi}(B) = 1$. We can also transform the sets A and B by any of the following transformations:

1. Translation: $x \mapsto x + c$, $c \in \mathfrak{R}$.

2. Dilation: $x \mapsto \lambda x$, $\lambda \neq 0$.

3. Inversion: $x \mapsto 1/x$.

These transformations will change the sets A and B, but finding an element between A and B will be equivalent to finding an element between the corresponding transformed sets. There are several cases to consider.

Case I. If $\operatorname{cof}(A) = \aleph_0$ and $\operatorname{coi}(B) = \aleph_0$, then this is the semi-$\eta_1$ case that was handled in Proposition 2.30.

Case II. $A = \emptyset$ and $B = \emptyset$. This case is trivial since any $x \in \mathfrak{R}$ will work.

Case III. $\operatorname{cof}(A) = 1$ and $\operatorname{coi}(B) = 1$. This case is also trivial since it is easy to find an element between any two given elements.

Case IV. $\operatorname{cof}(A) = \aleph_0$ and $B = \emptyset$. This will be our main case to prove. We do this below.

Case V. $\operatorname{cof}(A) = \aleph_0$ and $\operatorname{coi}(B) = 1$. By translation we can assume $B = \{0\}$. We then apply dilation by -1 and then inversion. Then net result is to be in the situation of Case IV.

Case VI. $\mathrm{coi}(B) = \aleph_0$ and $\mathrm{cof}(A) = 1$. In this case we translate to assume $A = \{0\}$. We then apply inversion to arrive at Case IV.

To deal with Case IV, let $\overline{a}_n \in \mathfrak{R}$ be a strictly increasing sequence, lifting to a strictly increasing sequence $a_n \in O \subseteq {}^\rho\mathbb{R}$. We have $v(a_n) \geq 0$ for all n. Consider the strictly decreasing sequence $\{\overline{\rho}^{-1/n}\} \subseteq {}^\rho\mathbb{R}$. We have

$$v(\overline{\rho}^{-1/n}) = -1/n < 0 \leq v(a_n) = 0 \tag{2.11}$$

for all $n \in \mathbb{N}$. Because the valuation is compatible with the ordering, (2.11) implies that for all n,

$$a_n < \overline{\rho}^{-1/n}$$

Because ${}^\rho\mathbb{R}$ is semi-η_1, we can find an element $x \in {}^\rho\mathbb{R}$, such that for all n

$$a_n < x < \overline{\rho}^{-1/n}$$

Applying the valuation to this inequality, we have for all n:

$$v(\overline{\rho}^{-1/n}) = -1/n \leq v(x) \leq v(a_n) = 0$$

which says that $v(x) = 0$. So $x \in O$ and because the sequence a_n was strictly increasing, $x \neq a_n$ for any n. Therefore we have $\overline{a}_n < \overline{x}$ for all n. This proves Case IV and finishes the proof of the proposition. □

Assuming the Continuum Hypothesis gives the following.

Theorem 2.32. $\mathfrak{R}$ is isomorphic to ${}^*\mathbb{R}$.

Proof: We know that ${}^*\mathbb{R}$ and $\mathfrak{R}$ are both real closed η_1-fields of cardinality $\mathfrak{c}$. We can therefore apply the isomorphism theorem, Theorem 2.5. □

2.4.5 An isomorphism theorem

The following isomorphism is due to Pestov and Diarra [23, Theorem 1.8] [6, Corollaire de la Proposition 8]).

Theorem 2.33. ${}^\rho\mathbb{R}$ is isomorphic both as an ordered field and as a valuation field to the Hahn field $\mathfrak{R}((\mathbb{R}))$.

This theorem does not require the Continuum Hypothesis. If we assume the Continuum Hypothesis, as usual, we can combine Theorem 2.33 with Theorem 2.32, Corollary 2.19 and Proposition 2.10 to get the following result.

Theorem 2.34. For any choice of ultrafilter, and any choice of infinitesimals, the fields ${}^\rho\mathbb{R}$ are all isomorphic as valuation fields.

2.4.6 Some remarks

The field ${}^\rho\mathbb{R}$ was introduced by A. Robinson [25]. Luxemburg mentions the isomorphism (Theorem 2.34) and mentions an argument for the proof [21, p. 196]. Luxemburg's argument relies on a theorem of Kaplansky [14, Theorem 7], but this method seems flawed as it does not account for our requirement of the Continuum Hypothesis. Finally, Pestov gives Theorem 2.33. As Pestov states, the proof of this theorem is actually due to Diarra. It appears that neither Pestov nor Diarra investigated the properties of the residue field, and Theorem 2.34 is a new result.

2.5 Extending functions

2.5.1 Extending functions to ${}^*\mathbb{R}$

We mentioned briefly in Example 2.20 how to extend a real function to a nonstandard function. We state more details of this extension, especially for functions that are given by power series.

Suppose $D \subseteq \mathbb{R}$ is a set. Then, by ${}^*D$, we mean the subset of ${}^*\mathbb{R}$ formed by sequences in D. Suppose $f : D \to \mathbb{R}$ is a function. We define ${}^*f : {}^*D \to {}^*\mathbb{R}$ as follows. If $x = [x_i] \in {}^*D$ we define

$$ {}^*f([x_i]) = [f(x_i)] \tag{2.12} $$

Since $x_i \in D$ for ω-almost all i, ${}^*f$ is well-defined.

We now suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by a power series with a radius of convergence $r > 0$. Then, for $x \in \mathbb{R}$:

$$ f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{for all } |x| < r \tag{2.13} $$

As before, this extends to a nonstandard function. If $x = [x_i] \in {}^*\mathbb{R}$ and $|x| < r$, then $|x_i| < r$ for ω-almost all i, and

$$\begin{aligned} {}^*f([x_i]) &= [f(x_i)] \\ &= \left[\sum_{n=0}^{\infty} a_n x_i^n\right] \end{aligned}$$

If we take as a definition that infinite sums in ${}^*\mathbb{R}$ are defined point wise, then (2.13) also makes sense for $x \in {}^*\mathbb{R}$.

Example 2.35. The exponential series. For any $x \in \mathbb{R}$, we have

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Because this formula is valid for all $x \in \mathbb{R}$, this formula is also valid for all $x \in {}^*\mathbb{R}$. We can therefore write, for $x = [x_i] \in {}^*\mathbb{R}$,

$$\begin{aligned} e^x &= [e^{x_i}] = \left[\sum_{n=0}^{\infty} \frac{x_i^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Example 2.36. The logarithm series. For $x \in \mathbb{R}$, the series

$$\begin{aligned} \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \\ &= -\sum_{n=1}^{\infty} \frac{x^n}{n} \end{aligned}$$

converges for $|x| < 1$. Therefore, this series also makes sense for $|x| < 1$, $x \in {}^*\mathbb{R}$.

2.5.2 Extending functions to ${}^\rho\mathbb{R}$

Now consider extending functions to ${}^\rho\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then as in (2.12), one can define ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$. We would like to define ${}^\rho f : {}^\rho\mathbb{R} \to {}^\rho\mathbb{R}$. If $\overline{x} \in {}^\rho\mathbb{R}$ and $x \in M_0$ projects to $\overline{x}$, one would like to define:

$${}^\rho f(\overline{x}) = \Pi\left({}^*f(x)\right)$$

where $\Pi : M_0 \to {}^\rho\mathbb{R}$ is the projection. For this definition to make sense, we need ${}^*f(x) \in M_0$ for all $x \in M_0$ and ${}^*f(x) - {}^*f(y) \in M_1$ whenever $x - y \in M_1$. To illustrate that these conditions will not in general be satisfied, consider the following examples.

Example 2.37. 1. Consider $\exp : {}^*\mathbb{R} \to {}^*\mathbb{R}$. $\rho \in M_0$, but $\exp(\rho) \notin M_0$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \tag{2.14}$$

Notice that for all $x \in M_0$, $x \neq 0$, $f(x) = 1$. But, $f(0) = 1$, so this function will not extend to a function $f : {}^\rho\mathbb{R} \to {}^\rho\mathbb{R}$ as in (2.12)

Thus, real functions cannot in general be extended to ${}^\rho\mathbb{R}$ functions.

2.5.3 Extending matrix functions

Consider the exponential of a matrix. If A is an $n \times n$ matrix with entries in $\mathbb{R}$, we can define

$$\begin{aligned} \exp A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}A^k \end{aligned} \tag{2.15}$$

If $A = [A_i]$ is a matrix with entries in ${}^*\mathbb{R}$, represented by matrices A_i with entries in $\mathbb{R}$. Then, we define the matrix series point wise as in (2.13).

$$\begin{aligned} \exp A &= [\exp A_i] \\ &= \left[I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \right] \\ &= \left[\sum_{k=1}^{\infty} \frac{1}{k!}A_i^k \right] \end{aligned} \tag{2.16}$$

This means that we are free to use matrix exponentials with matrices with entries in ${}^*\mathbb{R}$.

2.6 Solutions to polynomials

The linear algebraic groups we will study will be defined over $\mathbb{R}$ (or $\mathbb{Z}$). See Appendix C for a brief introduction on linear algebraic groups. We will often be concerned with the ${}^*\mathbb{R}$ and the ${}^\rho\mathbb{R}$-points of these groups. Therefore we will need to investigate the ${}^*\mathbb{R}$ solutions to polynomials in $\mathbb{R}[T]$. As in Appendix C, we will assume that T represents several variables so that

$$\mathbb{R}[T] = \mathbb{R}[T_1, \ldots, T_n]$$

For this section, we will fix $f \in \mathbb{R}[T]$, and we define the following sets

$$\begin{aligned}
V &= \{x \in \mathbb{R}^n \mid f(x) = 0\} \\
{}^*V &= \{x \in {}^*\mathbb{R}^n \mid f(x) = 0\} \\
{}^\rho V &= \{x \in {}^\rho\mathbb{R}^n \mid f(x) = 0\} \\
[V] &= \{[x_i] \in {}^*\mathbb{R}^n \mid x_i \in \mathbb{R}^n, f(x_i) = 0 \text{ for all } i\} \\
\overline{V} &= \{\overline{x_i} \in {}^\rho\mathbb{R}^n \mid x_i \in \mathbb{R}^n, f(x_i) = 0 \text{ for all } i\}
\end{aligned}$$

The following relationships between these sets are mostly trivial.

$${}^*V = [V] \qquad \overline{V} \subseteq {}^\rho V$$

Similarly, we also have

Proposition 2.38. $\overline{V} = {}^\rho V$.

Proof: This takes a bit of work and relies on Appendix D. We will denote by Π, the projections:

$$\Pi : M_0 \to {}^\rho\mathbb{R} \qquad \Pi : M_0^n \to {}^\rho\mathbb{R}^n$$

We define

$$Z = \{x \in {}^*\mathbb{R}^n \mid f(x) \in M_1\} \qquad U = \{x \in {}^*\mathbb{R}^n \mid {}^*d(x, {}^*V) \in M_1\}$$

Notice that we have the following.

$$\begin{aligned}
\Pi^{-1}\left({}^\rho V\right) &= Z \cap M_0^n \\
\Pi^{-1}\left(\overline{V}\right) &= U \cap M_0^n
\end{aligned}$$

Our goal is to show that $\Pi^{-1}\left({}^\rho V\right) = \Pi^{-1}\left(\overline{V}\right)$.

Notice that because $V \subseteq \mathbb{R}^n$ is defined by $f(x) = 0$, V is a closed set of $\mathbb{R}^n$. We apply Lemma D.5 to this set, and we see that the following semi-algebraic functions

$$x \mapsto d(x, V) \qquad x \mapsto f(x)$$

have the same zero sets (namely V). We now apply Theorem D.7 to the functions $d(x, V)$ and $f(x)$. This yields integers $N_1, N_2 \in \mathbb{N}$ and continuous semi-algebraic functions $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$(d(x,V))^{N_1} = h_1(x) f(x) \tag{2.17}$$

$$(f(x))^{N_2} = h_2(x) d(x,V) \tag{2.18}$$

for all $x \in \mathbb{R}^n$. We now apply Proposition D.6 to the semi-algebraic functions h_1, h_2, giving us $c \in \mathbb{R}$, $p \in \mathbb{N}$ such that for $i = 1, 2$,

$$|h_i(x)| \leq c\left(1 + |x|^2\right)^p \tag{2.19}$$

Combining (2.17), (2.18) and (2.19) gives us

$$(d(x,V))^{N_1} \leq c\left(1 + |x|^2\right)^p |f(x)| \tag{2.20}$$

$$|f(x)|^{N_2} \leq c\left(1 + |x|^2\right)^p d(x,V) \tag{2.21}$$

for all $x \in \mathbb{R}^n$. We emphasize that (2.20) and (2.21) hold for all $x \in \mathbb{R}^n$. Therefore, (2.20) and (2.21) will also hold for any sequence of real numbers. This gives us the corresponding inequalities for $x \in {}^*\mathbb{R}^n$:

$$(d(x,{}^*V))^{N_1} \leq c\left(1 + |x|^2\right)^p |f(x)| \tag{2.22}$$

$$|f(x)|^{N_2} \leq c\left(1 + |x|^2\right)^p d(x,{}^*V) \tag{2.23}$$

for all $x \in {}^*\mathbb{R}^n$.

We now show that $Z \cap M_0^n = U \cap M_0^n$. The key point is that M_1 is an ideal of M_0. First, suppose $x \in Z \cap M_0^n$. Then, $f(x) \in M_1$. Since $|x| \in M_0$, $c(1 + |x|^2)^p$ is also in M_0 (this is because M_0 is a ring). This means that the right hand side of inequality (2.22) is in $M_0 \cdot M_1 \subseteq M_1$. Therefore the left hand side is in M_1 which immediately implies that $d(x, {}^*V) \in M_1$. This means that $x \in U \cap M_0^n$ and shows that $Z \cap M_0^n \subseteq U \cap M_0^n$.

The proof that $U \cap M_0^n \subseteq Z \cap M_0^n$ is more or less identical using (2.23) $\square$

Corollary 2.39. Let G be a linear algebraic group, defined over $\mathbb{R}$. Then the projection $\Pi : M_0^n \to {}^\rho\mathbb{R}^n$ maps $G(M_0)$ onto $G({}^\rho\mathbb{R})$:

$$\Pi\left(G(M_0)\right) = G({}^\rho\mathbb{R})$$

Proof: Notice that $G(M_0) \subseteq G({}^*\mathbb{R})$ are defined by a finite set of polynomials in $\mathbb{R}[T]$. If $\{f_1, \ldots, f_n\} \subseteq \mathbb{R}[T]$ are the polynomials, we can define

$$f = \sum f_i^2$$

Then f also defines the algebraic set G. We can now directly apply Proposition 2.38. $\square$

CHAPTER 3

ASYMPTOTIC CONES

3.1 The construction

The asymptotic cone is a modified geometric ultrapower construction. We start with a metric space (X, d) and a nonprincipal ultrafilter ω. We then form the ultrapower ${}^*X$ as in Section 2.3.2. We can use the distance function on X to get a nonstandard distance function ${}^*d : {}^*X \times {}^*X \to {}^*\mathbb{R}$. This nonstandard distance is given by applying d to two sequences in X. Notice that ${}^*d$ satisfies all conditions of being a distance function except it takes values in ${}^*\mathbb{R}$ instead of in $\mathbb{R}$.

We now fix a positive infinite number $\lambda \in {}^*\mathbb{R}$. We define, for $x, y \in {}^*X$:

$$d_\infty(x, y) = \lim_\omega \left(\frac{{}^*d(x, y)}{\lambda} \right)$$

Thus, for all $x, y \in {}^*X$, we have $d_\infty(x, y) \in [0, \infty]$. By fixing a base point $\star \in {}^*X$, we can pick out a component:

$$X_\infty = \{x \in {}^*X \mid d_\infty(x, \star) < \infty\} \tag{3.1}$$

X_∞ is a pseudo-metric space with pseudo-distance d_∞. After we identify elements with distance zero, we obtain the metric space that we call the asymptotic cone of X:

$$\operatorname{Cone}_\omega X = \frac{X_\infty}{\sim}$$

Notice that if we start with a metric space, the asymptotic cone depends on the ultrafilter ω, the infinite number λ and the base point $\star$. The basic question is how $\operatorname{Cone}_\omega X$ changes when these inputs change. It is well known that in some cases, changing these inputs yields asymptotic cones that are not isometric [29].

3.2 Basic properties of $\mathrm{Cone}_\omega X$

Here we outline some basic properties of the asymptotic cone. For the most part, these are easy to prove and can be found in [15] or [17].

Proposition 3.1.

1. $\mathrm{Cone}_\omega(X \times Y) = \mathrm{Cone}_\omega X \times \mathrm{Cone}_\omega Y$.
2. If X is a geodesic space then $\mathrm{Cone}_\omega X$ is also a geodesic space.
3. If X is a CAT(0) space, then $\mathrm{Cone}_\omega X$ is also a CAT(0) space.
4. If X is a CAT(κ) space for $\kappa < 0$, then $\mathrm{Cone}_\omega X$ is a CAT($-\infty$) space (a metric tree).
5. If X is homogeneous, then $\mathrm{Cone}_\omega X$ is homogeneous.

Proposition 3.2. An element $f \in {}^*(\mathrm{Isom}\, X)$ determines a element of the nonstandard isometry group $\mathrm{Isom}({}^*X)$. Such an f determines an isometry of $\mathrm{Cone}_\omega X$ if and only if $d_\infty(f(\star), \star) < \infty$.

Proof: Let $f \in {}^*(\mathrm{Isom}\, X)$. Then f can be represented by a sequence $f_i \in \mathrm{Isom}\, X$. If $[x_i], [y_i] \in {}^*X$ then:

$$ {}^*d(f([x_i]), f([y_i])) = [d(f_i(x_i), f_i(y_i))] = [d(x_i, y_i)] = {}^*d([x_i], [y_i]) $$

which shows that f is a nonstandard isometry of ${}^*X$.

Clearly if f determines an isometry of $\mathrm{Cone}_\omega X$ then $d_\infty(f(\star), \star) < \infty$. Suppose that $d_\infty(f(\star), \star) < \infty$. Let $[x_i] \in X_\infty$. Then

$$ \begin{aligned} d_\infty(f(x), \star) &\leq d_\infty(f(x), f(\star)) + d_\infty(f(\star), \star) \\ &= d_\infty(x, \star) + d_\infty(f(\star), \star) < \infty \end{aligned} $$

Therefore, f preserves X_∞ and must give an isometry of $\mathrm{Cone}_\omega X$. □

3.3 Dependence on the base point

Theorem 3.3. If X is a homogeneous space, then the asymptotic cone is independent of the choice of base point in ${}^*X$.

Proof: Let $[x_i]$ and $[y_i]$ be two different base points in ${}^*X$. Since X is homogeneous, for each i there exists some $g_i \in \operatorname{Isom} X$ such that $g_i \cdot x_i = y_i$. By Proposition 3.2, $[g_i]$ defines a nonstandard isometry of ${}^*X$. Therefore $[g_i]$ defines an isometry

$$[g_i] : \mathrm{Cone}_\omega(X, [x_i]) \to \mathrm{Cone}_\omega(X, [y_i])$$

□

3.4 Examples

3.4.1 Bounded spaces

Proposition 3.4. If X is a bounded space, then $\mathrm{Cone}_\omega X$ is a point.

Proof: We just compute distance. If $x = [x_i], y = [y_i] \in X_\infty$ then:

$$\begin{aligned} d_\infty(x, y) &= \lim_\omega \left(\frac{{}^*d(x_i, y_i)}{\lambda} \right) \\ &\leq \lim_\omega \left(\frac{\operatorname{diam} X}{\lambda} \right) = 0 \end{aligned}$$

□

3.4.2 Hyperbolic spaces

$\mathbb{H}^n$ has strictly negative curvature and is therefore a CAT(κ) space for some $\kappa < 0$. By Proposition 3.1, we see that $\mathrm{Cone}_\omega \mathbb{H}^n$ is a metric tree. In fact, it is not too hard to see the following properties [7]:

1. $\mathrm{Cone}_\omega \mathbb{H}^n$ is a metric tree with uncountable branching at every point.

2. $\mathrm{Cone}_\omega \mathbb{H}^n$ and $\mathrm{Cone}_\omega \mathbb{H}^m$ are isometric for all n, m.

3.4.3 Euclidean spaces

The next example is $X = \mathbb{R}$. We start with $\mathbb{R}$, ω and λ. Our first step in the construction of $\mathrm{Cone}_\omega \mathbb{R}$ gives us the nonstandard real numbers ${}^*\mathbb{R}$ and the nonstandard distance ${}^*d$ on ${}^*\mathbb{R}$. Notice that ${}^*d$ can be written as ${}^*d(x,y) = |x-y|$. We also have the (possibly) infinite real distance d_∞ on ${}^*\mathbb{R}$:

$$d_\infty(x,y) = \lim_\omega \left(\frac{{}^*d(x,y)}{\lambda} \right)$$

We fix a base point in ${}^*\mathbb{R}$. By Theorem 3.3, we are free to choose any base point. An obvious choice is the point $0 \in \mathbb{R} \subseteq {}^*\mathbb{R}$. To determine the space $\mathbb{R}_\infty \subseteq {}^*\mathbb{R}$ defined in (3.1), notice that for $x \in {}^*\mathbb{R}$, ${}^*d(x,0) = |x|$.

$$\begin{aligned} \mathbb{R}_\infty &= \{x \in {}^*\mathbb{R} \mid d_\infty(x,0) < \infty\} \\ &= \{x \in {}^*\mathbb{R} \mid |x| < n\lambda \text{ for some } n \in \mathbb{N}\} \end{aligned}$$

Define a map $\phi : \operatorname{Cone}_\omega \mathbb{R} \to \mathbb{R}$. Defined as $\phi(x) = \operatorname{sign}(x) d_\infty(x,0)$, where $\operatorname{sign}(x) \in \{-1,0,1\}$ is defined as usual. Given $x, y \in \operatorname{Cone}_\omega X$

$$\begin{aligned} d(\phi(x),\phi(y)) &= |\phi(x) - \phi(y)| \\ &= |\operatorname{sign}(x)d_\infty(x,0) - \operatorname{sign}(y)d_\infty(y,0)| \\ &= \lim_\omega \left(\frac{|x-y|}{\lambda} \right) \\ &= \lim_\omega \left(\frac{{}^*d(x,y)}{\lambda} \right) \\ &= d_\infty(x,y) \end{aligned}$$

which shows that ϕ is an isometry and $\operatorname{Cone}_\omega \mathbb{R} = \mathbb{R}$. By combining this result with Proposition 3.1 we get the following result.

Proposition 3.5. $\operatorname{Cone}_\omega \mathbb{R}^n = \mathbb{R}^n$.

Remark 3.6. As was briefly mentioned in the introduction, when the spaces $(X, \frac{1}{\lambda_i}d)$ converge in the sense of Gromov-Hausdorff, the asymptotic cone is equal to this limit space. This is the case when $X = \mathbb{R}$. The spaces $(\mathbb{R}, \frac{1}{\lambda_i}d)$ converge to $\mathbb{R}$, showing another way to prove Proposition 3.5.

CHAPTER 4

SYMMETRIC SPACES

4.1 Basic definitions and properties

The general reference for nearly everything to do with symmetric spaces is Helgason's book on symmetric spaces [13]. Another very good reference is Eberlein's book on nonpositively curved manifolds [9]. All the results in this chapter that are not referenced or proved can be found in one of these references. One of the main points of this chapter is the statement of Theorem 4.6 and the distance formula of Proposition 4.4.

Definition 4.1. A Riemannian manifold P is a *symmetric space* if for each $p \in P$, there is an involutive isometry with p as an isolated fixed point. This isometry is called the *reflection* or *symmetry* at p.

The easiest examples of symmetric spaces are Euclidean spaces and spheres. In these cases, the symmetries are the usual reflections through points. Given a symmetric space P, we can apply the de Rham decomposition and write P as a product of irreducible symmetric spaces:

$$P = M_0 \times M_1 \times \cdots \times M_k$$

Each of these manifolds will either be compact, a Euclidean space, or noncompact and not a Euclidean space. We will be interested in finding the asymptotic cone of the symmetric space. By Proposition 3.1, to find the asymptotic cone of a symmetric space, it suffices to consider each irreducible component of P. By Proposition 3.4 and Proposition 3.5, we already know everything about asymptotic cones of compact spaces and Euclidean spaces. Therefore, we will be concerned only with irreducible symmetric spaces of *noncompact type*, meaning the irreducible symmetric space is noncompact and not Euclidean space.

4.2 Symmetric spaces of noncompact type

Let P be a symmetric space of noncompact type. Let G be the connected component of the identity in the isometry group. Then G is a Lie group and acts transitively on P. We fix a point $p \in P$ and let $K \subseteq G$ be the stabilizer of p. K is a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie algebra of K.

Let $\sigma_p : P \to P$ be the reflection at p. Then $g \mapsto \sigma_p g \sigma_p$ defines an involution $\sigma : G \to G$. This also gives an involution $d\sigma : \mathfrak{g} \to \mathfrak{g}$, of the Lie algebra. These involutions have the fixed point sets

$$\begin{aligned} \mathrm{Fix}(\sigma) &= K \\ \mathrm{Fix}(d\sigma) &= \mathfrak{k} \end{aligned}$$

Since $d\sigma$ is an involution, we can find the complement of $\mathfrak{k}$ in $\mathfrak{g}$:

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}$$

We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We know that G acts on P transitively which gives us the identification $P = G/K$. On the Lie algebra level, the Lie algebra maps onto the tangent space $\mathfrak{g} \to T_pP$. This map has kernel $\mathfrak{k}$, and thus we see we can identify the tangent space T_pP with $\mathfrak{p}$.

In $\mathfrak{g}$ we have the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$$

Using the Killing form and the identification $T_pP = \mathfrak{p}$, we define a positive definite inner product on T_pP.

$$\langle X, Y \rangle_p = B(X, Y)$$

Using the transitive group action, we can move this inner product around the symmetric space to get a Riemannian metric. It is known that with this metric, symmetric spaces of noncompact type satisfy the CAT(0) inequality [16, Section 2.1]).

Definition 4.2. Let P be a metric space. A *k-flat* in P is a complete, totally geodesic submanifold that is isometric to $\mathbb{R}^k$.

The following theorem can be found in [9, Section 2.10].

Theorem 4.3.

1. Every flat $\mathcal{F} \subseteq P$ through the base point $p \in P$ is the orbit of the exponential of an abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$:

$$\mathcal{F} = \exp(\mathfrak{a}) \cdot p$$

2. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are k-flats in P and p_1 and p_2 are points in $\mathcal{F}_1$ and $\mathcal{F}_2$, then there exists $g \in G$ such that $g \cdot \mathcal{F}_1 = \mathcal{F}_2$ and $g \cdot p_1 = p_2$.

3. If γ is a maximal geodesic of P, then there exists at least one maximal flat containing γ.

4.3 The symmetric space $P(n, \mathbb{R})$

The canonical example of a symmetric space of noncompact type is the space $P(n, \mathbb{R})$. By definition, $P(n, \mathbb{R})$ is the set of positive definite, symmetric $n \times n$ matrices of determinant one with coefficients in $\mathbb{R}$. A natural base point in $P(n, \mathbb{R})$ is the matrix I. Our goal in this section is to better understand $P(n, \mathbb{R})$ and, in particular, find a distance formula.

$SL(n, \mathbb{R})$ acts on $P(n, \mathbb{R})$. If $g \in SL(n, \mathbb{R})$ and $A \in P(n, \mathbb{R})$, we define the action

$$g \cdot A = gAg^t$$

It is an elementary fact from linear algebra that this is a transitive action. The stabilizer of the base point $I \in P(n, \mathbb{R})$ is $SO(n)$. The Lie algebra of $SL(n, \mathbb{R})$ is $\mathfrak{sl}(n, \mathbb{R})$, the set of all trace zero matrices. The Lie algebra of $SO(n)$ is $\mathfrak{so}(n)$, the set of skew symmetric matrices with trace zero. Notice that the exponential map,

$\exp : \mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R})$, is given by matrix exponentiation. The symmetry at $I \in P(n, \mathbb{R})$ is given by

$$\sigma_I(A) = A^{-1}$$

It is clear that σ_I is involutive and has I as an isolated fixed point. Using σ_I we obtain involutions $\sigma : SL(n, \mathbb{R}) \to SL(n, \mathbb{R})$ and $d\sigma : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$ which are easily seen to be given by the formulas:

$$\begin{aligned} \sigma(g) &= \sigma_I g \sigma_I = (g^{-1})^t \\ d\sigma(X) &= -X^t \end{aligned}$$

This allows us to decompose the Lie algebra $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}(n, \mathbb{R})$, where

$$\begin{aligned} \mathfrak{so}(n) &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^t = -X, \operatorname{tr} X = 0\} \\ \mathfrak{p}(n, \mathbb{R}) &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^t = X, \operatorname{tr} X = 0\} \end{aligned}$$

Given $X \in \mathfrak{sl}(n, \mathbb{R})$, $\operatorname{ad} X : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$ is given by

$$\operatorname{ad} X(Y) = [X, Y] = XY - YX$$

One can compute the Killing form and see that $B(X, Y) = 2n \operatorname{tr}(XY)$. Since replacing B with $B(X, Y) = \operatorname{tr}(XY)$ will only change the metric by a fixed scale, we do this. The metric at $I \in P(n, \mathbb{R})$ is therefore:

$$\langle X, Y \rangle_I = \operatorname{tr}(XY)$$

To find the metric at another point in $P(n, \mathbb{R})$, we use the group action to move the inner product defined at I around the symmetric space. If $g \cdot p = I$ then the metric at p is given by

$$\langle dg(X), dg(Y) \rangle_p = \langle X, Y \rangle_I$$

This definition ensures that $SL(n, \mathbb{R})$ acts on $P(n, \mathbb{R})$ via isometries. It can also be shown that σ_I is an isometry of $P(n, \mathbb{R})$.

We now determine the geodesics of $P(n, \mathbb{R})$. It is well known that the orbits of one parameter subgroups in the isometry group, which are tangent to $P(n, \mathbb{R})$, give geodesics through the base point in a symmetric space. One parameter subgroups in $SL(n, \mathbb{R})$ are seen in the Lie algebra as lines. Thus, the geodesics through I are of the form $\gamma(t) = \exp(tX)$ for some $X \in \mathfrak{p}(n, \mathbb{R})$.

To find a formula for length of such a geodesic, let $X \in \mathfrak{p}(n, \mathbb{R})$. Let $\gamma(t) = \exp(tX)$ be the corresponding geodesic. Notice that $\gamma'(0) = X$. To find length of a curve we use the metric on X and integrate:

$$d(I, \exp(X)) = \text{length}(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)} \, dt \tag{4.1}$$

To compute this integral, we use transvections. In particular, for each t, consider the isometry g_t, defined by

$$g_t = \sigma_I \circ \sigma_{\gamma(t/2)}$$

One can verify that

$$g_t(\gamma(u)) = \gamma(u - t)$$

and therefore, the family $\{g_t\}$ translate along the geodesic γ. Therefore,

$$\begin{aligned}
dg_t(\gamma'(t)) &= \gamma'(0) \\
\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)} &= \langle dg_t(\gamma'(t)), dg_t(\gamma'(t)) \rangle_{g_t(\gamma(t))} \\
&= \langle \gamma'(0), \gamma'(0) \rangle_{\gamma(0)} \\
&= \langle X, X \rangle_I = \text{tr}(X^2)
\end{aligned}$$

Thus, the distance in (4.1) becomes

$$\begin{aligned}
d(I, \exp(X)) &= \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)} \, dt \\
&= \int_0^1 \langle X, X \rangle_I \, dt \\
&= \langle X, X \rangle_I = \text{tr}(X^2)
\end{aligned}$$

To actually compute distances in $P(n, \mathbb{R})$, we notice that $\log : P(n, \mathbb{R}) \to \mathfrak{p}(n, \mathbb{R})$ is the inverse of the exponential map and is well-defined because $\exp : \mathfrak{p}(n, \mathbb{R}) \to P(n, \mathbb{R})$ is bijective. Therefore

$$d(A, I) = \sqrt{\text{tr}[(\log A)^2]} \tag{4.2}$$

Proposition 4.4. Given $A, B \in P(n, \mathbb{R})$, $d(A, B) = \sqrt{\mathrm{tr}[(\log AB^{-1})^2]}$.

Proof: Using the group action, we find $g \in SL(n, \mathbb{R})$ such that $g \cdot B = I$, or $gBg^t = I$. Therefore by (4.2),

$$\begin{aligned} d(A, B) &= d(g \cdot A, g \cdot B) = d(gAg^t, gBg^t) = d(gAg^t, I) \\ &= \sqrt{\mathrm{tr}((\log gAg^t)^2)} \end{aligned}$$

Define $X = \log(gAg^t)$. Then

$$d(A, B) = \sqrt{\mathrm{tr}(X^2)} \tag{4.3}$$

Notice that since $X = \log(gAg^t)$, $\exp(X) = gAg^t$. We rearrange this expression.

$$\begin{aligned} \exp(X) &= gAg^t = gAB^{-1}Bg^t = gAB^{-1}g^{-1}g^{-1t}g^t \\ &= gAB^{-1}g^{-1} \end{aligned} \tag{4.4}$$

Rearranging (4.4) gives

$$AB^{-1} = g^{-1}\exp(X)g = \exp(g^{-1}Xg)$$

Therefore,

$$\begin{aligned} g^{-1}Xg &= \log(AB^{-1}) \\ X &= g\log(AB^{-1})g^{-1} \end{aligned} \tag{4.5}$$

We now put equation (4.5) this into the formula (4.3):

$$\begin{aligned} d(A, B) &= \sqrt{\mathrm{tr}((X)^2)} \\ &= \sqrt{\mathrm{tr}(g(\log(AB^{-1}))g^{-1})^2} \\ &= \sqrt{\mathrm{tr}(g(\log(AB^{-1}))^2 g^{-1})} \\ &= \sqrt{\mathrm{tr}(\log(AB^{-1}))^2} \end{aligned}$$

The last equality is because trace is invariant under conjugation. □

Notice that in $\mathfrak{sl}(n, \mathbb{R})$ a maximal abelian subspace is the set of diagonal trace zero matrices. Theorem 4.3 implies the set of diagonal matrices in $P(n, \mathbb{R})$ is a maximal flat. Because $SL(n, \mathbb{R})$ must act transitively on maximal flats (again, Theorem 4.3), we see that any element of $P(n, \mathbb{R})$ can be diagonalized using an element of $SL(n, \mathbb{R})$. This is a well known result of elementary linear algebra.

Proposition 4.5. If $A \in P(n, \mathbb{R})$, then there exists some $g \in SL(n, \mathbb{R})$ such that gAg^t is diagonal.

4.4 The embedding

Let P be an irreducible symmetric space of noncompact type and let $p \in P$ be a point. Let G be the connected component of the isometry group of P. Let $K \subseteq G$ be the stabilizer of $p \in P$.

We have the following theorem that can be found in [8, p. 134].

Theorem 4.6. There is some n such that there is a diffeomorphism $P \to P(n, \mathbb{R})$ onto a complete totally geodesic submanifold of $P(n, \mathbb{R})$. It is possible to rescale the metric on P so that this map is an isometry. The group G embeds in $SL(n, \mathbb{R})$ and the group K embeds in $SO(n)$, and this group embedding respects the group action.

Thus, we can assume that every irreducible symmetric space of noncompact type is a submanifold of $P(n, \mathbb{R})$. To compute distances in P, it is enough to compute distances in $P(n, \mathbb{R})$. We also have the following.

Proposition 4.7. In the embedding of Theorem 4.6, we can assume that every element $A \in P \subseteq P(n, \mathbb{R})$ is diagonalizable by an element of K.

Proof: Suppose $P \subseteq P(n, \mathbb{R})$ is embedded and $\mathcal{F} \subseteq P$ is a maximal flat. Let $\mathcal{F}$ be a maximal flat in P and let $p \in \mathcal{F}$ be an arbitrary point. Let $\mathcal{D}$ be the flat of diagonal matrices in $P(n, \mathbb{R})$. Then, by Theorem 4.3, there exists some $g \in SL(n, \mathbb{R})$ such that

$$g \cdot \mathcal{F} \subseteq \mathcal{D} \qquad g \cdot p = I$$

By using this element g to modify the embedding, we can assume that there is a maximal flat of P contained in $\mathcal{D}$. Let $\mathcal{D}' \subseteq P$ be a maximal flat such that $\mathcal{D}' \subseteq \mathcal{D}$.

Now let $A \in P$ be arbitrary. Let γ be a geodesic connecting I and A. By Theorem 4.3, there is a group element $g \in G$ such that

$$g \cdot \gamma \in \mathcal{D}' \qquad g \cdot I = I$$

Since $g \cdot I = I$, we actually have $g \in K$, the compact subgroup. And, since $g \cdot A \in \mathcal{D}'$, $g \cdot A$ is diagonal. $\square$

CHAPTER 5

ASYMPTOTIC CONES OF SYMMETRIC SPACES

We now construct the asymptotic cone of an irreducible symmetric space of noncompact type P, and prove the main theorem regarding this asymptotic cone. We will assume that $P \subseteq P(n, \mathbb{R})$ and that this embedding satisfies Theorem 4.6 and Proposition 4.7. We take the base point to be $I \in P(n, \mathbb{R})$. We also fix a nonprincipal ultrafilter ω and an infinite number, $\lambda \in {}^*\mathbb{R}$. We let $\rho = e^{-\lambda}$, a positive infinitesimal in ${}^*\mathbb{R}$. Recall that this leads to the definitions of M_0 (2.5), M_1 (2.6) and the field ${}^\rho\mathbb{R}$ (2.7).

5.1 P_∞

Recall that P_∞ is defined in (3.1) as

$$P_\infty = \{x \in {}^*P \mid d_\infty(x, \star) < \infty\}$$

where

$$d_\infty(x, y) = \lim_\omega \left(\frac{{}^*d(x, y)}{\lambda} \right)$$

Also note, as mentioned in Example D.3, that for any ordered field K, $P(n, K)$ is a semi-algebraic set. We can consider the semi-algebraic sets $P(n, {}^*\mathbb{R})$ and $P(n, {}^\rho\mathbb{R})$. Notice that because $P \subseteq P(n, \mathbb{R})$, ${}^*P \subseteq {}^*(P(n, \mathbb{R}))$. The ultrafilter actually gives equality:

$${}^*(P(n, \mathbb{R})) = P(n, {}^*\mathbb{R})$$

Proposition 5.1.

$$P_\infty = {}^*P \cap P(n, M_0)$$

Proof: Take $A \in {}^*P$. Let $\{\alpha_1, \dots, \alpha_n\}$ be the eigenvalues of A. Because $\exp : \mathfrak{p} \to P$ is onto, there exists some $X \in {}^*\mathfrak{p}$ such that $\exp X = A$. By Lemma A.1, the eigenvalues of X are $\{\log \alpha_1, \dots, \log \alpha_n\}$. Similarly, the eigenvalues of X^2 are $\{(\log \alpha_1)^2, \dots, (\log \alpha_n)^2\}$. Because the trace of a matrix is the sum of its eigenvalues and by Proposition 4.4,

$$\begin{aligned}
d_\infty(A, I) < \infty \quad &\iff \quad \lim_\omega \left(\frac{{}^*d(A, I)}{\lambda} \right) = \lim_\omega \left(\frac{\sqrt{\operatorname{tr}((\log A)^2)}}{\lambda} \right) < \infty \\
&\iff \quad \exists C \in \mathbb{R}, \quad \operatorname{tr}((\log A)^2) < (C\lambda)^2 \\
&\iff \quad \exists C \in \mathbb{R}, \quad \sum (\log \alpha_i)^2 < (C\lambda)^2 \\
&\iff \quad \exists C \in \mathbb{R}, \forall i \quad (\log \alpha_i)^2 < (C\lambda)^2 \\
&\iff \quad \exists C \in \mathbb{R}, \forall i \quad -C\lambda < \log \alpha_i < C\lambda \\
&\iff \quad \exists C \in \mathbb{R}, \forall i \quad \rho^C = e^{-C\lambda} < \alpha_i < e^{C\lambda} = \rho^{-C} \qquad (5.1)
\end{aligned}$$

This shows that A is in P_∞ if and only if the eigenvalues of A are in $M_0 - M_1$. Lemma A.2 tells us that if $\alpha_i \in M_0 - M_1$ for all i, then the matrix entries of A satisfy $|A_{ij}| < n\rho^{-C} < \rho^{-(C+1)}$. Thus, if $A \in P_\infty$, then A has entries in M_0. Because A must be symmetric, positive definite and $\det A = 1$, $A \in P(n, M_0)$.

Similarly, if $A \in {}^*P \cap P(n, M_0)$, then all entries of A satisfy $|A_{ij}| < \rho^{-C}$ for some $C \in \mathbb{R}$. Therefore, since $A \in SL(n, {}^*\mathbb{R})$, Lemma A.2 implies that the eigenvalues $\{\alpha_1, \dots, \alpha_n\}$, of A must all satisfy

$$(n^2\rho^{-C})^{-(n-1)} < |\alpha_i| < n^2\rho^{-C}$$

Thus, there is some $C' \in \mathbb{R}$ (we can take $C' = nC$) so that all eigenvalues of A satisfy

$$\rho^{C'} < |\alpha_i| < \rho^{-C'}$$

In other words, all the eigenvalues of A are in $M_0 - M_1$. Therefore by (5.1), $A \in P_\infty$. $\square$

5.2 The group action

The action $G \curvearrowright P$ is a transitive isometric action. Therefore, the action $G^{\mathbb{N}} \curvearrowright P^{\mathbb{N}}$ defined point wise, must also be transitive. By applying the ultrafilter, we get a transitive action ${}^*G \curvearrowright {}^*P$. This "nonstandard" action is by nonstandard isometries. We want to determine the subgroup of ${}^*G$ that stabilizes the subset $P_\infty \subseteq {}^*P$.

Proposition 5.2. The stabilizer of P_∞ in ${}^*G$ is the group $G(M_0)$.

Proof: By Proposition 3.2, g stabilizes P_∞ if and only if $g \cdot I \in P_\infty$.

Assume first that $g \in {}^*G$ stabilizes P_∞. Then $g \cdot I = gg^t \in P_\infty$. Proposition 5.1 implies $gg^t \in SL(n, M_0)$. Therefore, for all i, j and for some $C \in \mathbb{R}$, $|(gg^t)_{ij}| < \rho^{-C}$. In particular, this is true when $i = j$:

$$\rho^{-C} > |(gg^t)_{ii}| = \left| \sum_{k=1}^{n} g_{ik} g_{ik} \right| = \sum_{k=1}^{n} (g_{ik})^2 \geq |g_{ik}|^2$$

Therefore, $g_{ik} \in M_0$ for all i, k and $g \in G(M_0)$.

Now assume that $g \in G(M_0)$. Then $g_{ij} \in M_0$ for all i, j and

$$(g \cdot I)_{ij} = (gg^t)_{ij} = \sum_{k=1}^{n} g_{ik} g_{jk} \tag{5.2}$$

Because M_0 is a ring and each $g_{ij} \in M_0$, the sum (5.2) must be in M_0. Thus, by Proposition 5.1, $g \cdot I = gg^t \in P_\infty$. □

5.3 Upgrading to ${}^\rho\mathbb{R}$

The previous section says that each $A \in P_\infty$ is a $n \times n$ symmetric matrix with coefficients in M_0 (Proposition 5.1). We can use the quotient map $\Pi : M_0 \to {}^\rho\mathbb{R}$ to induce a quotient $\Pi : P(n, M_0) \to P(n, {}^\rho\mathbb{R})$. Let $P({}^\rho\mathbb{R})$ be the image of P_∞ under this quotient map.

Proposition 5.3. The quotient map $P_\infty \to \mathrm{Cone}_\omega P$, obtained by identifying distance zero elements, factors through $P({}^\rho\mathbb{R})$

$$\begin{array}{ccc} P_\infty & \xrightarrow{\Pi} & P({}^\rho\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{Cone}_\omega P & \xrightarrow{=} & \mathrm{Cone}_\omega P \end{array}$$

Proof: We need to show that if $A, B \in P_\infty$ are such that $A - B$ has entries in M_1, then $d(A, B) = 0$ (where this is the pseudo-distance on $P(M_0)$). Suppose $A, B \in P_\infty$ and $A - B$ has entries in M_1. We first simplify the problem using the group action $G(M_0) \curvearrowright P_\infty$. There exists some $g \in G(M_0)$ such that $g \cdot B = I$. Then

$$(g \cdot A) - (g \cdot B) = gAg^t - gBg^t = g(A - B)g^t$$

Now, recall that M_1 is an ideal of M_0. Therefore, $A - B$ has entries in M_1 if and only if $g(A - B)g^t$ has entries in M_1. Using the transitive action $G(M_0) \curvearrowright P_\infty$, we translate the pair (A, B) to (A', I) such that $A' - I$ has entries in M_1. This translation is an isometry so $d(A, B) = d(A', I)$. We want to show that $d(A', I) = 0$. Next, using Proposition 4.7, we diagonalize A' using an element $k \in {}^*K$.

In other words, we need to show that if D is a diagonal matrix such that $D - I$ has entries in M_1 then $d(D, I) = 0$. Assume that $D - I$ has entries in M_1 and D is diagonal. This means that D is of the form $D = \mathrm{diag}(1 + \alpha_1, \ldots, 1 + \alpha_n)$, where $\alpha_i \in M_1$. We now compute distance:

$$\begin{aligned} d(D, I) &= \lim_\omega \frac{\sqrt{\mathrm{tr}(\log A)^2}}{\lambda} = \lim_\omega \sqrt{\sum_{k=1}^n \frac{(\log(1 + \alpha_k))^2}{(-\lambda)^2}} \\ &= \lim_\omega \sqrt{\sum_{k=1}^n \left(\frac{\log(1 + \alpha_k)}{\log \rho}\right)^2} = \lim_\omega \sqrt{\sum_{k=1}^n \left(\log_\rho(1 + \alpha_k)\right)^2} \\ &= \sqrt{\sum_{k=1}^n \left(\lim_\omega \log_\rho(1 + \alpha_k)\right)^2} \end{aligned} \tag{5.3}$$

Since each $\alpha_i \in M_1$, we can apply Lemma 2.22 and see that

$$\lim_\omega \left(\log_\rho(1 + \alpha_k)\right) = 0$$

for all k. Therefore $d(D, I) = 0$. □

The proof of Proposition 5.3 gives formula (5.3) for distance from the base point. This computation is valid in $P({}^\rho\mathbb{R})$. If we follow the computation through, we see that if $\{\overline{\alpha}_1 \ldots, \overline{\alpha}_n\}$ are the eigenvalues for $A \in P({}^\rho\mathbb{R})$, then

$$\begin{aligned} d(A, I) &= \sqrt{\sum_{i=1}^{n} \left(\lim_{\omega} \log_\rho(\alpha_i)\right)^2} \\ &= \sqrt{\sum \left(v(\overline{\alpha}_i)\right)^2} \end{aligned}$$

Corollary 5.4. If $A \in P({}^\rho\mathbb{R})$ has eigenvalues $\{\alpha_1, \ldots, \alpha_n\}$ then

$$d(A, I) = \sqrt{\sum \left(v(\alpha_i)\right)^2}$$

Corollary 5.5. $A \in P({}^\rho\mathbb{R})$ is distance zero to I if and only if all the eigenvalues of A have zero valuation.

Recall that ${}^*G \curvearrowright {}^*P$ by nonstandard isometries and therefore $G(M_0) \curvearrowright P_\infty$ is a nonstandard isometric action as well. Proposition 3.2 says that this action descends to an action $G(M_0) \curvearrowright \mathrm{Cone}_\omega P$.

Lemma 5.6. The action $G(M_0) \curvearrowright P_\infty$ induces an action $G({}^\rho\mathbb{R}) \curvearrowright P({}^\rho\mathbb{R})$. This action preserves the pseudo-distance on $P({}^\rho\mathbb{R})$.

Proof: We first check that this action is well-defined. Let $\overline{g} \in G({}^\rho\mathbb{R})$. By Corollary 2.39, $\overline{g}$ is in the image of the projection $G(M_0) \to G({}^\rho\mathbb{R})$. Suppose $g_1, g_2 \in G(M_0)$ both project to $\overline{g}$. To show that the action is well-defined, we have to show that if $\overline{A} \in P({}^\rho\mathbb{R})$ with $A \in P_\infty$, then

$$\Pi(g_1 \cdot A) = \Pi(g_2 \cdot A)$$

where $\Pi : P(n, M_0) \to P(n, {}^\rho\mathbb{R})$ is the projection. Since g_1 and g_2 both project to g, $g_1 - g_2$ has entries in M_1. Let $h = g_1 - g_2$. Since M_1 is an ideal of M_0, any matrix with entries in M_0, when multiplied by h, will have entries in M_1. This observations yields

$$\begin{aligned} (g_1 \cdot A) - (g_2 \cdot A) &= g_1 A g_1^t - g_2 A g_2^t \\ &= g_1 A g_1^t - (g_1 - h) A (g_1 - h)^t \\ &= g_1 A h^t + h A g_1^t + h A h^t \end{aligned} \tag{5.4}$$

where every term in the sum (5.4) must have entries in M_1. Therefore $g_1 \cdot A$ and $g_2 \cdot A$ must be equal in $P({}^\rho\mathbb{R})$ and the action is well-defined.

To see that the pseudo-distance is preserved, one only needs to see that the action $G(M_0) \curvearrowright P_\infty$ preserves the pseudo-distance and that the pseudo-distance in $P({}^\rho\mathbb{R})$ is actually computed in P_∞. $\square$

Thus we get an action $G({}^\rho\mathbb{R}) \curvearrowright P({}^\rho\mathbb{R})$ which induces a transitive action of $G({}^\rho\mathbb{R})$ on $\mathrm{Cone}_\omega P$.

Proposition 5.7. Every matrix in $P({}^\rho\mathbb{R})$ can be diagonalized by an element of $K({}^\rho\mathbb{R}) \subseteq G({}^\rho\mathbb{R})$.

Proof: Notice that if $g \in K$, then every matrix entry of g satisfies $|g_{ij}| \leq 1$ and $g \in G(M_0)$. Therefore, ${}^*K = K({}^*\mathbb{R}) = K(M_0)$.

If $\overline{A} \in P({}^\rho\mathbb{R})$ and $A \in P_\infty$ are such that $\Pi(A) = \overline{A}$, we apply Proposition 4.7. This gives us $k \in K({}^*\mathbb{R}) = K(M_0)$ such that $k \cdot A$ is diagonal. This clearly implies that $\overline{k} \cdot \overline{A}$ is diagonal in $P({}^\rho\mathbb{R})$. $\square$

5.4 The stabilizer of a point

5.4.1 Matrix valuations

Definition 5.8. If g is a matrix with entries in ${}^\rho\mathbb{R}$, we define the *matrix valuation of g* as:

$$v(g) = \min_{i,j}\{v(g_{ij})\}$$

Lemma 5.9. Let A, B be matrices with entries in ${}^\rho\mathbb{R}$. Then $v(AB) \geq v(A) + v(B)$

Proof:

$$\begin{aligned} v(AB) &= \min_{ij}\{v((AB)_{ij})\} = \min_{ij}\left\{v\left(\left|\sum_k A_{ik}B_{kj}\right|\right)\right\} \\ &\geq \min_{ijk}\{v(A_{ik}) + v(B_{kj})\} \geq \min_{ik}\{v(A_{ik})\} + \min_{jk}\{v(B_{kj})\} \\ &= v(A) + v(B) \end{aligned}$$

$\square$

Lemma 5.10. If $g \in O(n, {}^\rho\mathbb{R})$ then $v(g) = 0$. If $g, h \in GL(n, {}^\rho\mathbb{R})$ are such that $gh^{-1} \in O(n, {}^\rho\mathbb{R})$ then $v(g) = v(h)$.

Proof: $gg^t = I$ implies that the entries of g satisfy $|g_{ij}| \leq 1$. Taking valuations gives $v(g) \geq v(1) = 0$.

If $gh^{-1} \in O(n, {}^{\rho}\mathbb{R})$, then $v(gh^{-1}) \geq 0$. We now apply Lemma 5.9:

$$v(g) \;=\; v(gh^{-1}h) \geq v(gh^{-1}) + v(h) \geq v(h)$$

By symmetry, $v(g) = v(h)$. Notice that $v(I) = 0$. Therefore, for all $g \in O(n, {}^{\rho}\mathbb{R})$, $v(g) = v(I) = 0$. □

Lemma 5.11. If $g \in O(n, {}^{\rho}\mathbb{R})$ and $A \in GL(n, {}^{\rho}\mathbb{R})$, then $v(gAg^t) = v(A)$.

Proof: Applying Lemma 5.9:

$$v(gAg^t) \geq v(g) + v(A) + v(g^t) = v(A)$$

Similarly, since $A = g^t(gAg^t)g$,

$$v(A) = v(g^t(gAg^t)g) \geq v(g^t) + v(gAg^t) + v(g) = v(gAg^t)$$

and therefore $v(gAg^t) = v(A)$. □

Lemma 5.12. If $A \in P({}^{\rho}\mathbb{R})$ then

$$(n-1)v(A) \leq v(A^{-1}) \leq v(A)/(n-1)$$

Proof: By Proposition 5.7, we first diagonalize A with $g \in K({}^{\rho}\mathbb{R})$ so $gAg^t = D$ is diagonal. Lemma 5.11 implies that $v(A) = v(D)$.

$$\begin{aligned} D &= \operatorname{diag}(\alpha_1, \ldots, \alpha_n) \\ D^{-1} &= \operatorname{diag}(1/\alpha_1, \ldots, 1/\alpha_n) \end{aligned}$$

Computing the valuation of A and A^{-1}:

$$\begin{aligned} v(A) &= v(D) = \min\{v(\alpha_i)\} \\ v(A^{-1}) &= v(D^{-1}) = \min\{v(1/\alpha_i)\} = \min\{-v(\alpha_i)\} = -\max\{v(\alpha_i)\} \end{aligned}$$

Since the determinant of a matrix is the product of its eigenvalues, $\det A = 1$ implies $\prod \alpha_i = 1$. Taking the valuation of $\prod \alpha_i = 1$ gives $\sum v(\alpha_i) = 0$. Let

$m = \min\{v(\alpha_i)\}$ and $M = \max\{v(\alpha_i\}$. Notice $v(A) = m$ and $v(A^{-1}) = -M$. We now apply Lemma B.1 which implies

$$(n-1)m \leq -M \leq m/(n-1)$$

and proves the lemma. $\square$

Proposition 5.13. Let $A \in P({}^\rho\mathbb{R})$ and let $\{\alpha_1, \ldots, \alpha_n\}$ be the eigenvalues of A. Then the following are true (where a condition in all the eigenvalues is written as a condition in just α)

1. $$v(\alpha) \geq m \implies \begin{cases} m \leq 0 \\ m \leq v(\alpha) \leq -nm \\ v(A) \geq m \\ d(A, I) \leq -n^{3/2}m \end{cases}$$

2. $$v(\alpha) \leq M \implies \begin{cases} M \geq 0 \\ -nM < v(\alpha) \leq M \\ v(A) \geq -nM \\ d(A, I) \leq n^{3/2}M \end{cases}$$

3. $$v(A) \geq m \implies \begin{cases} m \leq 0 \\ m \leq v(\alpha) \leq -nm \\ d(A, I) \leq -n^{3/2}m \end{cases}$$

4. $$d(A, I) \leq d \implies \begin{cases} -d \leq v(\alpha) \leq d \\ v(A) \geq -d \end{cases}$$

Proof: This is an application of Lemma A.2 and Lemma B.1. Since $P = \exp(\mathfrak{p})$, Lemma A.1 implies that all eigenvalues of A are positive. Also by our assumptions, A is diagonalizable by an orthogonal matrix in $SO(n, {}^\rho\mathbb{R})$, and we can therefore apply Lemma A.2.

Since $\det A = 1$, $\prod \alpha_i = 1$. Taking the valuation of the equation $\prod \alpha_i = 1$ gives

$$0 = v(1) = v(\Pi\alpha_i) = \sum v(\alpha_i)$$

Recall that by Corollary 5.4,

$$d(A, I) = \sqrt{\sum v(\alpha_i)^2}$$

We are now set up to use Lemma B.1 with the following definitions:

$$d = d(A, I) \qquad m = \min\{v(\alpha_i)\} \qquad M = \max\{v(\alpha_i)\}$$

1. Since $\sum v(\alpha_i) = 0$, the lower bound on $v(\alpha)$ must be nonpositive. Therefore $m \leq 0$. We apply Lemma B.1 and see $v(\alpha) \leq -(n-1)m < -nm$. $v(\alpha) \geq m$ gives $\alpha < \overline{\rho}^{(m-\varepsilon)}$ for all $\varepsilon > 0, \varepsilon \in \mathbb{R}$. We now apply Lemma A.2 and see $|A_{ij}| < n\overline{\rho}^{(m-\varepsilon)}$ for all i, j. Taking valuations gives, for all $\varepsilon > 0$,

   $$v(A_{ij}) \geq v(n\overline{\rho}^{(m-\varepsilon)}) = m - \varepsilon$$

   and therefore $v(A_{ij}) \geq m$.

   Again, applying Lemma B.1,

   $$d(A, I) \leq -n^{3/2}m$$

2. Since $\det A = 1$, we have as before $\sum v(\alpha_i) = 0$ and therefore $M \geq 0$. We now apply Lemma B.1 and immediately see that $v(\alpha) \geq -(n-1)M$ and therefore $v(\alpha) > -nM$. This gives $|\alpha| \leq \overline{\rho}^{(-nM-\varepsilon)}$ for all $\varepsilon > 0$. We now apply Lemma A.2 and get $|A_{ij}| \leq n\overline{\rho}^{(-nM-\varepsilon)}$. Taking valuations gives $v(A_{ij}) \geq -nM - \varepsilon$ for all $\varepsilon > 0$. Therefore, $v(A_{ij}) \geq -nM$.

   To compute $d(A, I)$ we apply Lemma B.1:

   $$d(A, I) \leq n^{3/2}M$$

3. If $v(A) \geq m$, then every entry of A must satisfy $v(A_{ij}) \geq m$. This implies that $A_{ij} < \overline{\rho}^{(m-\varepsilon)}$ for all $\varepsilon > 0$. We now apply Lemma A.2 and see that all

eigenvalues satisfy $|\alpha| \leq n^2 \overline{\rho}^{(m-\varepsilon)}$. Taking valuations gives $v(\alpha) \geq m - \varepsilon$ for all $\varepsilon > 0$. Therefore we have $v(\alpha) \geq m$. We can now apply part 1 of this lemma and get the desired results.

4. This is easy because

$$(d(A, I))^2 = \sum v(\alpha_i)^2 = d^2$$

Therefore, for all eigenvalues, $-d \leq v(\alpha) \leq d$. We can apply part 1 of this lemma and see that we must have $v(A) \geq -d$.

□

5.4.2 The stabilizer

We now use the notion of matrix valuation to compute the stabilizer group of our base point in $G({}^\rho\mathbb{R})$. To begin, we have the following immediate consequence of Proposition 5.13.

Corollary 5.14. Let $A \in P({}^\rho\mathbb{R})$. Then $d(A, I) = 0$ if and only if $v(A) \geq 0$.

Proposition 5.15. For any $g \in G({}^\rho\mathbb{R})$, $d(gg^t, I) = 0$ if and only if $v(g) = 0$.

Proof: Let $A = gg^t$. Assume that $v(g) = 0$. In this case we have $v(gg^t) \geq v(g) + v(g^t) = 0$ and we apply Corollary 5.14 which implies $d(A, I) = 0$.

Suppose now that $d(A, I) = 0$. We diagonalize A using an element $a \in K({}^\rho\mathbb{R})$. So $aAa^t = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. Since $0 = d(A, I)^2 = \sum v(\alpha_i)^2$ we have $v(\alpha_i) = 0$ for each i. We define $b = \mathrm{diag}(1/\sqrt{\alpha_1}, \ldots, 1/\sqrt{\alpha_n})$. Notice that $v(b) = 0$ and $baAa^tb^t = I$. Therefore, if $A = g \cdot I$, we have $(bag)(bag)^t = I$ which implies $bag \in SO(n, {}^\rho\mathbb{R})$. Lemma 5.10 implies that $v(g) = v((ba)^{-1})$. Since $a \in SO(n, {}^\rho\mathbb{R})$, we have $v((ba)^{-1}) = v(b^{-1}) = 0$. Thus $v(g) = 0$. □

Recall that $O \subseteq {}^\rho\mathbb{R}$ is the valuation ring (2.1) and $G(O)$ is the set of O-points of G. We have the following immediate consequence of Proposition 5.15.

Corollary 5.16. Under the action $G({}^\rho\mathbb{R}) \curvearrowright \mathrm{Cone}_\omega(P)$,

$$\mathrm{Stab}(I, G({}^\rho\mathbb{R})) = G(O) = \{g \in G({}^\rho\mathbb{R}) \mid v(g) = 0\}$$

We have identified the asymptotic cone as the homogeneous space:

$$\mathrm{Cone}_\omega P \cong \frac{G({}^{\rho}\mathbb{R})}{G(O)} \tag{5.5}$$

5.5 The isometry

Consider two different ultrafilters ω_1, ω_2, leading to two (isomorphic) nonstandard real fields ${}^*\mathbb{R}_1$ and ${}^*\mathbb{R}_2$. We then pick infinite numbers $\lambda_1 \in {}^*\mathbb{R}_1$ and $\lambda_2 \in {}^*\mathbb{R}_2$ determining infinitesimals $\rho_1 = e^{-\lambda_1}$ and $\rho_2 = e^{-\lambda_2}$. This leads to two (isomorphic) valuation fields ${}^{\rho_1}\mathbb{R}$ and ${}^{\rho_2}\mathbb{R}$. These valuations fields have valuation rings $O_1 \subseteq {}^{\rho_1}\mathbb{R}$ and $O_2 \subseteq {}^{\rho_2}\mathbb{R}$. By Theorem 2.34, there is a valuation preserving isomorphism

$$\phi : {}^{\rho_1}\mathbb{R} \to {}^{\rho_2}\mathbb{R}$$

Since ϕ is valuation preserving, $\phi(O_1) = O_2$.

Next, we introduce our irreducible symmetric space of noncompact type, $P \cong G/K$. We now construct asymptotic cones of P using both sets of data and apply (5.5):

$$\mathrm{Cone}_1 P \cong \frac{G({}^{\rho_1}\mathbb{R})}{G(O_1)} \qquad \mathrm{Cone}_2 P \cong \frac{G({}^{\rho_2}\mathbb{R})}{G(O_2)}$$

Using the isomorphism $\phi : {}^{\rho_1}\mathbb{R} \to {}^{\rho_2}\mathbb{R}$, we get a group isomorphism, also denoted by ϕ:

$$\phi : G({}^{\rho_1}\mathbb{R}) \to G({}^{\rho_2}\mathbb{R})$$

Because ϕ is valuation preserving, ϕ maps the subgroup $G(O_1)$ onto the subgroup $G(O_2)$. We now define $\phi_* : \mathrm{Cone}_1 P \to \mathrm{Cone}_2 P$ as

$$\phi_*(g \cdot I) = \phi(g) \cdot I$$

To see this is well-defined, suppose $g_1 \cdot I = g_2 \cdot I$ in $\mathrm{Cone}_1 P$. Then $g_2^{-1} g_1 \in G(O_1)$. Therefore $\phi(g_2^{-1} g_1) = \phi(g_2)^{-1}\phi(g_2) \in G(O_2)$ (because ϕ maps $G(O_1)$ onto $G(O_2)$). Therefore $\phi(g_1) \cdot I = \phi(g_2) \cdot I$ and ϕ_* is well-defined.

Theorem 5.17. $\phi_* : \mathrm{Cone}_1 P \to \mathrm{Cone}_2 P$ is an isometry.

Proof: First note that because $\phi(G(O_1) = G(O_2)$, ϕ_* maps the base point of $\mathrm{Cone}_1 P$ to the base point of $\mathrm{Cone}_2 P$. Next, we check that ϕ_* is an isometry at the base point $I \in \mathrm{Cone}_1 P$. We need to check that $d(\phi_*(x), \phi_*(I)) = d(x, I)$ for all $x \in \mathrm{Cone}_1 P$. Let $g \in G({}^{\rho_1}\mathbb{R})$ be such that $x = g \cdot I$. By Corollary 5.4:

$$d(x, I) = \sqrt{\sum v((\alpha_i))^2}$$

where α_i are the eigenvalues of the matrix gg^t. Notice that if $\alpha \in {}^{\rho_1}\mathbb{R}$ is an eigenvalue of gg^t, then $\phi(\alpha) \in {}^{\rho_2}\mathbb{R}$ is an eigenvalue of $\phi(g)\phi(g)^t$. And, because ϕ is valuation preserving, $v(\alpha) = v(\phi(\alpha))$. Thus, we must have

$$\begin{aligned} d(x, I) &= \sqrt{\sum v((\alpha_i))^2} = \sqrt{\sum v((\phi(\alpha_i)))^2} \\ &= d(\phi_*(x), I) \end{aligned}$$

Thus, ϕ_* is an isometry at the base point $I \in \mathrm{Cone}_1 P$.

Next, consider two arbitrary points $x, y \in \mathrm{Cone}_1 P$. Using the isometric group action $G({}^{\rho_1}\mathbb{R}) \curvearrowright \mathrm{Cone}_1 P$, we can translate to the base point:

$$(x, y) \overset{g}{\rightsquigarrow} (g \cdot x, g \cdot y) = (I, z)$$

Because the action is isometric, $d(x, y) = d(I, z)$. Also notice that because $g \cdot x = I$, $\phi(g) \cdot \phi_*(x) = I$, the base point in $\mathrm{Cone}_2 P$. Similarly, $\phi(g) \cdot \phi_*(y) = \phi_*(g \cdot y)$. Therefore, we have

$$\begin{aligned} d(\phi_*(x), \phi_*(y)) &= d(\phi(g) \cdot \phi_*(x), \phi(g) \cdot \phi_*(y)) \\ &= d(I, \phi_*(g \cdot y)) = d(I, \phi_*(z)) \\ &= d(I, z) = d(x, y) \end{aligned}$$

and therefore, ϕ_* is an isometry. □

Theorem 5.18. Let P be a symmetric space. Then the asymptotic cone of P is independent of base point, scale factors and ultrafilter.

Proof: Let P be an arbitrary symmetric space. Applying the de Rham decomposition:

$$P = P_1 \times \cdots \times P_k \times \mathbb{R}^m \times Q$$

where each P_i is an irreducible symmetric space of noncompact type, $m \in \mathbb{N}$ and Q is compact. By Proposition 3.1,

$$\mathrm{Cone}_\omega P = \mathrm{Cone}_\omega P_1 \times \cdots \times \mathrm{Cone}_\omega P_k \times \mathrm{Cone}_\omega \mathbb{R}^m \times \mathrm{Cone}_\omega Q$$

Proposition 3.5 says $\mathrm{Cone}_\omega \mathbb{R}^m = \mathbb{R}^m$ and Proposition 3.4 says that $\mathrm{Cone}_\omega Q$ is a point. We now have

$$\mathrm{Cone}_\omega P = \mathrm{Cone}_\omega P_1 \times \cdots \times \mathrm{Cone}_\omega P_k \times \mathbb{R}^m \tag{5.6}$$

Theorem 5.17 and Theorem 3.3 together say that each $\mathrm{Cone}_\omega P_i$ is independent of base point, scale factors and ultrafilter. Combining this with (5.6) gives the theorem. □

CHAPTER 6

OPEN QUESTIONS

Although we have answered the main question in Theorem 5.18, there are many questions left open. For this section, we will assume that P is a symmetric space of noncompact type.

1. As mentioned in the introduction, Kleiner and Leeb showed that the asymptotic cone of P is a Euclidean building [17]. Bruhat and Tits show how Euclidean buildings arise from "valued root data" in an algebraic group [4]. Valued root data in an algebraic group can arise from a valuation field. Is the building $\mathrm{Cone}_\omega P$, the building that arises from the valued root data from the algebraic group G and the field ${}^\rho\mathbb{R}$?

2. The Tits boundary of a Euclidean building is a spherical building. Tits has shown that spherical buildings correspond to algebraic groups and fields [30]. Is the Tits boundary of $\mathrm{Cone}_\omega P$ the spherical building associated to G and the field ${}^\rho\mathbb{R}$?

3. As showed by Roitman, if one assumes the negation of the Continuum Hypothesis, there are infinitely many nonisomorphic nonstandard real fields [26]. Our proof that ${}^\rho\mathbb{R}$ is independent of ultrafilter and infinitesimal relied on Theorem 2.5, which relies on the Continuum Hypothesis. If one assumes the negation of the Continuum Hypothesis, can one obtain nonisomorphic fields, ${}^\rho\mathbb{R}$?

4. If one assumes the negation of the Continuum Hypothesis, can one obtain nonisometric asymptotic cones, $\mathrm{Cone}_\omega P$?

APPENDIX A

SOME MATRIX RESULTS

Let R be either the field $\mathbb{R}$ or $^*\mathbb{R}$. See Section 2.5 for a discussion on series of matrices in $^*\mathbb{R}$.

Lemma A.1. Let A be a $n \times n$ matrix with entries in R. If α is an eigenvalue of A, then e^α is an eigenvalue of $\exp(A)$. Similarly, if $\log(A)$ makes sense and α is an eigenvalue of A, then $\log \alpha$ is an eigenvalue of $\log(A)$.

Proof: This is straightforward using the definition of the exponential map. If α is an eigenvalue with eigenvector v then $Av = \alpha v$. Then

$$\begin{aligned} \exp(A)v &= \left(\sum \frac{A^k}{k!}\right) v = \sum \frac{\alpha^k v}{k!} \\ &= \left(\sum \frac{\alpha^k}{k!}\right) v = e^\alpha v \end{aligned}$$

This computation also makes it clear that $\log \alpha$ is an eigenvalue of $\log(A)$. □

Lemma A.2. Let A be a symmetric with entries in R.

1. If all the eigenvalues of A satisfy $|\alpha| \leq C$ then all the entries of A satisfy $|A_{ij}| \leq nC$.

2. If the entries of A satisfy $|A_{ij}| \leq C$ then all the eigenvalues of A satisfy

   $$|\alpha| \leq n^2 C$$

   Furthermore, if $A \in SL(n, R)$ then we also have

   $$(n^2C)^{-(n-1)} \leq |\alpha| \leq n^2 C$$

   and

   $$|\log |\alpha|| \leq (n-1)\log(n^2C)$$

Proof: Since A is symmetric, there exists some $B \in SO(n, R)$ such that $BAB^t = D$ is diagonal. Or equivalently, $A = B^t DB$.

1. We write out the entries of the product $A = B^t DB$. Remember that the norm of the entries of a real orthogonal matrix are bounded by one.

$$\begin{aligned} |A_{ij}| &= \left|(B^t DB)_{ij}\right| = \left|\sum_k B_{ki} D_{kk} B_{kj}\right| \\ &= \sum_k |B_{ki}||D_{kk}||B_{kj}| \\ &\leq \sum_k |D_{kk}| \leq nC \end{aligned}$$

2. We have $BAB^t = D$ is diagonal with the eigenvalues on the diagonal. We write out the entries of this matrix product:

$$\begin{aligned} |\alpha_i| &= |D_{ii}| = \left|\sum_{k,l} B_{il} A_{lk} B_{ik}\right| \\ &\leq \sum_{k,l} |A_{lk}| \leq n^2 C \end{aligned}$$

   If $A \in SL(n, \mathbb{R})$ then $1 = \det A = \Pi \alpha_i$, which implies that

$$|\alpha_i| = (\Pi_{j \neq i} \alpha_j)^{-1} \geq (n^2 C)^{-(n-1)}$$

   Taking the log of $(n^2 C)^{-(n-1)} \leq |\alpha| \leq n^2 C$ finishes the lemma.

$\square$

APPENDIX B

A RESULT ON N-TUPLES

Here, we let R be one of the real closed fields $\mathbb{R}$, ${}^*\mathbb{R}$ or ${}^\rho\mathbb{R}$.

Lemma B.1. Let $a = (a_1, \ldots, a_n) \in R^n$ let $d > 0$, $d \in R$. Let $m = \min\{a_i\}$ and $M = \max\{a_i\}$. Suppose that a satisfies the following conditions

$$\begin{aligned}\sum_{i=1}^{n} a_i &= 0 \\ \sum_{i=1}^{n} a_i^2 &= d^2\end{aligned}$$

Then

1. $m \leq -d/n^{3/2}$
2. $M \geq d/n^{3/2}$
3. $M \leq -(n-1)m$
4. $m \geq -(n-1)M$

Proof: Note that for all i, we have $m \leq a_i \leq M$. We split the sums into positive and negative parts:

$$\sum_{a_i \geq 0} a_i + \sum_{a_i < 0} a_i = 0 \quad \Longrightarrow \quad \sum_{a_i \geq 0} a_i = \sum_{a_i < 0} -a_i$$

Looking at each of these sums, noting that each sum must contain between 1 and $n-1$ terms:

$$M \leq \sum_{a_i \geq 0} a_i \leq (n-1)M \tag{B.1}$$

and

$$-m \leq \sum_{a_i<0} -a_i \leq -(n-1)m \tag{B.2}$$

Putting together (B.1) and (B.2):

$$\begin{aligned} M &\leq -(n-1)m \\ -m &\leq (n-1)M \end{aligned}$$

We now use $d^2 = \sum a_i^2$

$$\begin{aligned} d^2 &= \sum_{a_i \geq 0} a_i^2 + \sum_{a_i<0} a_i^2 \\ &\leq (n-1)M^2 + (n-1)m^2 \\ &\leq (n-1)^3 m^2 + (n-1)m^2 \\ &= (n-1)((n-1)^2+1)m^2 \\ &\leq n^3 m^2 \end{aligned}$$

This now gives the first inequality. The other inequality is obtained similarly by using the n-tuple $-a$. □

APPENDIX C

LINEAR ALGEBRAIC GROUPS

The basic references for this section are [3], [27], and [28]. Fix a field K and a subfield $k \subseteq K$. We will assume that the characteristic of K is zero. Let $V = K^n$. Let $K[T] = K[T_1, \ldots, T_n]$ be the polynomial algebra. Given a set of polynomials $I = \{f_1, \ldots, f_k\} \subseteq K[T]$, we can talk about the zero set of I. Such a set is called an *algebraic set*. If $I \subseteq k[T]$, we say that the algebraic set V is *defined over* k.

Given two algebraic sets $V_1 \subseteq K^n$ and $V_2 \subseteq K^m$, a map $\phi : V_1 \to V_2$ is called a *morphism* if the coordinate functions are given by polynomials. If the defining polynomials have coefficients in k, then we say that the morphism is defined over k.

Definition C.1. A group G, is a *linear algebraic group* if G is an algebraic set in K^n for some n, and the map $G \times G \to G$ defined by $(x, y) \mapsto xy^{-1}$ is a morphism.

If G is defined over k and the morphism is defined over k, then we say that G is defined over k. If this is the case, then it makes sense to talk about the group of k-rational points of G, or the k-points of G:

$$G(k) = G \cap k^n$$

If H and G are algebraic groups and $\phi : H \to G$ is a homomorphism that is also a morphism of algebraic sets, then we say that ϕ is a group morphism. If a group morphism is defined over k, then we say that ϕ is a k-morphism.

Example C.2. 1. K, as an additive group, is defined by the zero polynomial.

2. K^*, the nonzero elements of K, as a multiplicative group can be defined as a subset of K^2 and the polynomial $xy - 1$.

3. $SL(n, K) \subseteq K^{n^2}$ is defined by the polynomial determined by $\det(A) = 1$.

4. $SO(n, K) \subseteq K^{n^2}$ is defined by the polynomial representing $\det A = 1$ (as above for $SL(n, K)$) and polynomials for the relation $AA^t = I$.

5. The diagonal group, $D(n, K) \subseteq K^{n^2+1}$ is also an algebraic group. It has polynomials $x_{ij} = 0$ for $i \neq j$. $D(n, K)$ is an algebraic set in K^{n^2+1} and not K^{n^2} in order to get a polynomial representing the inequality $\det A \neq 0$. See [3] for the details.

6. $\phi : D(n, K) \to K^*$ given by $\phi(g) = x_{11}$ is a morphism.

7. $\det : GL(n, K) \to K^*$ is a morphism.

8. The map $K^* \to SL(2, K)$, defined by

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

is a morphism.

APPENDIX D

REAL ALGEBRAIC GEOMETRY

This appendix comes from [2] where all proofs can be found. R designates a real closed field, which for us will be either $\mathbb{R}$, ${}^*\mathbb{R}$ or ${}^\rho\mathbb{R}$. For $x = (x_1, \ldots, x_n) \in R^n$, we define

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2} \tag{D.1}$$

Notice that for $x \in R$, $|x|$ is an element of R. We give R (and R^n) the topology from the ordering on R. As in Appendix C, let $T = [T_1, \ldots, T_n]$.

Definition D.1. An *algebraic set* of R^n is a set of the form

$$\{x \in R^n \mid f(x) = 0, \forall f \in I\}$$

were $I \subseteq R[T]$ is a finite subset.

Definition D.2. A *semi-algebraic set* of R^n is a set of the form

$$\{x \in R^n \mid f(x) = 0, g(x) > 0, \forall f \in I, \forall g \in J\}$$

where $I, J \subseteq R[T]$ are finite subsets.

Example D.3. For an ordered field K, let $P(n, K)$ be the set of positive definite symmetric matrices with determinant one. $P(n, K)$ is a semi-algebraic set with polynomials representing these conditions. Note that the condition of being a positive definite matrix is defined by polynomial inequalities.

Definition D.4. Let $A \subseteq R^n$ and $B \subseteq R^m$ be semi-algebraic sets. A map $f : A \to B$ is a *semi-algebraic map* if its graph in R^{n+m} is a semi-algebraic set, i.e., if

$$\{(x, y) \in R^n \times R^m \mid y = f(x)\}$$

is a semi-algebraic set.

The easiest examples of semi-algebraic maps are polynomials and the map (D.1). Another important example of a semi-algebraic function is the distance function.

Lemma D.5. Let $A \subseteq R^n$ be a nonempty semi-algebraic set.

1. Then for every $x \in R^n$, the distance between x and A

   $$d(x, A) = \inf \{|x - y|\}$$

   is well-defined as an element of R,

2. The function $x \mapsto d(x, A)$ from R^n to R is continuous, semi-algebraic, vanishes on the closure of A and is positive elsewhere.

The main point in the proof is that the function (D.1) is a semi-algebraic function.

Proposition D.6. Let $A \subseteq R^n$ be a closed semi-algebraic set and $f : A \to R$ a continuous semi-algebraic function. Then there exists $c \in R$, $p \in \mathbb{N}$ such that for every $x \in A$,

$$|f(x)| \leq c\left(1 + |x|^2\right)^p$$

Theorem D.7. Let A be a locally closed semi-algebraic set. Let f and g two continuous semi-algebraic functions from A to R such that $f^{-1}(0) \subseteq g^{-1}(0)$. Then, there exists an integer $N > 0$ and a continuous semi-algebraic function $h : A \to R$, such that $g^N = hf$ on A.

For us Theorem D.7, together with Proposition D.6 will be the important keys for Section 2.6. A nice corollary of Theorem D.7 is Łojasiewicz's inequality:

Corollary D.8. Let A be a closed and bounded semi-algebraic set and f and g two continuous semi-algebraic functions from A to R, such that $f^{-1}(0) \subseteq g^{-1}(0)$. Then, there exists an integer $N > 0$ and a constant $c \in R$ such that $|g|^N \leq c|f|$ on A.

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