

SCHUBERT CALCULUS ON FLAG VARIETIES AND SESHADRI CONSTANTS ON JACOBIANS

by

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A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

May 2001

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THE UNIVERSITY OF UTAH GRADUATE SCHOOL

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ABSTRACT

This dissertation contains two chapters.

In Chapter 1 we discuss a new and effective way of doing intersection theory on flag manifolds. Namely we do Schubert calculus on flag manifolds and flag bundles via equivariant cohomology and localization. The basic idea is to locate the flag manifold as a fixed-point component for a torus action on a larger ambient space, then apply Atiyah-Bott localization theorem to relate residues on the flag manifold to residues on some simpler manifold through equivariant maps. From these we get our Schubert formulas, which are very effective in computation. We will also give some applications.

In Chapter 2 we discuss Seshadri constants on Jacobian of algebraic curves. We find the exact values of Seshadri constants of Jacobian of hyperelliptic curves, as well as of curves with genus three and four. For higher genus curves we conclude that if the Seshadri constants of their Jacobian are less than 2, then the curves must be hyperelliptic.

This dissertation was written under the direction of Aaron Bertram, my Ph.D. thesis advisor.

To my parents

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ACKNOWLEDGMENTS

This dissertation was written under the direction of Aaron Bertram, my Ph.D. thesis advisor. I hereby give my thanks to him.

I also want to thank all my supervisory committee members for their help and encouragement.

CHAPTER 1

SCHUBERT CALCULUS ON FLAG MANIFOLDS

1.1 Introduction and Preliminaries

1.1.1 Introduction

In this project we discuss a new and effective way of doing intersection theory on flag manifolds. Namely we do Schubert calculus on flag manifolds and flag bundles via equivariant cohomology and localization. The basic idea is to locate the flag manifold as a fixed-point component for a torus action on a larger ambient space, then apply the Atiyah-Bott localization theorem to relate residues on the flag manifold to residues on some simpler manifold (projective space in our case) through equivariant maps.

Let us briefly recall the formula for $Fl(1, 2, V)$ discussed in Bertram's paper ([4]), from where our main methods and techniques come.

The basic diagram is:

$$\begin{array}{ccc}
 \bar{N}_{0,0}(\mathbf{P}^n, 1) & \xrightarrow{u} & \mathbf{P}(V \otimes W^*) \\
 \uparrow i & & \uparrow j \\
 Fl(1, 2, V) & \xrightarrow{\pi} & \mathbf{P}(V) \\
 \downarrow & & \\
 G(2, V) & &
 \end{array}$$

Here V and W are complex vector spaces of rank $n + 1$ and 2. The morphisms are as follows: First the “graph space”

$$u : \bar{N}_{0,0}(\mathbf{P}^n, 1) \rightarrow \mathbf{P}(V \otimes W^*)$$

is simply the blow-up along the Segre embedding $\mathbf{P}(V) \times \mathbf{P}(W^*)$. Then there are two embeddings:

$$j : \mathbf{P}(V) := \mathbf{P}(V) \times \{w_1^*\} \rightarrow \mathbf{P}(V \otimes W^*)$$

where $\{w_1^*\}$ is a point in W^* , and

$$i : Fl(1, 2, V) \rightarrow \bar{N}_{0,0}(\mathbf{P}^n, 1)$$

which locates the flag variety $Fl(1, 2, V)$ as a fixed-point locus under a torus action. Finally

$$\pi : Fl(1, 2, V) \rightarrow \mathbf{P}(V)$$

is just the forgetful map, which is also the restriction of u .

Now Let h be the hyperplane class on $\mathbf{P}(V)$, ψ the relative hyperplane class from the projection $\pi : Fl(1, 2, V) \rightarrow \mathbf{P}(V)$. Let $\sigma(q_1, q_2)$ be a cohomology class pulled back from $G(2, V)$, represented by a symmetric polynomial in the Chern roots $-q_i$ of the universal subbundle $S \subset V \otimes \mathcal{O}_{G(2,V)}$.

The following formula, obtained from applying localization theorem to the standard $\mathbf{C}^*$ action on W , allows one to compute Schubert calculus on $G(2, V)$ by reading off the appropriate coefficient:

$$\int_{Fl(1,2,V)} \pi^*(h^b) \cup \psi^a \cup \sigma(q_1, q_2) = \text{coeff. of } t^{-a-2} \text{ in } \int_{\mathbf{P}(V)} \frac{h^b \cup \sigma(h, h+t)}{(h+t)^{n+1}}.$$

Also in the relative version it produces the Porteous formula in Segre classes.

The above basic diagram is the special (degree one) case of the diagram Bertram uses to produce a new computation of one-point Gromov-Witten invariants. In this project we extend the idea to higher rank cases.

Let $\dim(W) = m \geq 3$, and consequently we do Schubert calculus on the partial flag manifold $Fl(1, 2, \dots, m, V)$. The main difference between our cases and the $m = 2$ case is that there are more than one (actually $m - 1$) loci that we need to blow up sequentially. Consequently in the end there are more than one (actually $(m - 1)!$) disjoint fixed-point components, each isomorphic to $Fl(1, 2, \dots, m, V)$ but with different equivariant Euler classes, that map to the same $\mathbf{P}(V)$. But

our methods work out and lead to the first Schubert formula (see below). More importantly, after a good approximation outside the boundary divisors it turns out that an error-term estimation is also in our favor, thus we get our second Schubert formula in general cases. The relative versions will also be discussed.

For the simplicity of illustration, we initially discuss the situation of $m = 3$, then discuss the general formulas in the last section of this chapter. Here we briefly state our main formulas in the most general cases. (For a detailed explanation see Theorem 7 and 8 in Section 1.4.)

Let h be the hyperplane class in $\mathbf{P}(V)$, let

$$\pi : Fl(1, 2, \dots, m, V) \rightarrow \mathbf{P}(V) \quad (i < m)$$

be the forgetful map and let ψ_i be the relative hyperplane class for the projection

$$Fl(1, 2, \dots, i+1, V) \rightarrow Fl(1, 2, \dots, i, V)$$

pulled back to $Fl(1, 2, \dots, m, V)$.

Schubert Formula 1:

$$\sum_{\{I|I_1=i\}} \pi_* \left(\frac{1}{\prod_{0 \leq j < k \leq m-1} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s)} \right) = \frac{1}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}}$$

The second formula involves cohomology classes pulled back from the Grassmannian under the projection

$$p : Fl(1, 2, \dots, m, V) \rightarrow G(m, V).$$

Represent such a cohomology class as a symmetric polynomial $\tau(q_1, \dots, q_m)$ in the Chern roots $-q_i$ of the universal subbundle $S \subset V \otimes \mathcal{O}_{G(m,V)}$.

Schubert Formula 2:

$$\begin{aligned} & \sum_{\{I|I_1=i\}} \pi_* \left(\frac{\pi^*(h^b) \cup p^* \tau(q_1, \dots, q_m)}{\prod_{0 \leq j < k \leq m-1} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s)} \right) \\ &= \frac{h^b \cup \tau(h + t_1 - t_i, \dots, h + t_m - t_i)}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}} + \text{irrelevant terms} \end{aligned}$$

We also discuss two applications: The first application is about counting planes, the analogue of one-point Gromov-Witten invariants in degree one case. We will

look at class of the locus of hypersurfaces which contain projective planes, and find the number of planes if class has the right dimension. We will also consider the number of planes satisfying certain linear conditions on certain hypersurfaces. In the second application we will produce a Porteous-like formula using the relative version formula, which turns out to be neat in the form of Segre classes.

We organize this chapter in the following order: In the rest of this section we will go over some basic facts about equivariant cohomology and compute some useful examples. Also we will fix some notations. Then in Section 1.2 we will set up our main formulas and associated topics in the case of $\dim(W) = m = 3$. This section contains the main techniques and results. In Section 1.3 we will use our formulas to do a few applications, mainly to illustrate the effectiveness of our methods. Finally in Section 1.4 we will discuss general cases. We will explain that all the corresponding results, though they may look complicated, follow the same principles from those in Section 1.2.

1.1.2 Preliminaries and notations

We will recall some basic facts needed in this paper. Most of them are about equivariant cohomology. We will also compute some examples which will be used in next two chapters.

For group action, we always assume the group $T = (\mathbf{C}^*)^k$ is a complex torus throughout this paper.

Definition 1 The universal principal T bundle is the T -fibering

$$ET := (\mathbf{C}^\infty - \{0\})^k \rightarrow (\mathbf{C}P^\infty)^k =: BT$$

Here ET is contractable on which T acts freely, and the quotient $BG = ET/T$ is called the classifying space.

Let X be a compact complex manifold equipped with a T action. Since T acts on $X \times ET$, we construct the quotient $X_T := X \times_T ET$. Note that $\pi_X : X_T \rightarrow ET/T = BT$ is a T bundle with fiber X .

Example 1 If T acts trivially on X , then $X_T = X \times BT$. On the other hand, if T acts freely on X , then $X_T = X/T \times BT$.

Definition 2 The T -equivariant cohomology of X is the cohomology of X_T :

$$H_T^*(X, \mathbf{Q}) := H^*(X_T, \mathbf{Q}).$$

Example 2 Since $T = (\mathbf{C}^*)^k$, we have $H^*(BT) = \mathbf{Q}[\lambda_1, \lambda_2, \dots, \lambda_k]$. Since $H_T^*(\text{point}) = H^*(BT)$, we see that $H_T^*(X)$ is a $H^*(BT)$ -module by pulling back from a point.

Example 3 If T acts trivially on X , then $H_T^*(X) = H^*(X) \otimes H^*(BT)$. On the other hand if T acts freely on X , then $H_T^*(X) = H^*(X/G)$ with trivial $H^*(BT)$ module structure.

Definition 3 An equivariant vector bundle on X (with a T action) is a vector bundle V over X such that the T action on X lifts to an linearized action of V . Then $V_T \rightarrow X_T$ is a vector bundle of same rank. We define the equivariant chern class $c_k^T(V) := c_k(V_T) \in H^*(X_T)$.

We will frequently need to compute equivariant chern classes, especially equivariant Euler classes (the top chern class of an equivariant normal bundle). We compute several examples below that will be used later in this paper.

Example 4 Let $T = (\mathbf{C}^*)$ act on $\mathbf{C}^2$ by

$$t \cdot (a, b) = (a, tb).$$

Denote by $0 = (1, 0)$ and $\infty = (0, 1)$ in $\mathbf{P}^1$. They are the only fixed points of the action. We would like to compute the equivariant Euler class of the normal bundle

$$c_{top}^T(N_{\{0\}/\mathbf{P}^1}) = c_1^T(T_{\mathbf{P}^1}|_{\{0\}})$$

by determining the weight on the tangent line.

By looking at the tautological space of the universal bundle $\mathcal{O}_{\mathbf{P}^1}(-1)$ we see that (by looking at the action of T on the line in $\mathcal{O}_{\mathbf{P}^1}(-1)$ over 0 and ∞) $\mathcal{O}_{\mathbf{P}^1}(-1)$

has weights 0 and 1 at 0 and ∞ . Consequently $\mathcal{O}_{\mathbf{P}^1}(1)$ has weights 0 at 0 and -1 at ∞ . Now from the Euler sequence (equivariant if one writes in the following way)

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \rightarrow \mathcal{O}_{\mathbf{P}^1}(1) \otimes \mathbf{C}^2 \rightarrow T_{\mathbf{P}^1} \rightarrow 0$$

we can compute the equivariant Euler class of the normal bundle as follows: First $\mathcal{O}_{\mathbf{P}^1}(1) \otimes \mathbf{C}^2$ has weight, $(0+0)$ and $(1+0)$ (weights of coordinates of $\mathbf{C}^2$ plus the weight of $\mathcal{O}_{\mathbf{P}^1}(1)$) at 0 and weight $(0-1)$ and $(1-1)$ at ∞ . Since the action on $\mathcal{O}_{\mathbf{P}^1}$ is trivial, it always cancel the weight 0 part in $\mathcal{O}_{\mathbf{P}^1} \otimes \mathbf{C}^2$. So $T_{\mathbf{P}^1}$ has weight 1 at 0 and -1 at ∞ . In summary we see the following table:

bundle	weight at 0	weight at ∞
$\mathcal{O}_{\mathbf{P}^1}(-1)$	0	1
$\mathcal{O}_{\mathbf{P}^1}(1)$	0	-1
$T_{\mathbf{P}^1}$	1	-1

From here we see that

$$c_{top}^T(N_{\{0\}/\mathbf{P}^1}) = c_1^T(T_{\mathbf{P}^1}|_{\{0\}}) = t.$$

This example gives a proof from the definition how to compute the weight of certain equivariant classes. One can also see that $T_{\mathbf{P}^1}|_{\{0\}}$ has weight 1 at 0 since it is the weight on the second coordinate, and weight -1 at ∞ since it is the weight on the first coordinate by considering the action as $t \cdot (a, b) = (t^{-1}a, b)$ (on $\mathbf{P}^1$ the two actions are the same).

Example 5 In this paper we will mainly consider the case when $T = (\mathbf{C}^*)^m$ acts on an m -dimensional vector space W (fix a basis $\langle e_1, \dots, e_m \rangle$) as follows

$$(t_1, t_2, \dots, t_m) \cdot (w_1, w_2, \dots, w_m) = (t_1^{-1}w_1, t_2^{-1}w_2, \dots, t_m^{-1}w_m)$$

such that the dual action on the dual space W^* is

$$(t_1, t_2, \dots, t_m) \cdot (w'_1, w'_2, \dots, w'_m) = (t_1 w'_1, t_2 w'_2, \dots, t_m w'_m).$$

The action has m fixed points in $\mathbf{P}(W^*)$ which are e_i^* ($i = 1, 2, \dots, m$). In view of the above example, we give the answer of the following computations quickly:

(1) We want to compute

$$c_{top}^T(N_{\{0\}/\mathbf{P}(W^*)}) = c_{top}^T(T_{\mathbf{P}(W^*)}|_{\{0\}}).$$

Here $0 := (1, 0, \dots, 0) = e_1^*$ in $\mathbf{P}(W^*)$. As before we can see that $\mathcal{O}_{\mathbf{P}(W^*)}(-1)$ has weight $(1, \dots, 0)$ at e_1^* and weights $(0, \dots, 0, 1, 0, \dots, 0)$ at e_i^* (The nonzero factor corresponds the weight on t_i). Find the corresponding weights on $\mathcal{O}_{\mathbf{P}(W^*)}(1)$ and looking at the Euler sequence:

$$0 \rightarrow \mathcal{O}_{\mathbf{P}(W^*)} \rightarrow \mathcal{O}_{\mathbf{P}(W^*)}(1) \otimes \mathbf{C}^m \rightarrow T_{\mathbf{P}(W^*)} \rightarrow 0$$

We see that $T_{\mathbf{P}(W^*)}$ has weight $(0, 0, \dots, 0), (-1, 1, 0, \dots, 0), \dots, (-1, \dots, 0, 1)$ at 0. So consequently

$$c_{top}^T(N_{\{0\}/\mathbf{P}(W^*)}) = c_{top}^T(T_{\mathbf{P}(W^*)}|_{\{0\}}) = \prod_{i=2}^m (t_i - t_1).$$

(2) We also want to look at the point $\infty := (0, \dots, 0, 1) \in \mathbf{P}((W^*)^*) = \mathbf{P}(W)$. In this case we just switch the sign of all weights on $\mathcal{O}_{\mathbf{P}(W^*)}(-1)$, and the computation shows that $T_{\mathbf{P}(W)}$ has weight $(-1, 0, \dots, 0, 1), \dots, (0, \dots, 0, -1, 1)$, and $(0, \dots, 0)$ at ∞ . This shows

$$c_{top}^T(N_{\{\infty\}/\mathbf{P}(W)}) = c_{top}^T(T_{\mathbf{P}(W)}|_{\{\infty\}}) = \prod_{i=1}^{m-1} (t_m - t_i).$$

If $f : X \rightarrow Y$ is a T -equivariant morphism of compact complex manifolds with T -actions. Then f induces a map (still denoted by) $f : X_T \rightarrow Y_T$. We can define the equivariant pull-back and (in case f is proper) the equivariant push-forward from the induced map.

Next we want to recall Atiyah-Bott localization theorem, which is essential to set up our main formulas.

Let $T = (\mathbf{C}^*)^m$ be a complex torus. Let X be a compact complex manifold with a T -action. Let $F_1, \dots, F_n \subset X$ be the (smooth) connected components of the fixed-point locus. Let $i_k : F_k \rightarrow X$ denote their embeddings. Notice that the normal bundles $N_{F_k/X}$ canonically have induced T -action. Denote by $t_I := \{t_1, t_2, \dots, t_m\}$.

Theorem 1 ([2]) With notations as above:

- (1) The equivariant Euler class $c_{top}^T(N_{F_k/X})$ is invertible in $H^*(F_k, \mathbf{Q})(t_I)$.
- (2) If $c_T \in H_T^*(X, \mathbf{Q})$ is a torsion-free equivariant class, then it uniquely decomposes in $H_T^*(X, \mathbf{Q}) \otimes \mathbf{Q}(t_I)$ as:

$$c_T = \sum_{k=1}^n (i_k)_* \frac{i_k^*(c_T)}{c_{top}^T(N_{F_k/X})}$$

The localization theorem says we need only know the contributions from the fixed-point loci. On the other hand, let $f : X \rightarrow Y$ be an equivariant map of compact complex manifolds with T -actions. Suppose $i_k : F_k \rightarrow X$ ($k = 1, 2, \dots, n$) and $j : G \rightarrow Y$ are components of the fixed-point loci with the property that F_k 's are the only fixed-point components that map to G . The following corollary says we can compute the residues through reduction to Y .

Corollary 1 If c_T is an equivariant cohomology class on X , then

$$\sum_{k=1}^n (f|_F)_* \left(\frac{i_k^* c_T}{c_{top}^T(N_{F_k/X})} \right) = \frac{j^* f_* c_T}{c_{top}^T(N_{G/Y})}$$

Proof. The free part of C_T is a push-forward of classes from the fixed-point loci of X by the localization theorem. Since only F_k 's map to G , only their contributions are needed when considering $j^* f_* c_T$ and $i^* c_T$. So we may assume that $C_T = \sum_{k=1}^{m-1} (i_k)_* (b_k)_T$ where $(b_k)_T \in H^*(F_k, \mathbf{Q})[t_I, t_I^{-1}]$.

Notice that

$$(b_k)_T = \frac{i_k^* (i_k)_* (b_k)_T}{c_{top}^T(N_{F_k/X})} = \frac{i_k^* c_T}{c_{top}^T(N_{F_k/X})}.$$

Now from the diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow i_k & & \uparrow j \\ F_k & \xrightarrow{f|_{F_k}} & G \end{array}$$

and the formula for $(b_k)_T$ the corollary follows since

$$j^* f_* c_T = \sum_{k=1}^{m-1} j^* f_*(i_k)_*(b_k)_T = \sum_{k=1}^{m-1} j^* j_*(f|_{F_k})_*(b_k)_T = \sum_{k=1}^{m-1} c_{top}^T(N_{G/Y})(f|_{F_k})_*(b_k)_T$$

■

Notations: We will fix some notations throughout this paper:

1. All varieties and vector spaces are over the complex number field $\mathbf{C}$.
2. V is an $(n + 1)$ -dimensional vector space. W is an m -dimensional vector space.
 $\mathbf{P}(V)$ and $\mathbf{P}(W)$ are n - and $(m - 1)$ - dimensional projective spaces $\mathbf{P}^n$ and $\mathbf{P}^{m-1}$.
 $\mathbf{P}(W^*)$ is the dual projective space.
3. $\mathbf{P}(V \otimes W^*) = \mathbf{P}(\text{Hom}(W, V))$ is the projective space of the corresponding vector space $\text{Hom}(W, V)$. Note that $\dim(\mathbf{P}(V \otimes W^*)) = m(n + 1) - 1$.
4. The Grassmannian space $G(k, V)$ is the space of all k -dimensional subspaces in V . Note that $\dim(G(k, V)) = k(n + 1 - k)$
5. The flag variety $Fl(1, 2, \dots, m, V)$ is the space of all flags
 $V_1 \subset V_2 \subset \dots \subset V_m \subset V$ where $\dim(V_i) = i$.
Note that $\dim(Fl(1, 2, \dots, m, V)) = m(n + 1) - \frac{m(m+1)}{2}$.
6. In the relative setting:
 V is a rank- $(n + 1)$ vector bundle over a smooth projective variety X .
Replace “space” by “bundle” in all above notations.
Also note that all dimensions are relative (over X).

1.2 Schubert Calculus on $FL(1, 2, 3, V)$

When $m = \dim(W) \geq 3$, we need to blow up more than once to desingularize all lower rank loci. Consequently we will do Schubert calculus on the flag manifold $Fl(1, 2, \dots, m, V)$. In this section we are going to discuss the case $m = 3$, and leave the general cases to section 1.4. This section describe the main techniques of our project.

The basic diagram for $m = 3$ will be:

$$\begin{array}{ccc}
N_2 & \xrightarrow{u} & \mathbf{P}(V \otimes W^*) \\
\uparrow i & & \uparrow j \\
Fl(1, 2, 3, V) & \xrightarrow{\pi} & \mathbf{P}(V) \\
\downarrow & & \\
G(3, V) & &
\end{array}$$

Briefly speaking, we want to do intersection theory on the partial flag variety $Fl(1, 2, 3, V)$, especially consider classes which are pull-back from Grassmannian $G(3, V)$. To make effective ways of computation in view of our methods, we first locate the flag variety as a fixed-point component of the standard torus action in the ambient space N_2 , which is obtained by two blow-ups from the mapping space $\mathbf{P}(V \otimes W^*)$. Then we observe the flag variety $Fl(1, 2, 3, V)$ maps to some fixed-point component and the whole diagram is compatible with the group action. Therefore we need to express the computation in the form of some residues, and use the Localization theorem to relate the residues on flag varieties to those on simpler spaces (in our case, projective spaces).

To discuss this more carefully, we need to give a more detailed Main Diagram.

1.2.1 Main diagram and explanation

We explain in detail all the necessary steps then give the Main Diagram. Basically we look at the low-rank loci in the mapping space

$$\mathbf{P}(V) \times \mathbf{P}(W^*) = \mathbf{P}(Hom(W, V)).$$

In first step we blow-up the rank one locus (the Segre embedding), which makes the proper transform of the rank two locus smooth. Then in second step we blow-up this proper transform to get the ambient space N_2 . We find that the needed flag manifolds are inside the intersection of two exceptional divisors and, after considering the “standard” torus action, are exactly the fixed loci.

Fix a basis of W as $\langle w_1, w_2, w_3 \rangle$ with the dual basis $\langle w_1^*, w_2^*, w_3^* \rangle$.

We get the Main Diagram through the following steps:

Step 0: First we have two vector spaces V and W with dimensions $n+1$ and 3. Denote by $\mathbf{P}(V) := \mathbf{P}(V) \times (w_1^*)$. Consider the following which is the right column of the diagram

$$\mathbf{P}(V) \rightarrow M_1 \rightarrow M_2 \rightarrow \mathbf{P}(V \otimes W^*).$$

Here $M_2 = \{\phi \in \text{Hom}(W, V) | \dim(\text{Im}\phi) \leq 2\}$ is the rank two locus (with codimension $n-1$), and the rank one locus $M_1 = \{\phi \in \text{Hom}(W, V) | \dim(\text{Im}\phi) \leq 1\}$ is the Segre embedding $\mathbf{P}(V) \times \mathbf{P}(W^*)$. Note that M_2 is singular along M_1 .

Step 1: Now we blow up the rank one part $\mathbf{P}(V) \times \mathbf{P}(W^*)$ to get $u_1 : N_1 \rightarrow \mathbf{P}(V \otimes W^*)$. Note that the exceptional divisor E_1 is a projective bundle $\mathbf{P}(V/l \otimes K^*)$ over $\mathbf{P}(V) \times \mathbf{P}(W^*)$, where l and K are image and kernel of the corresponding map in $V \otimes W^*$. Let M_2' be the proper transform of M_2 . It is a projective bundle $\mathbf{P}(p_V \otimes Q^*)$ where p_V and Q are image and quotient (in W) of the corresponding map in $V \otimes W^*$, so in particular it is smooth. The above description of two bundles (fiber-wise) shows that every element of the intersection of E_1 and M_2' consists a pair of flags: $(l \subset p_V)$ in $Fl(1, 2, V)$ and $Q^* \cap K^* \subset K^*$ in $Fl(1, 2, W^*)$. From these it is easy to check that

$$M_2' \cap E_1 = Fl(1, 2, V) \times Fl(1, 2, W^*).$$

Also one component of preimage of $\mathbf{P}(V)$ is $Fl(1, 2, V)$. (See below.) So we have the middle column

$$Fl(1, 2, V) \rightarrow Fl(1, 2, V) \times Fl(1, 2, W^*) \rightarrow M_2' \rightarrow N_1.$$

Step 2: We have to blow up again (along M_2'). This time we get $u_2 : N_2 \rightarrow N_1$. We denote the exceptional divisor as D_2 and also the preimage of E_1 as D_1 . Note that as in step 1, the intersection of D_1 and D_2 consists of pairs of flags in $Fl(1, 2, 3, V)$ (dimension of images increases) and $Fl(1, 2, W^*)$ (dimension of kernels decreases). So we check that

$$D_1 \cap D_2 = Fl(1, 2, 3, V) \times Fl(1, 2, W^*),$$

which has same image under u_2 as D_1 . Inside $D_1 \cap D_2$ we find two flag varieties: $i_{F_1} : F_1 = Fl(1, 2, 3, V) \times \{0\} \rightarrow N_2$ and $i_{F_2} : F_2 = Fl(1, 2, 3, V) \times \{\infty\} \rightarrow N_2$.

Here $\{0\}$ corresponds the point $\{(*, 0, 0) \subset (*, *, 0) \subset W^*\}$ in $FL(1, 2, W^*)$, and $\{\infty\}$ corresponds the point $\{(*, 0, 0) \subset (*, 0, *) \subset W^*\}$ in $FL(1, 2, W^*)$. Note that this means there are also two $Fl(1, 2, V)$ in the second column that map to $\mathbf{P}(V)$.

Thus we have the left column

$$F_i \rightarrow Fl(1, 2, 3, V) \times Fl(1, 2, W^*) \rightarrow D_j \rightarrow N_2. \quad (i, j = 1, 2).$$

With a little bit of abuse of notation, we will denote by π the forgetful map, from both F_1 and F_2 to $\mathbf{P}(V)$.

Finally we provide the Main Diagram in the rank 3 case:

Main Diagram:

$$\begin{array}{ccccc}
 N_2 & \xrightarrow{u_2} & N_1 & \xrightarrow{u_1} & \mathbf{P}(V \otimes W^*) \\
 \uparrow i_3 & & \uparrow k_3 & & \uparrow j_3 \\
 D_2 & \longrightarrow & M'_2 & \longrightarrow & M_2 \\
 \uparrow i_2 & & \uparrow k_2 & & \uparrow j_2 \\
 D_1 \cap D_2 & \longrightarrow & M'_2 \cap E_1 & \longrightarrow & P(V) \times P(W^*) \\
 \uparrow i_1 & & \uparrow k_1 & & \uparrow j_1 \\
 Fl(1, 2, 3, V) & \longrightarrow & Fl(1, 2, V) & \longrightarrow & \mathbf{P}(V)
 \end{array}$$

Note that all vertical maps are embeddings.

We denote $i = i_3 \circ i_2 \circ i_1$, $j = j_3 \circ j_2 \circ j_1$, $k = k_3 \circ k_2 \circ k_1$, and let $u = u_1 \circ u_2$. Also note there is a family of surfaces $\pi' : N'_2 \rightarrow N_2$ and an evaluation map $e : N'_2 \rightarrow \mathbf{P}(V)$. (See Proposition 3 for details.)

1.2.2 Main formulas

Now consider the linearized action of $T = (\mathbf{C}^*)^2$ on W :

$$(t, u) \cdot (w_1, w_2, w_3) \rightarrow (w_1, t^{-1}w_2, u^{-1}w_3).$$

This action induces actions through each step of our diagram, such that all maps in the diagram are T -equivariant.

Note that $(w_1^*) \in W^*$ is one of the fixed points of T -action. So $\mathbf{P}(V) := \mathbf{P}(V) \times (w_1^*)$ is one of the fixed-point components in $\mathbf{P}(V \otimes W^*)$. The important thing here is that the two flag varieties F_i above are two fixed-point components of the T action on N_2 , and are the only such components that map to the $\mathbf{P}(V)$ above. So we can apply the localization theorem to do Schubert calculus on flag varieties through reduction to the projective space $\mathbf{P}(V)$. We need, however, to investigate and compute all terms involved (equivariant top chern class on normal bundles, push forwards, and error terms).

Let $h \in H^2(\mathbf{P}(V), \mathbf{Z})$ be the hyperplane class. Let $H := e^*(h)$ and define the equivariant chern class $h^T := c_1^T(\mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1))$. The following proposition will compute some important terms that are needed in the main formulas.

Proposition 1 From the above setting:

- (1) $\mathbf{P}(V) \subset \mathbf{P}(V \otimes W^*)$ is a component of the fixed-point locus.
- (2) F_1 and F_2 are the only fixed-point components in N_2 to map to $\mathbf{P}(V)$.
- (3) H extends to the equivariant class $u^*(h^T)$.
- (4) The equivariant Euler class of F_1 in N_2 is

$$tu(u-t)(t-\psi_1)(u-t-\psi_2).$$

Similarly the equivariant Euler class of F_2 in N_2 is

$$tu(t-u)(u-\psi_1)(t-u-\psi_2) = tu(u-t)(u-\psi_1)(u-t+\psi_2).$$

Here ψ_1 and ψ_2 are the relative canonical classes $c_1(\omega_{\pi_1})$ and $c_1(\omega_{\pi_2})$, where $\pi_1 : Fl(1, 2, 3, V) \rightarrow Fl(2, 3, V)$ and $\pi_2 : Fl(1, 2, 3, V) \rightarrow Fl(1, 3, V)$ are forgetful maps.

- (5) The equivariant Euler class of $\mathbf{P}(V)$ in $\mathbf{P}(V \otimes W^*)$ is

$$(h+t)^{n+1}(h+u)^{n+1}.$$

Proof. (1)-(3) are straightforward. But notice (2) means the formula we will get has more than one terms on the left side.

For (4) We need to understand three terms:

Step1: $c_{top}^T(N_{F_1/D_1 \cap D_2}) = c_{top}^T(N_{\{0\}/Fl(1,2,W^*)})$ which is $tu(u-t)$.

Let $W_1^* = (w_1^*, w_2^*)^*$ be the two-dimensional subspace of W^* . From the embeddings

$$\{0\} \rightarrow Fl(1, W_1^*) \rightarrow Fl(1, 2, W^*)$$

We know that

$$c_{top}^T(N_{\{0\}/Fl(1,2,W^*)}) = c_{top}^T(N_{\{0\}/Fl(1,W_1^*)}) \cdot c_{top}^T(N_{Fl(1,W_1^*)/Fl(1,2,W^*)}|_{\{0\}})$$

The first term is just $c_1^T(N_{\{0\}/(\mathbf{P}^1)^*}) = t$ by Example 5 in Section 1 (see also Example 4 for detailed proof). For the second term, the restriction $N_{Fl(1,W_1^*)/Fl(1,2,W^*)}|_{\{0\}}$ is just $N_{\{(0,0,*)\}/(\mathbf{P}^2)}$ (think it as $((\mathbf{P}^2)^*)^* = \mathbf{P}^2$). Since the T -action on $(\mathbf{P}^2)$ has the weight $(1, -t, -u)$, or $(u, u-t, 1)$ as we are looking at $\{(0, 0, *)\}$ (also by Example 5), we know that the equivariant top chern class $c_2^T(N_{\{(0,0,*)\}/(\mathbf{P}^2)^*}) = u \cdot (u-t)$.

Step2: $c_1^T(N_{D_1 \cap D_2/D_2}|_{F_1}) = c_1^T(N_{D_1/N_2}|_{F_1})$ which is $(t - \psi_1)$.

Since D_1 is the pull-back of the divisor E_1 on N_1 which intersects the blow-up locus M_2' transversely, by Riemman-Hurwitz we have $N_{D_1/N_2} = u_2^*(N_{E_1/N_1})$. Since $E_1 \subset N_1$ is the exceptional divisor of the blow-up of $\mathbf{P}(V \otimes W^*)$ along the Segre embedding M_1 , we have $N_{E_1/N_1} = \mathcal{O}_{\mathbf{P}(N_{M_1/\mathbf{P}(V \otimes W^*)})}(-1)$. Restrict it to $\mathbf{P}(V) \times \{0\}$, and we see that we need to compute $c_1^T(\mathcal{O}_{\mathbf{P}(T\mathbf{P}(V))}(-1))$, the first equivariant chern class of $\mathcal{O}(-1)$ on the projectivized tangent bundle. That bundle, however, is just $e_1 : Fl(1, 2, V) \rightarrow \mathbf{P}(V)$. So its c_1^T is just $t - \psi_1$. Here t , as shown in Step 1, is the $c_1^T(T_{\{0\}/\mathbf{P}^1})$ with regarding to $(*, 0, 0) \subset (*, *, 0)$ in $Fl(1, W_1^*) = (\mathbf{P}^1)^*$, and ψ is simply $c_1(\mathcal{O}_{e_1}(1))$. It is easy to check that ψ is actually $c_1(\omega_{\pi_1})$ by comparing the degrees on the fibers of the following two different projections:

$$\begin{array}{ccc} Fl(1, 2, V) & \xrightarrow[e_1]{} & Fl(1, V) \\ \downarrow \pi_1 & & \\ Fl(2, V) & & \end{array}$$

When pulling back to $Fl(1, 2, 3, V)$, ψ_1 is actually the relative canonical class of the forgetful map $Fl(1, 2, 3, V) \rightarrow Fl(2, 3, V)$ (by adding the three-dimensional subspace), but we will still denote it by same ψ_1 .

Step3: $c_1^T(N_{D_2/N_2}|_{F_1})$ which is $(u - t - \psi_2)$.

Since D_2 is the exceptional divisor of the blow-up of N_1 along M'_2 , we know that $N_{D_2/N_2} = \mathcal{O}_{\mathbf{P}(N_{M'_2/N_1})}(-1)$. We see that the restriction of the projective normal bundle to $Fl(1, 2, V)$ is just the $\mathbf{P}^{n-2}$ -bundle $\mathbf{P}(V/p_V)$ over $Fl(1, 2, V)$, which is (naturally) just $Fl(1, 2, 3, V)$. By looking at the following diagram:

$$\begin{array}{ccc} Fl(1, 2, 3, V) & \xrightarrow[e_2]{} & Fl(1, 2, V) \\ \downarrow \pi_2 & & \\ Fl(1, 3, V) & & \end{array}$$

We see that $c_1^T(N_{D_2/N_2}|_{F_1}) = c_1^T(\mathcal{O}_{e_2}(-1)) = u - t - \psi_2$. Here again the equivariant part $u - t$ as $c_{top}^T(N_{P(W^*)/P(W^*)}) = c_1^T(T_{\mathbf{P}^1}|_\infty)$ where $\mathbf{P}^1 = \mathbf{P}(< e_2, e_3 >)$ and $\infty = (0, 1)$. Also one can check $\psi_2 = c_1(\mathcal{O}_{e_2}(1))$ is also relative canonical class $c_1(\omega_{\pi_2})$.

For (5) We need to use the Euler sequences for the tangent bundles of $\mathbf{P}(V)$ and $\mathbf{P}(V \otimes W^*)$. From the diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ & \mathcal{O}_{\mathbf{P}(V)} & \equiv & \mathcal{O}_{\mathbf{P}(V)} & & & \\ & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & \mathcal{O}(1)^{n+1} & \longrightarrow & \mathcal{O}(1)^{n+1} \otimes W^*|_{\mathbf{P}(V)} & \longrightarrow & N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{\mathbf{P}(V)} & \longrightarrow & T_{P(V \otimes W^*)}|_{\mathbf{P}(V)} & \longrightarrow & N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

we see that the chern polynomials satisfy

$$\begin{aligned} c^T(N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)}) &= c^T(T_{P(V \otimes W^*)}|_{\mathbf{P}(V)})/c^T(T_{\mathbf{P}(V)}) \\ &= c^T(\mathcal{O}(1)^{n+1} \otimes W^*|_{\mathbf{P}(V)})/c^T(\mathcal{O}(1)^{n+1}) \\ &= c^T(W^*(1)/\mathcal{O}(1))^{n+1} \end{aligned}$$

So the equivariant Euler class of $N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)}$ is

$$[c_{top}^T(W^*(1)/\mathcal{O}(1))]^{n+1} = [(h+t)(h+u)]^{n+1} = (h+t)^{n+1} \cdot (h+u)^{n+1}$$

where the equivariant part comes by considering $(*, 0, 0) \in W^*$. ■

Now we can write down our basic formula based on the Localization Theorem (and Corollary 1), Proposition 1 and our Main Diagram:

Theorem 2 (Schubert Formula 1 on $Fl(1, 2, 3, V)$) For any equivariant class c_T on N_2 , we have:

$$\begin{aligned} \pi_* \left(\frac{i_{F_1}^* c_T}{tu(u-t)(t-\psi_1)(u-t-\psi_2)} \right) + \pi_* \left(\frac{i_{F_2}^* c_T}{tu(u-t)(u-\psi_1)(u-t+\psi_2)} \right) \\ = \frac{j^* u_* c_T}{(h+t)^{n+1}(h+u)^{n+1}} \end{aligned}$$

Remark: We can use the formula to do a lot of computations, by choosing different class c_T . Naturally we want $i_{F_1}^* c_T$ to be the form $H^b \cup \sigma \cup \tau$ where $\sigma(p_1, p_2)$ (resp., $\tau(q_1, q_2, q_3)$) is any chern class pulled back from Grassmannian $G(2, V)$ (resp. $G(3, V)$) and expressed as a symmetric polynomial in the chern roots of the universal sub-bundle. Also, for convenient calculation, we want that $i_{F_1}^* c_T$ and $i_{F_2}^* c_T$ have the same form. That means the equivariant class should not remember the rank two spaces. So we would not consider the σ part for the moment, and always let $i_{F_1}^* c_T = i_{F_2}^* c_T = H^b \cup \tau$.

The simplest case, H^b , is already interesting because our formula encodes all the information about intersection numbers on the flag manifold. Note that the Picard number of Flag variety $Fl(1, 2, 3, V)$ is three for $n \geq 3$. Since H , ψ_1 and ψ_2 are clearly linearly independent, they generate $Pic(Fl(1, 2, 3, V))$.

Proposition 2 All the products $H^b \cup \psi_1^i \cup \psi_2^j$ ($b+i+j = 3n-3$) can be determined from the formula by letting $i_{F_1}^* c_T = H^b$.

Proof. The the integral form of Theorem 2 with H^b being the class is:

$$\begin{aligned} & \int_{F_1} \frac{H^b}{tu(u-t)(t-\psi_1)(u-t-\psi_2)} + \int_{F_2} \frac{H^b}{tu(u-t)(u-\psi_1)(u-t+\psi_2)} \\ &= \int_{\mathbf{P}(V)} \frac{h^b}{(h+t)^{n+1}(h+u)^{n+1}}. \end{aligned}$$

By multiplying suitable power of $u-t$ (in fact $(u-t)^{3n-1-b}$), and inverting all terms in the denominator, we can see all the terms $H^b \cup \psi_1^i \cup \psi_2^j$ with fixed b are just combinations of coefficients of a known power series of t and u (from right hand side). So we just read off the coefficients one by one. Also note $H^b = 0$ for all $b > n$. So there are only $n+1$ cases. ■

Example 6 Some data for small n .

(1). $\dim(V) = n+1 = 3$, then $\dim(Fl(1, 2, 3, V)) = 3$. The intersections are:

$$\begin{array}{lll} H^3 = 0 & H^2\psi_1 = 1 & H^2\psi_2 = -2 \\ H\psi_1^2 = -3 & H\psi_1\psi_2 = 3 & H\psi_2^2 = 0 \\ \psi_1^3 = 6 & \psi_1\psi_2^2 = \psi_1^2\psi_2 = -3 & \psi_2^3 = 6 \end{array}$$

In this case $\psi_2 = -3H - 2\psi_1$.

(2). $\dim(V) = n+1 = 4$, then $\dim(Fl(1, 2, 3, V)) = 6$. Some intersections are:

$$\begin{array}{lll} H^3\psi_1^3 = 0 & H^3\psi_1^2\psi_2 = 1 & H^3\psi_1\psi_2^2 = -3 \\ H^3\psi_2^3 = 6 & H^2\psi_1^4 = 0 & H^2\psi_1^3\psi_2 = -4 \\ H^2\psi_1^2\psi_2^2 = 8 & H^2\psi_1\psi_2^3 = -8 & H^2\psi_2^4 = 0 \end{array}$$

1.2.3 Dealing with error terms

Note that in the above case, there are no error terms because in this case $c_T = (u_*(h^T))^b$. This is not the case when $i_{F_1}^* c_T = H^b \cup \tau$. But as in the rank 2 case, an approximation outside the boundary divisors D_1 and D_2 can be made, and fortunately as in that case, the error terms (there are four) will not contribute to the computations of key terms.

Consider the projection

$$\mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \rightarrow \mathbf{P}(V \otimes W^*).$$

Note that there exists a rational map (evaluation map)

$$\alpha : \mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \dashrightarrow \mathbf{P}(V)$$

by sending an element in $\mathbf{P}(W)$ to its image through the corresponding element in $V \otimes W^*$. The map α is not defined on any kernel of an element from the lower rank locus M_2 .

To resolve it, first blow up the $\mathbf{P}^1$ -bundle

$$\{(\phi, \ker(\phi)) \subset \mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \mid \phi \in M_1\}$$

over the Segre embedding M_1 , and we get $\tilde{N}_1$. Now in $\tilde{N}_1$ the locus where the induced rational map is undefined is the section of M_2' (since it has rank 2). Blow up the section again, and we get $\tilde{N}_2$, which has an evaluation morphism (denoted by e) to $\mathbf{P}(V)$.

As the result we get the following diagram (on top of our Main Diagram):

$$\begin{array}{ccccc} \tilde{N}_2 & \xrightarrow{\tilde{u}_2} & \tilde{N}_1 & \xrightarrow{\tilde{u}_1} & \mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \\ \downarrow \pi' & & \downarrow & & \downarrow \\ N_2 & \xrightarrow{u_2} & N_1 & \xrightarrow{u_1} & \mathbf{P}(V \otimes W^*) \end{array}$$

Proposition 3 There is a natural equivariant map:

$$\Phi : \pi'_* e^* \mathcal{O}_{\mathbf{P}(V)}(1) \rightarrow W^* \otimes u^*(\mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1))$$

which is an isomorphism away from $D_1 \cup D_2 \subset N_2$.

Proof. Since we have

$$e^* \mathcal{O}_{\mathbf{P}(V)}(1) = \tilde{u}^*(\mathcal{O}_{\mathbf{P}(V \otimes W^*) \times \mathbf{P}(W)}(1, 1))(-E'_1 - E'_2)$$

where $\tilde{u} = \tilde{u}_1 \circ \tilde{u}_2$ and E'_i are two exceptional divisors on $\tilde{N}_2$. So there is a natural map Φ . Note that these two bundles are the same on where the original rational

map α is defined. So Φ is not an isomorphism only on the loci that are pull-back of those low rank loci — the full preimage of M_2 , which is exactly $D_1 \cup D_2$. $\blacksquare$

Corollary 2 If $i_{F_1}^* c_T = \tau(q_1, q_2, q_3)$, then $u_*(c_T + c'_T) = \tau(h, h + t, h + u)$. Here c'_T is push-forward of an equivariant class on $D_1 \cup D_2$.

Proof. This follows from above proposition since

- 1). The equivariant chern roots of $W^* \otimes \mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1)$ are h , $h + t$ and $h + u$.
- 2). The restriction of $\pi'_* e^* \mathcal{O}_{\mathbf{P}(V)}(1)$ on $Fl(1, 2, 3, V)$ is the pullback of the dual universal subbundle S^* on $G(3, V)$.

To see 2), first notice that $S^* = (\pi_1)_* \pi^*(\mathcal{O}_{\mathbf{P}(V)}(1))$. Also the projection $Fl(1, 3, V) \rightarrow G(3, V)$ is just the projectivized bundle $\mathbf{P}(S^*)$. While the projection $N_2|_{\pi^{-1}(Fl(1, 2, 3, V))} \rightarrow Fl(1, 2, 3, V)$ is not a $\mathbf{P}^2$ -bundle (each fiber contains three components — the original $\mathbf{P}(W)$ plus two copies of $\mathbf{P}^2$ from each blow-up), it has a morphism to the $\mathbf{P}^2$ bundle over $Fl(1, 2, 3, V)$ (the fiber product). Moreover, the evaluation map e factors through π via the following diagram from where our claim follows:

$$\begin{array}{ccccc}
 N_2|_{\pi^{-1}(Fl(1, 2, 3, V))} & \xrightarrow{e} & & & \mathbf{P}(V) \\
 \downarrow \pi'| & & & & \downarrow = \\
 Fl(1, 2, 3, V) \times_{G(3, V)} Fl(1, 3, V) & \longrightarrow & Fl(1, 2, 3, V) & \xrightarrow{e_1} & \mathbf{P}(V) \\
 \downarrow & & \downarrow & & \\
 Fl(1, 3, V) & \xrightarrow{\pi_1} & G(3, V) & &
 \end{array}$$

$\blacksquare$

Unlike the rank 2 case, we first put the error terms on the left side of the formula. The reason is that we can see quickly from their denominators that they will not contribute to the terms we want to compute.

Lemma 1 The error terms in the formula are of the form:

- (1) from D_1 : $Er_1 = \frac{i_{F_1}^* c'_T}{tu(u-t)(u-t-\psi_2)}$ (similarly for Er'_1 on F_2) and
- (2) from D_2 : $Er_2 = \frac{i_{F_2}^* c'_T}{tu(u-t)(t-\psi_1)}$ (similarly for Er'_2 on F_2).

Proof. We just need to see the change of denominators.

Since Er_1 supports on D_1 , the denominator is

$$\begin{aligned} c_{top}^T(N_{F_1/D_1}) &= c_{top}^T(N_{F_1/D_1 \cap D_2}) \cdot c_{top}^T(N_{D_1 \cap D_2/D_1}) \\ &= ut(u-t)c_{top}^T(N_{D_2/N_2}) = ut(u-t)(u-t-\psi_2). \end{aligned}$$

Similarly since Er_2 supports on D_2 , that denominator is

$$\begin{aligned} c_{top}^T(N_{F_1/D_2}) &= c_{top}^T(N_{F_1/D_1 \cap D_2}) \cdot c_{top}^T(N_{D_1 \cap D_2/D_2}) \\ &= ut(u-t)c_{top}^T(N_{D_1/N_2}) = ut(u-t)(t-\psi_1). \end{aligned}$$

■

Theorem 3 (Schubert Formula 2 on $Fl(1, 2, 3, V)$)

$$\begin{aligned} &\pi_*\left(\frac{\pi^*(h^b) \cup \tau}{tu(u-t)(t-\psi_1)(u-t-\psi_2)}\right) + \pi_*\left(\frac{\pi^*(h^b) \cup \tau}{tu(u-t)(u-\psi_1)(u-t+\psi_2)}\right) \\ &= \frac{\tau(h, h+t, h+u)}{(h+t)^{n+1}(h+u)^{n+1}} + \text{irrelevant terms} \end{aligned}$$

Proof. The theorem says that all the terms $H^b \cup \psi_1^i \cup \psi_2^j \cup \tau$ can be determined by looking at the corresponding term (with same denominator) on the right side of the formula. That is, error terms do not contribute at all.

To see this, look at the formula

$$\begin{aligned} &\pi_*\left(\frac{H^b \cup \tau}{tu(u-t)(t-\psi_1)(u-t-\psi_2)}\right) + \pi_*\left(\frac{H^b \cup \tau}{tu(u-t)(u-\psi_1)(u-t+\psi_2)}\right) \\ &+ \pi_*(E_1) + \pi_*(E'_1) + \pi_*(E_2) + \pi_*(E'_2) = \frac{\tau(h, h+t, h+u)}{(h+t)^{n+1}(h+u)^{n+1}} \end{aligned}$$

It suffices to show all $H^b \cup \psi_1^i \cup \psi_2^j \cup \tau$ terms come with denominator $t^a u^b$ where $a \geq 2$ and $b \geq 2$, and all error terms have are either $O(\frac{1}{t})$ or $O(\frac{1}{u})$.

We check these by straightforward computations. For the regular two terms, their combination cancels all of $O(\frac{1}{t})$ and $O(\frac{1}{u})$ in the denominators (since there are more $u-t$ exponents in the denominators, they must form some power of $u-t$ to cancel that, which leads to the desired form). For the error terms: those from

D_1 do not count because of dimension reasons (since they collapse to M_1), and one check those from D_2 have either $O(\frac{1}{t})$ or $O(\frac{1}{u})$. ■

We also need to determine which class on $Fl(1, 2, 3, V)$ will not change the degree of τ after pushing forward to $\mathbf{P}(V)$. That is, which class has degree one on the fiber of the projection $Fl(1, 2, 3, V) \rightarrow G(3, V)$, which is $Fl(1, 2, 3)$.

Lemma 2 Degree of $H^2 \cup \psi_1$ is one on the fiber of the projection.

Proof. This can be seen geometrically, since ψ_1 , after restricts to a fiber, is just the relative canonical class $c_1(\mathcal{O}(1))$ of a $\mathbf{P}^1$ bundle (ψ_2 restricts to the trivial class). We can also check this by computations in the following example. ■

Example 7 If $\text{codim}(\tau) = 3n - 6$ (maximum), here are some computations:

$b = 3$: $H^3 \cup \tau = 0$ (doesn't dependent on τ)

$b = 2$: $H^2 \cup \psi_2 \cup \tau = -2H^2 \cup \psi_1 \cup \tau$, and

$$e_*(H^2 \cup \psi_1 \cup \tau) = \text{coeff. of } \frac{1}{-2t^2u^2} \text{ in } \int_{\mathbf{P}(V)} \frac{h^2 \cdot \tau(h, h+t, h+u) \cdot (u-t)^2}{(h+t)^{n+1}(h+u)^{n+1}}.$$

In particular, if $\tau = \sigma_3^{n-2}$, then $H^2 \cup \psi_1 \cup \sigma_3^{n-2} = 1$, $H^2 \cup \psi_2 \cup \sigma_3^{n-2} = -2$.

1.2.4 Relative version of $FL(1, 2, 3, V)$.

Here is the new setting: Let $f : V \rightarrow X$ be a rank- $(n+1)$ vector bundle over a projective variety. Let $\mathbf{P}(V) \rightarrow X$ be the projective bundle. Still let W be a vector space of dimension $m = 3$. We do the same diagram as before, only to remember everything is on relative base (i.e., over X):

$$\begin{array}{ccc} N_2 & \xrightarrow{u} & \mathbf{P}(V \otimes W^*) \\ \uparrow i & & \uparrow j \\ FL(1, 2, 3, V) & \xrightarrow{\pi} & \mathbf{P}(V) \\ \downarrow & & \downarrow f \\ G(3, V) & & X \end{array}$$

The first thing changed is the equivariant Euler class of $\mathbf{P}(V)$. Let α_i be the chern roots of V ($i = 1, 2, \dots, n+1$).

Proposition 4 The equivariant Euler class of $\mathbf{P}(V)$ in $\mathbf{P}(V \otimes W^*)$ is

$$\prod_{i=1}^{n+1} (h + \alpha_i + t) \cdot \prod_{i=1}^{n+1} (h + \alpha_i + u).$$

Proof. Again this comes from computation of the Euler sequences for the tangent bundles to $\mathbf{P}(V)$ and $\mathbf{P}(V \otimes W^*)$. Use the same diagram in the proof of Part (5) of Proposition 1, only to notice its relative setting. That is, $c_t(V) = \prod_{k=1}^{n+1} (1 + \alpha_k)$. ■

The following is the basic formula in this relative setting:

Theorem 4 (Relative Version of Schubert Formula 1) Let c_T be an equivariant class on N_2 , then:

$$\begin{aligned} & \pi_* \left(\frac{i_{F_1}^* c_T}{tu(u-t)(t-\psi_1)(u-t-\psi_2)} \right) + \pi_* \left(\frac{i_{F_2}^* c_T}{tu(u-t)(u-\psi_1)(u-t+\psi_2)} \right) \\ &= \frac{j^* u_* c_T}{\prod_{i=1}^{n+1} (h + \alpha_i + t) \cdot \prod_{i=1}^{n+1} (h + \alpha_i + u)}. \end{aligned}$$

As in standard case (where V is a vector space instead of a vector bundle), the above formula can give us many applications by computing various intersections. Pushing forward $H_b \cup \psi_1^i \cup \psi_2^j$ to $\mathbf{P}(V)$ then to X , we can compute their intersection (in the form of chern roots) by inverting the denominator of the right hand side and looking at the appropriate coefficients. When error terms come, note that the approximation and error terms are the same as before, and more importantly, error terms do not contribute when the class involve τ (pull back from $G(3, V)$). So we can also do relative Schubert calculus by pushing forward $H_b \cup \psi_1^i \cup \psi_2^j \cup \tau$ to $\mathbf{P}(V)$ then to X .

There are many interesting computations when τ varies among all symmetric polynomials. We will do a few applications in the next section.

1.3 Some Applications

We will do two applications by using our formulas from the previous section to illustrate the effectiveness of our methods. The point is to translate the original problems into some computations of certain coefficients. That means we need to know what is the base variety X , the vector bundle V along with its characteristic classes, and which class τ pulled-back from the Grassmannian $G(3, V)$.

1.3.1 Hypersurfaces that contain planes

Let X be the projective space of degree d hypersurfaces in $\mathbf{P}(V)$ where $\dim(V) = n+1$. Then $X = \mathbf{P}^{\binom{n+d}{d}-1}$. Let $V(1) = V \otimes \mathcal{O}_X(1)$ be the direct sum of $n+1$ copies of $\mathcal{O}_X(1)$. Consider the incidence variety $I_{d,n} = \{(\mathbf{P}^2, Y) \in \mathbf{P}(V) \times X \mid \mathbf{P}^2 \subset Y\}$. We would like to compute the class of $(p_2)_*(I_{d,n})$ (which we call $P_{d,n}$) as an element in $A_*(X)$. Here p_i is the projection to the i -th factor ($i = 1, 2$). Since $A_*(X)$ is generated by $l = c_1(\mathcal{O}_X(1))$, we know the answer will be in the form of $q_{d,n} \cdot l^k$ where $q_{d,n}$ is the degree. So we have to find a suitable τ .

Let S be the universal subbundle in $G(3, V)$. Let $f : \mathbf{P}(V) \rightarrow X$ and $\pi : Fl(1, 2, 3, V) \rightarrow \mathbf{P}(V)$ be the projection (see diagram in section 1.2.4).

Proposition 5 $P_{d,n} = f_*\pi_*(H^2\psi_1\tau)$ where $\tau = c_{top}(Sym^d(S^*))$.

Proof. First observe that the incidence variety $\{(\mathbf{P}^2, \mathbf{P}^n) \mid \mathbf{P}^2 \subset \mathbf{P}^n\}$ is just the Grassmannian $G(3, V)$ (planes in $\mathbf{P}^n$). For a generic degree d hypersurfaces Y , let $I_Y := \{\mathbf{P}^2 \subset Y\}$ be the incidence variety. Then the natural map $I_Y \rightarrow G(3, V)$ is an embedding.

If S is the tautological rank 3 subbundle on $G(3, V)$, then the fiber S_P over a plane P is the three-dimensional subspace of V whose projectivization is P . An equation of Y gives a section s of the rank $r := \binom{d+2}{2}$ -bundle $Sym^d(S^*)$. Then I_Y is just the zero scheme $Z(s)$ of s .

Let $C_Z G$ be the normal cone of the embedding $i : Z(s) \rightarrow G(3, V)$. Then the above construction produces the class $s^*[C_Z G] \in A_{3(n-2)-r}(I_Y)$. A standard

result ([12]) shows that $i_*(s^*[C_Z G]) \in A_{3(n-2)-r}(G(3, V))$ is just the Euler class $c_{top}(Sym^d(S^*) \cap G(3, V))$.

Since we want to pull back the appropriate class τ (in this case $c_{top}(Sym^d(S^*))$) from Grassmannian $G(3, V)$ to flag variety $Fl(1, 2, 3, V)$ and then push it forward to $\mathbf{P}(V)$ (with $H^2\psi_1$ to keep the degree), we are done if the expected codimension is zero, i.e., $3(n-2) = r$. In this case the $\mathcal{O}(1)$ simply does not matter, and $p_2^*(Y) = I_Y$ for generic Y . So $P_{d,n}$ satisfies the proposition.

In general, if the expected codimension $3(n-2) - r$ is positive, then we should get a class in the form $q_{d,n}l^{3(n-2)-r}$. Follow the same principle as above, one checks that the twist $\mathcal{O}(1)$ produces the hyperplane section l with the correct codimension. ■

The following lemma is straightforward.

Lemma 3 Let p_1, p_2 , and p_3 be chern roots of S^* , then

$$c_{top}(Sym^d(S^*)) = \prod_{i+j+k=d} (ip_1 + jp_2 + kp_3).$$

Now we can give the formula for our computation:

Theorem 5 All notations as above:

$$P_{d,n} = \text{coeff. of } \frac{1}{-2t^2u^2} \text{ in } \int_{\mathbf{P}(V)} \frac{h^2 \cdot \prod_{i+j+k=d} (dh + jt + ku + l) \cdot (u - t)^2}{(h + t)^{n+1}(h + u)^{n+1}}.$$

Proof. Similarly as in Example 7, we know that

$$\pi_*(H^2\psi_2\tau) = \text{coeff. of } \frac{1}{t^2u^2} \text{ in } \frac{h^2 \cdot \tau(h, h + t, h + u) \cdot (u - t)^2}{\prod_{i=1}^{n+1} (h + \alpha_i + t) \cdot \prod_{i=1}^{n+1} (h + \alpha_i + u)}$$

and

$$\pi_*(H^2\psi_1\tau) = \frac{-1}{2} \pi_*(H^2\psi_2\tau)$$

Note that since V is not a trivial bundle, we use the formula in the relative version. In fact, since V is just a trivial bundle twisted by a line bundle $\mathcal{O}(1)$, we

can make our computation a little easier by assume V is a trivial bundle and adjust the term $c_{top}(Sym^d(S))$. Namely, we would add the hyperplane class to each linear term of $c_{top}(Sym^d(S^*))$, such that

$$c_{top}(Sym^d(S^*)) = \prod_{i+j+k=d} (ip_1 + jp_2 + kp_3 + l).$$

Now instead dealing with all Segre class of V , we see that there is only one non-zero Segre class—the trivial class $S_0 = f^*(h^n)$. So all we need to do is just do integral on $\mathbf{P}(V)$ using the formula from Theorem 1. One can easily check that this way of computation has the same answer as if one use the relative version throughout the whole computation.

But now the theorem follows immediately once we simplify the formula at the beginning of the proof: First by assuming all $\alpha_i = 0$ we get the desired denominator. Then we substitute τ with the new formula of $c_{top}(Sym^d(S^*))$ above and notice that

$$ih + j(h + t) + k(h + u) + l = dh + jt + ku + l$$

since $i + j + k = d$. ■

Remark: Notice that the right side of the formula, as usual, is homogeneous. Since $\text{rank}(Sym^d(S)) = \binom{d+2}{2}$, it is easy to see the $\frac{1}{t^2 u^2}$ term come as $l^{\binom{d+2}{2} - 3(n-2)}$ term, i.e., the exponent is $\text{rank}(Sym^d(S) - \dim(G(3, V)))$. So depending on the values of d and n , there are three cases to consider:

Case 1. $\binom{d+2}{2} = 3(n-2)$.

This is the special case in which the answer is $q_{d,n} \cdot l^0$. This means the general hypersurface in this case contains finitely many projective planes, and the degree, $q_{d,n}$, is the number of planes. Since there is no l term appears in the answer, we can actually forget the $\mathcal{O}_X(1)$ part and just let V be the trivial bundle. We give some examples below for some d (and n).

Example 8 $d = 1$. Then $n = 3$. This is the trivial case $\mathbf{P}^2 \subset \mathbf{P}^3$. Natural, the formula gives the answer $q_{1,3} = 1$.

Example 9 $d = 2$. Then $n = 4$. The first nontrivial case turns out to be trivial (at least from the result). Here we are considering quadric threefolds in $\mathbf{P}^4$. But the following claim says there will not be finitely many planes.

Claim: If a quadric hypersurface in $\mathbf{P}^4$ contains a plane, then it contains a one-dimensional family of planes.

In fact, we can always diagonalize the corresponding quadratic form. If the rank is not maximal, one can easily construct a one-dimensional family of planes. On the other hand, if it contains a plane, say with equation $x_3 = x_4 = 0$, then clearly the corresponding 5×5 matrix can not have maximal rank.

Since the dimension count claims there are only finitely many planes in a general quadric threefold in $\mathbf{P}^4$, the only possible answer is $q_{2,4} = 0$, which is exactly what the computation shows.

Example 10 $d > 2$. Note that d can not be a multiple of 3. Unfortunately for the next case ($d = 4, n = 7$) the answer is already very large for a hand computation. We give some data here:

$$d = 4 \quad n = 7 \quad q_{4,7} = 3297280$$

$$d = 5 \quad n = 9 \quad q_{5,9} = 420760566875$$

$$d = 7 \quad n = 14 \quad q_{7,14} = 279101475496912988004267637$$

$$d = 8 \quad n = 17 \quad q_{8,17} = 1876914105621812001806757234042994688$$

Case 2. $\binom{d+2}{2} > 3(n-2)$.

In this case the class $P_{d,n}$ has positive codimension $\binom{d+2}{2} - 3(n-2)$. We list here the following data array.

d	n	$\binom{d+2}{2} - 3(n-2)$	$P_{d,n}$
1	3	0	1
2	3	3	$20l^3$
3	3	7	$220l^7$
4	3	12	$1540l^{12}$
5	3	18	$7770l^{18}$
6	3	25	$30856l^{25}$
2	4	0	0
3	4	4	$3675l^4$
4	4	9	$293300l^9$
5	4	15	$9364075l^{15}$
3	5	1	$3402l$
4	5	6	$8754732l^6$
5	5	12	$2547516517l^{12}$
4	6	3	$31886848l^3$
5	6	9	$169739006052l^9$
4	7	0	3297280
5	7	6	$2502288860940l^6$
5	8	3	$5573769695835l^3$
5	9	0	420760566875
7	14	0	279101475496912988004267637
8	17	0	1876914105621812001806757234042994688

Case 3. $\binom{d+2}{2} < 3(n-2)$.

In this case the formula gives the answer 0 — for dimension reasons. For each hypersurface in this case, it contains some positive dimensional families of planes.

This is not the whole story. Since generally each hypersurfaces in this case contains family of planes. We can ask the opposite question: what is the class of planes satisfying certain linear conditions, i.e., meeting certain codimension subspaces. We will be specially interested in when the answer is finite — number of the planes intersect certain codimension subspaces on a general hypersurface in such cases.

To be successful, those hypersurfaces can not have a family of planes of dimension large than $n-2$ (e.g., hyperplanes). Another way to look at is: from the right side of the formula in the Theorem above, the part of all terms containing t or u has degree $\binom{d+2}{2} - 1 - 2n$. To assure it can have a term of $\frac{1}{t^2 u^2}$, we see that its degree is at least -4 . So combine the condition in this case we see that d and n must satisfy the following inequality:

$$2n - 3 \leq \binom{d+2}{2} \leq 3n - 6$$

We simply multiply extra h^a to compute the corresponding invariant. Notice a is actually fixed to assure the answer is a constant. In fact $a = 3(n-2) - \binom{d+2}{2}$. Notice that $a \leq n-3$. The reason is we already have h^2 in the formula (which means two-plane intersects codimension two subspaces— actually no condition) and a term dh (from $c_{top}^T(Sum^d(S))$) which means the class of hypersurface it self. So we could at most multiply h^{n-3} to imply the extra linear conditions.

Now we can compute for suitable situations. For example if the number of planes passing through a general point on the hypersurface is finite, we must have $\binom{d+2}{2} = 2n - 3$ and we will multiply h^{n-3} . Some computations are as follows:

Example 11 Here are some numbers in this case. Notice that similarly as in Case 1, d and n are somewhat related. The codim column means linear condition (meet which codimension subspaces), the number is simply $3(n-2) - \binom{d+2}{2} + 3$. We will specifically mention 2-plane, lines or points.

n	d	$3(n-2) - \binom{d+2}{2}$	codimension	$P_{d,n}$
6	3	2	line	1134
7	4	0	codim 3	3297280
8	4	3	2-plane	5969920
9	4	6	point	258048
9	5	0	codim 3	420760566875
10	5	3	codim 4	1206991940000
11	5	6	2-plane	216419448000
12	5	9	point	2772576000
12	6	2	codim 5	6304179785228043264

1.3.2 Porteous formula

One simple and very useful case is when τ is or contains high power of σ_3 , since it will make the computation of the right side much easier.

The following is the Porteous formula when τ is a power of σ_3 . Similar formula holds for general cases.

The theorem uses the same set up as in the relative version of $Fl(1, 2, 3, V)$ case: Let $f : V \rightarrow X$ be a rank- $(n+1)$ vector bundle over a smooth projective variety X , with chern roots α_i ($i = 1, 2, \dots, n+1$).

Theorem 6 Let $\tau = \sigma_3^k$ where $k \geq n-2$, then

$$f_* e_*(H^2 \cup \psi_1 \cup \tau) = \begin{vmatrix} s_{k-n+2} & s_{k-n+3} & s_{k-n+4} \\ s_{k-n+1} & s_{k-n+2} & s_{k-n+3} \\ s_{k-n} & s_{k-n+1} & s_{k-n+2} \end{vmatrix}.$$

Here s_i 's are the Segre classes of V on X .

Proof. Since error terms do not contribute, by looking at the correct term on the left side of the formula ($H^2 \cup \tau$ being the numerator) after multiplying $(u - t)^2$, we see that

$$e_*(H^2 \cup \psi_1 \cup \tau) = \text{coefficient of } \frac{1}{-2t^2u^2} \text{ in } \frac{h^2 \cdot [h \cdot (h + t) \cdot (h + u)]^k \cdot (u - t)^2}{\prod_{i=1}^{n+1}(h + \alpha_i + t) \cdot \prod_{i=1}^{n+1}(h + \alpha_i + u)}.$$

Now denote $s_{(y^{-1})}(\pi^*V) = 1 + y^{-1}s_1(\pi^*V) + y^{-2}s_2(\pi^*V) + \dots$, then straight computation shows that

$$\frac{1}{\prod_{i=1}^{n+1}(h + \alpha_i + t) \cdot \prod_{i=1}^{n+1}(h + \alpha_i + u)} = \frac{s_{(h+t)^{-1}}(\pi^*V)}{(h + t)^{n+1}} \cdot \frac{s_{(h+u)^{-1}}(\pi^*V)}{(h + u)^{n+1}}$$

So the right side, after simplifying, is

$$h^{k+2} \cdot (u - t)^2 \cdot [(h + t)^{k-n-1} s_{(h+t)^{-1}}(\pi^*V)] \cdot [(h + u)^{k-n-1} s_{(h+u)^{-1}}(\pi^*V)]$$

whose $\frac{1}{t^2u^2}$ term comes from:

$$h^{k+2} \cdot u^2 \cdot \left(\frac{1}{t^2} \text{ term from } A\right) \cdot \left(\frac{1}{u^4} \text{ term from } B\right),$$

$$h^{k+2} \cdot (-2ut) \cdot \left(\frac{1}{t^3} \text{ term from } A\right) \cdot \left(\frac{1}{u^3} \text{ term from } B\right), \text{ and}$$

$$h^{k+2} \cdot t^2 \cdot \left(\frac{1}{t^4} \text{ term from } A\right) \cdot \left(\frac{1}{u^2} \text{ term from } B\right).$$

Here we denote $A := (h + t)^{k-n-1} s_{(h+t)^{-1}}(\pi^*V)$ and $B := (h + u)^{k-n-1} s_{(h+u)^{-1}}(\pi^*V)$.

It turns out to be

$$\begin{aligned} & h^{k+2} \cdot [2(s_{k-n+3} - 3s_{k-n+2}h + 3s_{k-n+1}h^2 - s_{k-n}h^3) \\ & \cdot (s_{k-n+1} - s_{k-n}h) - 2(s_{k-n+2} - 2s_{k-n+1}h + s_{k-n}h^2)^2]. \end{aligned}$$

Push it forward to X and notice that $f_*(h^k) = s_{k-n}$, the theorem follows. $\blacksquare$

1.4 General Cases: $Fl(1, 2, 3, \dots, m, V)$.

In this section we will discuss general cases when $m \geq 3$, consequently we will do Schubert calculus on $Fl(1, 2, 3, \dots, m, V)$ using equivariant cohomology and localization theorem.

We will use a similar basic diagram below (as in Section 2), though the detailed Main Diagram is much more complicated.

$$\begin{array}{ccc}
 N_{m-1} & \xrightarrow[u]{} & \mathbf{P}(V \otimes W^*) \\
 \uparrow i & & \uparrow j \\
 F_I & \xrightarrow[\pi]{} & \mathbf{P}(V) = \mathbf{P}(V) \times (w_i^*)
 \end{array}$$

1.4.1 Main diagram and explanation

Fix a basis of W as $\langle w_1, w_2, \dots, w_m \rangle$, with dual basis $\langle x_1, x_2, \dots, x_m \rangle$.

We need to do a series of blow-ups — $(m-1)$ times, that is. Although the Main Diagram looks a little complicated, it becomes clear if one notice the following properties:

1. Let M_k^l denote the proper transform of the rank- k locus M_k in $\mathbf{P}(V \otimes W^*)$ after l blow-ups. Recall that $\text{codim}(M_k) = (m-k)(n+1-k)$. Then notice that M_k is singular along M_{k-1} , so do M_k^l along M_{k-1}^l for $l < k$. Also M_k^k , the exceptional divisor for k -th blow up, is smooth. After that, all M_k^l just the pull back of the divisor M_k^{l-1} for $l > k$.

2. Notice all the squares below the diagonal in the Main Diagram are fiber squares, and all the squares above the diagonal in the Main Diagram are blow-ups. This actually uniquely determines the whole diagram.

3. We will be interested, as in $m = 3$ case, the fixed-points loci in the final blow-up, which turn out to be $m!$ components that are all isomorphic to $Fl(1, 2, 3, \dots, m, V)$. They are all contained in the intersection of all M_k^{m-1} , which turns out to be $Fl(1, 2, 3, \dots, m, V) \times Fl(1, 2, 3, \dots, m-1, W^*)$.

4. For each step (after k -th blow up), all the diagram, formulas and computation are the same as though $m = k$. So our formulas and proofs can be seen inductively through each step.

5. To make our formulas more symmetric, we will use the group action which is homogeneous on each coordinate.

The following is the Main Diagram in general cases:

Main Diagram:

$$\begin{array}{ccccc}
N_{m-1} & \cdots \longrightarrow & N_k & \cdots \longrightarrow & \mathbf{P}(V \otimes W^*) \\
\uparrow & & \uparrow & & \uparrow \\
M_{m-1}^{m-1} & \cdots \longrightarrow & M_{m-1}^k & \cdots \longrightarrow & M_{m-1} \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\bigcap_{\alpha=k+1}^{m-1} M_{\alpha}^{m-1} & \cdots \longrightarrow & M_{k+1}^k & \cdots \longrightarrow & M_{k+1} \\
\uparrow & & \uparrow & & \uparrow \\
\bigcap_{\alpha=k}^{m-1} M_{\alpha}^{m-1} & \cdots \longrightarrow & \bigcap_{\beta=k}^{k+1} M_{\beta}^k & \cdots \longrightarrow & M_k \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\bigcap_{\alpha=1}^{m-1} M_{\alpha}^{m-1} & \cdots \longrightarrow & \bigcap_{\beta=1}^{k+1} M_{\beta}^k & \cdots \longrightarrow & M_1 \\
\uparrow & & \uparrow & & \uparrow \\
Fl(1, 2, \dots, m, V) \cdots & \longrightarrow & Fl(1, 2, \dots, k, V) \cdots & \longrightarrow & \mathbf{P}(V)
\end{array}$$

Notations: Almost all notations and terms are similar when compared to the Main Diagram in Section 2.

All vertical maps are embeddings.

We still denote $u : N_{m-1} \rightarrow \mathbf{P}(V \otimes W^*)$, $i_I : Fl(1, 2, \dots, m, V) \rightarrow N_{m-1}$ (see next section for the definition of index I), $j : \mathbf{P}(V) \rightarrow \mathbf{P}(V \otimes W^*)$, and $\pi : Fl(1, 2, \dots, m, V) \rightarrow \mathbf{P}(V)$.

Denote by $D_k = M_k^{m-1}$ ($k = 1, 2, \dots, m-1$), the preimage of exceptional divisors in all blow-ups.

1.4.2 Main formulas

Now consider the standard (homogeneous) torus action of $T = (\mathbf{C}^*)^m$ on W^* :

$$(t_1, t_2, \dots, t_m) \cdot (x_1, x_2, \dots, x_m) \rightarrow (t_1 x_1, t_2 x_2, \dots, t_m x_m).$$

This action induces actions through each step of our diagram, such that all maps in the diagram are T -equivariant.

Notice that

$$\bigcap_{\beta=1}^k M_{\beta}^k = Fl(1, 2, \dots, k+1, V) \times Fl(1, 2, \dots, k, W^*).$$

In particular the intersection of the $m-1$ exceptional divisors on N_{m-1} is:

$$\bigcap_{k=1}^{m-1} D^k = Fl(1, 2, \dots, m, V) \times Fl(1, 2, \dots, m-1, W^*).$$

Let $I = \{i_1, i_2, \dots, i_m\}$ be a permutation of m numbers. Let

$$\Lambda_I := [(x_{i_1}) \subset (x_{i_1}, x_{i_2}) \subset \dots \subset (x_{i_1}, x_{i_2}, \dots, x_{i_m}) = W^*]$$

be the $m!$ fixed points in the full flag variety $Fl(1, 2, \dots, m-1, W^*)$. Then the fixed-point loci for the action of T in N_{m-1} are all contained in the intersection of exceptional divisors and are of the form $F_I := \Lambda_I \times Fl(1, 2, \dots, m, V)$. It is a disjoint union of $m!$ components. For each fixed-point locus $\mathbf{P}(V) = \mathbf{P}(V) \times (w_i^*)$ in $\mathbf{P}(V \otimes W^*)$, there are $(m-1)!$ fixed-point components F_I in N_{m-1} (with $i_1 = i$ for index I) that map to it. So as in section 2 we can apply the Localization theorem to relate residues through reduction to the projective space $\mathbf{P}(V)$.

Let $h \in H^*(\mathbf{P}(V), \mathbf{Z})$ be the hyperplane class. Let $H := e^*(h)$ and define the equivariant chern class $h^T := c_1^T(\mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1))$.

Proposition 6 From the above setting:

- (1) $\mathbf{P}(V) \subset \mathbf{P}(V \otimes W^*)$ is a component of the fixed-point locus.
- (2) F_I 's are all fixed-point components in N_{m-1} that map to $\mathbf{P}(V)$.
- (3) H extends to the equivariant class $u^*(h^T)$.
- (4) The equivariant Euler class of F_I in N_{m-1} is

$$\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s).$$

Here ψ_s 's are the relative canonical classes $c_1(\omega_{\pi_s})$, where $\pi_s : Fl(1, 2, \dots, m, V) \rightarrow Fl(1, 2, \dots, \hat{s}, \dots, m, V)$ are forgetful maps ($1 \leq s \leq m-1$).

(5) The equivariant Euler class of $\mathbf{P}(V) = \mathbf{P}(V) \times (w_i^*)$ in $\mathbf{P}(V \otimes W^*)$ is

$$\prod_{s \neq i} (h + t_s - t_i)^{n+1}.$$

Proof. (1)-(3) are straightforward.

For simplicity we assume $I = \{1, 2, \dots, m-1\}$ in part (4) and (5).

For (4) We need to calculate the following terms:

Step 1:

$$c_{top}^T(N_{F_I / \bigcap_{k=1}^{m-1} D_k}) = c_{top}^T(N_{\{0_I\} / Fl(1, 2, \dots, m-1, W^*)})$$

which is

$$\prod_{1 \leq j < k \leq m} (t_k - t_j).$$

We prove this by induction. Consider the following $(m-1)$ -dimensional subspace $W_{m-1} = \langle e_0, e_1, \dots, e_{m-2} \rangle$. Then the embedding

$$\{0\} \rightarrow Fl(1, 2, \dots, m-2, W_{m-1}^*) \rightarrow Fl(1, 2, \dots, m-1, W^*)$$

shows that

$$\begin{aligned} c_{top}^T(N_{\{0\} / Fl(1, 2, \dots, m-1, W^*)}) &= \\ c_{top}^T(N_{\{0\} / Fl(1, 2, \dots, m-2, W_{m-1}^*)}) \cdot c_{top}^T(N_{Fl(1, 2, \dots, m-2, W_{m-1}^*) / Fl(1, 2, \dots, m-1, W^*)} | \{0\}) \end{aligned}$$

But the first term is

$$\prod_{0 \leq j < k \leq m-2} (t_k - t_j)$$

by induction (see Proposition 1 in section 2 for the case $m = 3$). The second term is actually $c_{top}^T(N_{\{(0, \dots, 0, *)\} / (\mathbf{P}^{m-1})})$ (think it as $((\mathbf{P}^{m-1})^*)^* = \mathbf{P}^{m-1}$). Since the T -action on $\mathbf{P}^{m-1}$ has weight $(t_{m-1}, t_{m-1} - t_1, \dots, t_{m-1} - t_{m-2})$ at $(0, \dots, 0, *)$, we see that the equivariant Euler class

$$c_{top}^T(N_{\{(0, \dots, 0, *)\} / (\mathbf{P}^{m-1})}) = \prod_{k=1}^{m-2} (t_{m-1} - t_k)$$

Combine these terms gets the desired formula.

Step 2: $c_{top}^T(N_{D_s / N_{m-1}} |_{F_I})$ which is $t_{s+1} - t_s - \psi_s$.

We simply notice D_k is pull back of M_k^k . So working in the $(k-1)$ -th step of the Basic Diagram and restricting the normal bundle to one fixed-point locus $Fl(1, 2, \dots, k+1, V)$, we can see easily that the normal bundle we need to compute is just $c_1^T(\mathcal{O}(-1))$ of the horizontal part in the following diagram:

$$\begin{array}{ccc} Fl(1, 2, \dots, k+1, V) & \xrightarrow[e_k]{} & Fl(1, 2, \dots, k, V) \\ \downarrow \pi_k & & \\ Fl(1, 2, \dots, k-1, k+1, V) & & \end{array}$$

Define $\psi_k := c_1^T(\mathcal{O}_{e_k}(1))$ and check that $\psi_k = c_1^T(\omega\pi_k)$ we get the nonequivariant part, the equivariant part come from the proof in Step 1. Pull all the way back to N_{m-1} we have our formula.

Part (5) Use the same diagram as in part (5) of the proof of Proposition 1 in Chapter 2, we see that the chern polynomials satisfy

$$\begin{aligned} c^T(N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)}) &= c^T(T_{P(V \otimes W^*)}|_{\mathbf{P}(V)})/c^T(T_{\mathbf{P}(V)}) \\ &= c^T(\mathcal{O}(1)^{n+1} \otimes W^*|_{\mathbf{P}(V)})/c^T(\mathcal{O}(1)^{n+1}) \\ &= c^T(W^*(1)/\mathcal{O}(1))^{n+1}. \end{aligned}$$

So the equivariant Euler class of $N_{\mathbf{P}(V)/\mathbf{P}(V \otimes W^*)}$ is

$$[c_{top}^T(W^*(1)/\mathcal{O}(1))]^{n+1} = \prod_{k=2}^m (h + t_k - t_1)^{n+1}.$$

■

Now we can write down our basic formula in general case based on the Localization theorem and Proposition 6:

Theorem 7 (Schubert Formula 1) For an equivariant class c_T on N_{m-1} :

$$\sum_{\{I|l_1=i\}} \pi_* \left(\frac{i_{F_I}^* c_T}{\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s)} \right) = \frac{j^* u_* c_T}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}}$$

As usual, we can first use this formula to compute all the intersections on $\text{Pic}(Fl(1, 2, \dots, m, V))$, without worrying about error terms. Note that $H, \psi_1, \dots, \psi_{m-1}$ generate $\text{Pic}(Fl(1, 2, \dots, m, V))$.

Proposition 7 All the products $H^b \cup \psi_1^{s_1} \cup \cdots \cup \psi_{m-1}^{s_{m-1}}$ ($b + s_1 + \cdots + s_{m-1} = mn - \frac{m(m-1)}{2}$) can be determined from the formula by letting $i_{F_I}^* c_T = H^b$.

Proof. The the integral form of the above Theorem with H^b being the class is:

$$\begin{aligned} \sum_{\{I|i_1=i\}} \int_{F_I} \frac{H^b}{\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^m (t_{i_s} - t_{i_{s-1}} - \psi_s)} \\ = \int_{\mathbf{P}(V)} \frac{h^b}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}}. \end{aligned}$$

By multiplying suitable power of $\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j})$, and invert all terms in the denominator, we can see all the terms $H^b \cup \psi_1^{s_1} \cup \cdots \cup \psi_{m-1}^{s_{m-1}}$ with fixed b are just coefficients of a known polynomial (from right hand side). Since $H^b = 0$ for all $b > n$, again there are only $n + 1$ cases. ■

Example 12 Some data for small m and n .

$\dim(W) = m = 4$, and $\dim(V) = n + 1 = 4$, then $\dim(Fl(1, 2, 3, 4, V)) = 6$. Some intersections are (compare with Example 6 in Section 2):

$$\begin{aligned} H^3 \psi_1^3 &= 0 & H^3 \psi_1^2 \psi_2 &= 1 & H^3 \psi_1^2 \psi_3 &= -2 \\ H^3 \psi_1 \psi_2^2 &= -3 & H^3 \psi_1 \psi_2 \psi_3 &= 3 & H^3 \psi_1 \psi_3^2 &= 0 \\ H^3 \psi_2^3 &= 6 & H^3 \psi_2^2 \psi_3 &= -3 & H^3 \psi_2 \psi_3^2 &= -3 \\ H^3 \psi_3^3 &= 6 \end{aligned}$$

1.4.3 Dealing with error terms

Note that in above case, there are no error terms because in this case $c_T = (u_*(h^T))^b$. This is not the case when $i_{F_I}^* c_T = H^b \cup \tau$. As in Section 1.2, an approximation outside the boundary divisors $D_1, D_2, \dots, D_{m-1}$ can be made, and the error terms will not contribute to the terms that we want to compute.

Look at the following diagram (on top of Main Diagram):

$$\begin{array}{ccccc} \tilde{N}_{m-1} \cdots & \longrightarrow & \tilde{N}_k \cdots & \longrightarrow & \mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \\ \downarrow \pi' & & \downarrow & & \downarrow \\ N_{m-1} \cdots & \longrightarrow & N_k \cdots & \longrightarrow & \mathbf{P}(V \otimes W^*) \end{array}$$

There exists a rational map (evaluation map)

$$\alpha : \mathbf{P}(V \otimes W^*) \times \mathbf{P}(W) \dashrightarrow \mathbf{P}(V)$$

by sending an element in $\mathbf{P}(W)$ to its image through the corresponding map in $V \otimes W^*$. α is not defined on any kernel of an element in the lower rank locus M_{m-1} .

To resolve it, blow up sequentially along the $\mathbf{P}^{m-1-k}$ -bundle $\{(\phi, \ker(\phi)) \subset \tilde{N}_{k-1} | \phi \in M_k^{k-1}\}$ for $k = 1, 2, \dots, m-2$. Each time the induced rational map is undefined over $M_{k+1}^k \subset \tilde{N}_k$. Finally blow up the section of M_{m-1}^{m-2} to get $\tilde{N}_{m-1}$ which has an evaluation morphism (denoted by e) to $\mathbf{P}(V)$.

Proposition 8 There is a natural equivariant map:

$$\Phi : \pi'_* e^* \mathcal{O}_{\mathbf{P}(V)}(1) \rightarrow u^*(W^* \otimes \mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1))$$

which is an isomorphism away from $D_1 \cup D_2 \cup \dots \cup D_{m-1} \subset N_{m-1}$.

Proof. This follows directly from the construction above, as the natural map (as in Proposition 3 in Section 2) is the identity on where the original rational map can be defined. The whole exceptional locus in N_{m-1} is exactly $D_1 \cup D_2 \cup \dots \cup D_{m-1}$. ■

Corollary 3 If $i_{F_1}^* c_T = \tau(q_0, q_1, \dots, q_{m-1})$, then

$$u_*(c_T + c'_T) = \tau(h, h + t_1, \dots, h + t_{m-1}).$$

Here c'_T is push-forward of an equivariant class on $D_1 \cup D_2 \cup \dots \cup D_{m-1}$.

Proof. This follows from the above proposition since

- 1). The equivariant chern roots of $W^* \otimes \mathcal{O}_{\mathbf{P}(V \otimes W^*)}(1)$ are $h, h + t_1, \dots, h + t_{m-1}$, and
- 2). $\pi'_* e^* \mathcal{O}_{\mathbf{P}(V)}(1)$ is the pull back of the dual tautological bundle S^* from $G(m, V)$. ■

Next we show error terms do not contribute.

Lemma 4 The error terms from D_k are of the form:

$$E_k = \frac{i_{F_I}^* c_T'}{\prod_{1 \leq j < l \leq m} (t_{i_l} - t_{i_j}) \cdot \prod_{j \neq k} (t_{i_{j+1}} - t_{i_j} - \psi_j)}$$

Proof. We just need to see the change of the denominator. The only term missing from there is $t_{i_{k+1}} - t_{i_k} - \psi_k$, which is $c_{top}^T(N_{D_k/N_{m-1}}|_{F_I})$. ■

Theorem 8 (Schubert Formula 2)

$$\begin{aligned} & \sum_{\{I| i_1=i\}} \pi_* \left(\frac{\pi^*(h^b) \cup \tau(q_1, \dots, q_m)}{\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s)} \right) \\ &= \frac{h^b \cup \tau(h + t_1 - t_i, \dots, h + t_m - t_i)}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}} + \text{irrelevant terms} \end{aligned}$$

Proof. From the formula

$$\begin{aligned} & \sum_I \int_{F_I} \frac{H^b \cup \tau}{\prod_{0 \leq j < k \leq m-1} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^m (t_{i_s} - t_{i_{s-1}} - \psi_s)} \\ &+ \sum_I \sum_{k=1}^{m-1} \int_{D_{i_k}} E_{i_k} = \int_{\mathbf{P}(V)} \frac{h^b}{\prod_{s \neq i} (h + t_s - t_i)^{n+1}}. \end{aligned}$$

It suffices to show that all $H^b \cup \prod_{k=1}^{m-1} \psi_k^{i_k} \cup \tau$ terms come with denominator $\prod_{k=1}^{m-1} t_k^{a_k}$ where $a_k \geq 2$ for all k , and all error terms are $O(\frac{1}{t_j})$ for some j . We check these exactly as in the proof of Theorem 3. ■

We also need to determine which class $H^b \cup \prod_{k=1}^{m-1} \psi_k^{i_k}$ will not change the degree of τ after pushing forward to $\mathbf{P}(V)$. That is, which class has degree 1 on the fiber of the projection $Fl(1, 2, \dots, m, V) \rightarrow G(m, V)$, which is $Fl(1, 2, \dots, m-1, m)$.

Lemma 5 Degree of $H^{m-1} \cup \prod_{k=1}^{m-2} \psi_k^{m-1-k}$ is one on the fiber of the projection.

Proof. This can be seen geometrically, since each ψ_k , after restricts to a fiber, is just the relative canonical class $c_1(\mathcal{O}(1))$ of a $\mathbf{P}^{m-1-k}$ bundle (ψ_{m-1} restricts to the trivial class). We can also check this by computations in the following example. ■

Example 13 In the case of $m = 4$, if $\text{codim}(\tau) = m(n + 1 - m) = 4(n - 3)$ (maximum), some computations are as follows:

$b = 4$: $H^4 \cup \tau = 0$ (doesn't dependent on τ)

$b = 3$: $H^3 \cup \psi_1^2 \cup \psi_2 \cup \tau = \frac{1}{6}$ of the constant term of

$$\int_{\mathbf{P}(V)} \frac{h^3 \cdot \tau(h, h + t, h + u) \cdot \prod_{1 \leq j < l \leq 4} (t_l - t_j)^2}{\prod_{k=2}^4 (h + t_k - t_1)^{n+1}}$$

In particular, if $\tau = \sigma_4^{n-3}$, then $H^3 \cup \psi_1^2 \cup \psi_2 \cup \sigma_4^{n-3} = 1$.

1.4.4 Relative version of $FL(1, 2, \dots, m, V)$.

Here is the new setting: Let $f : V \rightarrow X$ be a rank- $(n + 1)$ vector bundle over a projective variety. Let $\mathbf{P}(V) \rightarrow X$ be the projective bundle. Still let W be a vector space of dimension $m \geq 3$. We do the same diagram of the relative case in section 2 (i.e., over X).

The only term changed is the equivariant Euler class of $\mathbf{P}(V)$. Let α_i be the chern roots of V ($i = 1, 2, \dots, n + 1$). The following Proposition has same proof as in Section 1.2.

Proposition 9 The equivariant Euler class of $\mathbf{P}(V)$ in $\mathbf{P}(V \otimes W^*)$ is

$$\prod_{k \neq i} \prod_{j=1}^{n+1} (h + \alpha_j + t_k - t_i).$$

The following is the basic formula in this relative setting:

Theorem 9 Let c_T be an equivariant class on N_{m-1} , then:

$$\begin{aligned} \sum_{\{I | i_1 = i\}} \pi_* \left(\frac{i_{F_I}^* c_T}{\prod_{1 \leq j < k \leq m} (t_{i_k} - t_{i_j}) \cdot \prod_{s=1}^{m-1} (t_{i_{s+1}} - t_{i_s} - \psi_s)} \right) \\ = \frac{j^* u_* c_T}{\prod_{k \neq i} \prod_{j=1}^{n+1} (h + \alpha_j + t_k - t_i)}. \end{aligned}$$

As in standard case, pushing forward $H_b \cup \prod \psi_k^{i_k}$ or $H_b \cup \prod \psi_k^{i_k} \cup \tau$ to $\mathbf{P}(V)$ then to X , we can compute their intersection with chern roots by inverting the denominator

of the right hand side and looking at the appropriate coefficients. Again, the error terms do not contribute to the computation when the class involves τ (pull-back from $G(m, V)$).

CHAPTER 2

SESHADRI CONSTANTS ON JACOBIANS OF CURVES

2.1 Introduction and Statement of Theorem

Let X be a smooth complex projective variety. Let L be a numerically effective (nef) line bundle on X .

Definition 4 The Seshadri constant of L at a point $p \in X$ is the real number

$$\epsilon(L, p) = \inf \left\{ \frac{C \cdot L}{\text{mult}_p C} \mid p \in C \subset X \right\}.$$

Here the infimum is taken over all reduced curves C passing through p , and $\text{mult}_p C$ is the multiplicity of C at p .

Let $f : Bl_p X \rightarrow X$ be the blow-up of X at the point p and $E = f^{-1}(p)$ be the exceptional divisor. It is easy to see that the following is an equivalent definition of the Seshadri constant.

Definition 5

$$\epsilon(L, p) = \sup \{ \epsilon \mid f^*L - \epsilon E \text{ is nef} \}.$$

Here the $\mathbf{R}$ -divisor $f^*L - \epsilon E$ is nef means that $f^*L \cdot C' \geq \epsilon E \cdot C'$ for every irreducible curve $C' \subset Bl_p X$.

One can also define the global Seshadri constant.

Definition 6

$$\epsilon(L) = \inf \{ \epsilon(L, p) \mid p \in X \}.$$

The Seshadri constant indicates how far an ample divisor is from the boundary of the ample cone near point p , thus measures positivity, or ampleness locally. The study of Seshadri constants has drawn increasing interests during recent years (see [9] and [11] for some applications).

We first list some properties of Seshadri constants.

1. Seshadri constants have general upper bounds. Let $\dim(X) = n$. Then $\epsilon(L, p) \leq \sqrt[n]{L^n}$ (because $(f^*L - \epsilon E)^n \geq 0$ as in Definition 5).
2. The first nontrivial property of Seshadri constants is Seshadri's criterion ([15]), which says that L is ample if and only if $\epsilon(L) > 0$. Also if L is very ample then obvious $\epsilon(L) \geq 1$.
3. However, there is no uniform lower bound for Seshadri constants. For any $n \geq 2$ and $\delta > 0$, there is a smooth n -dimensional projective variety X and an ample line bundle L on X such that $\epsilon(L, p) < \delta$ for some $p \in X$. ([10])
4. If the Seshadri constant does not achieve its upper bound (property 1), then it is a d -th root of a rational number for some $1 \leq d \leq n - 1$. ([20])

Usually it is difficult to compute exact values of Seshadri constants. In many cases one would rather try to give some specific bounds (especially lower bounds). For example [11] shows that for any ample line bundle L on a surface, $\epsilon(L, p) \geq 1$ for very general points $p \in X$. For general dimension n , the similar result $\epsilon(L, p) \geq \frac{1}{n}$ has been shown in [10]. Another interesting aspect of the study of Seshadri constants is the rationality problem. For example from property 4 above, one sees that on a surface, either $\epsilon(L, p) = \sqrt{L^2}$ or it is rational. A noticeable fact is that there has been no known example of irrational Seshadri constants.

We will focus on abelian varieties (actually special cases of those, i.e., Jacobians of curves). Let (A, Θ) be a principally polarized abelian variety of dimension g . That is, A is a complex torus and Θ is an ample divisor with $h^0(A, \mathcal{O}_A(\Theta)) = 1$. Since abelian varieties are homogeneous spaces, we can define $\epsilon := \epsilon(\Theta, 0) = \epsilon(\Theta, p)$ for any p . We list some important results in this case:

1. [18] shows that $\epsilon \geq 1$ and the equality holds if and only if $A = E \times B$ where E is an elliptic curve. So if the abelian variety is indecomposable, then $\epsilon > 1$.

2. [17] shows that general elements in the moduli space of principally polarized abelian varieties of dimension g have Seshadri constants very close to their maximum upper bound. Specifically, $\epsilon \geq \frac{2\frac{1}{g}}{4} \cdot \sqrt[g]{\Theta^g}$.

3. On the other hand there are some special abelian varieties, namely Jacobian, which have relatively small Seshadri constants. Let C be a smooth complex algebraic curves with genus $g = g(C) \geq 2$. Denote by $(J(C), \Theta)$ its Jacobian (Recall $J(C) = \text{Pic}^0(C)$). The following are known (cf [17]):

- (a). $1 < \epsilon \leq \sqrt{g}$.
- (b). If C is hyperelliptic then $\epsilon \leq \frac{2g}{g+1}$
- (c). In particular if $g = 2$ (then C is hyperelliptic), $\epsilon = \frac{4}{3}$.

The problem becomes very interesting even when $g = 3$. The point here is to see if the Seshadri constants can be their maximum, i.e. $\sqrt{g}$, thus most time irrational, or always less than their maximum – thus more likely rational. While all the existing examples suggest the later, we investigate this problem in detail, mainly look at the cases when $\epsilon \leq 2$.

Our main result is the following theorem:

Theorem 10 Assume the Picard number of $J(C)$ is one. Then

- (1) If C is hyperelliptic, then $\epsilon = \frac{2g}{g+1}$.
- (2) If $g = 3$ and C is not hyperelliptic, then $\epsilon = \frac{12}{7}$.
- (3) If $g = 4$ and C is not hyperelliptic, then $\epsilon = 2$.
- (4) If $g \geq 5$ and C is not hyperelliptic, then $\epsilon \geq 2$.

Part (4) of the theorem can be restated as:

Corollary 4 If $g \geq 5$ and $\epsilon < 2$, then C is hyperelliptic.

Remark:

(1) For the ease of calculation on Neron-Severi group of C_2 , we need that it is generated by a fiber and the diagonal, i.e., its Picard number is 2. That is true if

C is of general moduli. We need this condition throughout this paper. But this restriction, however, seems not essential.

(2) We can also locate all the special curves that give relatively small ratios in cases (1)–(3).

2.2 Proof of Theorem: Hyperelliptic Case

The following observation, while straightforward, points out where we want to find special curves that give the exact values of Seshadri constants.

Lemma 6 If C' is an irreducible curve in $J(C)$ such that $\frac{C' \cdot \Theta}{\text{mult}_0 C'} \leq 2$, then for any divisor D with $D \equiv k\Theta$ and $\text{mult}_0 D \geq 2k$, we have $C' \subset D$.

If C is hyperelliptic, then the case of $k = 1$ in Lemma 6 reads $D \equiv \Theta$ and $\text{mult}_0 D \geq 2$ which we denote as $(*)$, we have:

Proposition 10 Let $u : C_d \rightarrow J(C)$ be the Abel-Jacobi map. Then

$$\bigcap_{(*)} D = u(C_2).$$

Proof. Let L be a hyperelliptic line bundle on C . Let p_0 be a ramification point of the g_2^1 , so $L = \mathcal{O}_C(2p_0)$. We fix a translation of Abel-Jacobi map $u : C_d \rightarrow J(C)$ by sending $Y \in C_d$ to $Y - \deg(Y) \cdot p_0 \in J(C)$, and for simplicity we ignore the p_0 part for representation of points in $J(C)$ in our proof. Also recall $\phi : C_{g-3} \rightarrow C_{g-1}, Y \rightarrow Y + L$ maps C_{g-3} birationally and surjectively to $\text{Sing}(\Theta)$ in our case.

For any $Y \in C_{g-3}$, define $D_Y = \Theta - Y$. If translates $Y + 2p_0 \in \text{Sing}(\Theta)$ to $0 \in D_Y$. Thus $D_Y \equiv \Theta$ and $\text{mult}_0 D \geq 2$. So we need to show $\bigcap D_Y = u(C_2)$.

It is obvious that $u(C_2) \subset \bigcap D_Y$ since for any point $(p, q) \in C_2$ we can rewrite it as $(p + q + Y) - Y \in D_Y$ for any $Y \in C_{g-3}$.

On the other side, any points in $\bigcap D_Y$ can be represented as $D - Y$ for some $D \in C_{g-1}$ and $Y \in C_{g-3}$. Also since it is in the intersection, for any $F \in C_{g-3}$, there exists $E \in C_{g-1}$ such that $D - Y = E - F$, i.e., $D - Y + F$ is (equivalent to) an

effective divisor for any F . We claim $D - Y$ itself must be effective, and since it has degree 2, it is in $u(C_2)$.

Pick an representation of $D - Y$ such that Y contains no ramification point of g_2^1 . First assume D contains no hyperelliptic pair. If $D - Y$ is not effective, pick $p \in Y$ but $p \notin D$, and let $L = \mathcal{O}_C(p + p')$. Choose $F = Y - p + p'$. Then $D - Y + F = D - p + p'$. On the other hand the linear system $|D - p + p'|$ is empty since otherwise it must contain multiple of hyperelliptic pairs and base points, which will leads to $p \in D$.

If D has some hyperelliptic pairs. Cancel as many points in $D - Y$ as possible until either $D - Y$ is effective or D runs out of hyperelliptic pairs and reduce to a similar situation in first case. ■

If C is hyperelliptic, and $rk(NS(C_2)) = 2$, then $NS(C_2)$ is generated by a fiber F and the diagonal Δ . There is a rational curve, call it $\mathbf{P}^1$, which consists of hyperelliptic pairs $\{(p, q) \in C_2 | \mathcal{O}_C(p + q) = L\}$. Also denote $u^*(\Theta)$ still as Θ . We list the numerical properties of $NS(C_2)$ below.

Lemma 7 Notation as above, we have:

- (1) $\Theta = (g + 1)F - \frac{1}{2}\Delta$ and $\mathbf{P}^1 = 2F - \frac{1}{2}\Delta$.
- (2) $F^2 = 1, F \cdot \Delta = 2, \Delta^2 = 4 - 4g$.

The Abel-Jacobi map $u : C_2 \rightarrow J(C)$ contracts $\mathbf{P}^1$ and is isomorphic outside $\mathbf{P}^1$. Now let C'' be an irreducible curve in C_2 not contracted by u and $C' = u(C'')$. Then

$$\frac{C'' \cdot \Theta}{C'' \cdot \mathbf{P}^1} = \frac{C' \cdot \Theta}{\text{mult}_0 C'}$$

So our Theorem in hyperelliptic case follows from the following Proposition:

Proposition 11 Among all irreducible curves in C_2 not contracted by u , Δ is the only curve with minimum ratio $\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{2g}{g+1}$.

Proof. Since $\Delta \cdot \Theta = 4g$ and $\Delta \cdot \mathbf{P}^1 = 2g + 2$, we have $\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{4g}{2g+2} = \frac{2g}{g+1}$

Let $C_0 = aF + b\Delta \subset C_2$ be an irreducible curve not contracted by u . Then $C_0 \cdot \mathbf{P}^1 = a + (2g + 2)b \geq 0$ and $C_0 \cdot \Theta = (a + 4b)g > 0$. Then

$$\frac{\Delta \cdot \Theta}{\Delta \cdot \mathbf{P}^1} = \frac{(a + 4b)g}{a + (2g + 2)b} > \frac{2g}{g + 1} \iff a > 0$$

But if $a \leq 0$, then we must have $b > 0$ since $C_0 \cdot \Theta = (a + 4b)g > 0$. Now we have $C_0 \cdot \Delta = 2a + b(4 - 4g) < 0$. Since both C_0 and Δ are irreducible, $C_0 = \Delta$. ■

Remark: A little more detailed calculation shows Δ is actually the only curve whose corresponding ratio is less than two.

2.3 Proof of Theorem: Nonhyperelliptic Case

If C is nonhyperelliptic, then choose the case $k = 2$ in Lemma 6 which reads $D \equiv 2\Theta$ and $\text{mult}_0 D \geq 4$, i.e., the base locus of $|2\Theta|_{00}$. We need the following result of Welters:

Proposition 12 (Welters [21]) $Bs(|2\Theta|_{00}) = \lambda(C \times C)$. Here $\lambda : C \times C \rightarrow J(C)$, $\lambda(p, q) = p - q$ is the difference map.

Remark: Welters' theorem is true for all curves with $g = 3$ or $g \geq 5$. For $g = 4$ the base locus has two more isolated points, which will not affect our proof since we are looking at curves inside the base locus.

In this case we look at the NS group in $C \times C$. It is generated by fibers F_1, F_2 and the diagonal Δ . We list their numerical properties below.

Lemma 8 Notation as above, then:

- (1) $\lambda^*\Theta = (g - 1)(F_1 + F_2) + \Delta$
- (2) $F_i^2 = 0$, $F_1 \cdot F_2 = F_i \cdot \Delta = 1$, $\Delta^2 = 2 - 2g$, $i = 1, 2$.

Since C is nonhyperelliptic, the difference map λ contracts the diagonal Δ to $0 \in J(C)$ and is isomorphic outside Δ . So let C'' be an irreducible curve in $C \times C$ not contracted by λ and $C' = u(C'')$. Then

$$\frac{C'' \cdot \lambda^*\Theta}{C'' \cdot \Delta} = \frac{C' \cdot \Theta}{\text{mult}_0 C'}$$

So our Theorem in nonhyperelliptic case follows from the following Proposition:

Proposition 13 Notation as above:

(1) If $g = 3$, the minimum ratio $\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta}$ is $\frac{12}{7}$ for curves in $C \times C$, and is achieved by one curve.

(2) If $g = 4$, the minimum ratio $\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta}$ is 2 for curves in $C \times C$, and is achieved by more than one curves.

(3) If $g \geq 5$, then $\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} \geq 2$ for all curves in $C \times C$ not contract by λ .

Proof. (1) $g = 3$: In this case, the canonical system embeds C as an plane quartic. Let $\mathcal{O}_C(1)$ be its hyperplane section. Consider the curve $C_0 = \{(p, q) | \mathcal{O}_C(p + q + 2r) = \mathcal{O}_C(1) \text{ for some } r \in C\} \subset C_2$. Write $C_0 = aF + b\Delta$. Since $C_0 \cdot \Delta = 56$ (twice the number of bitangent) and $C_0 \cdot F = 10$ (degree of the ramification divisor of dual curve's g_3^1), we can solve a and b and get $C_0 = 16F - 3\Delta$. C_0 is irreducible since it is isomorphic to C via $p + q \rightarrow r$. Pull it back to $C \times C$ we get a curve $C''_0 = 16(F_1 + F_2) + 6\Delta$. Now

$$\frac{C''_0 \cdot \lambda^* \Theta}{C''_0 \cdot \Delta} = \frac{[16(F_1 + F_2) - 6\Delta] \cdot [2(F_1 + F_2) + \Delta]}{[16(F_1 + F_2) - 6\Delta] \cdot \Delta} = \frac{96}{56} = \frac{12}{7}$$

To claim $\frac{12}{7}$ is the minimum ratio, let $C'' = aF_1 + bF_2 + c\Delta$ be any irreducible curve in $C \times C$ not contracted by λ . If $C'' \neq C''_0$, then $C'' \cdot C''_0 = 10(a+b) + 56c \geq 0$. So if $c \geq 0$,

$$\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} = \frac{3(a+b)}{a+b-4c} \geq 3 > \frac{12}{7}.$$

If $c < 0$,

$$\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} = \frac{3(a+b)}{a+b-4c} \geq \frac{3(a+b)}{a+b+\frac{5}{7}(a+b)} = \frac{7}{4} > \frac{12}{7}.$$

This shows the only curve achieves the minimum ratio $\frac{12}{7}$ is C''_0 .

(2) $g = 4$: In this case C has two g_3^1 's. Let L be one g_3^1 . Consider the curve $C_0 = \{(p, q) | |L - p - q| > 0\} \subset C_2$. Since $C_0 \cdot F = 2$, and $C_0 \cdot \Delta = 12$ (degree of ramification divisor of L), we find that $C_0 = 3F - \frac{1}{2}\Delta$. Lift to $C \times C$ to get $C''_0 = 3(F_1 + F_2) - \Delta$. Calculation as above shows that

$$\frac{C''_0 \cdot \lambda^* \Theta}{C''_0 \cdot \Delta} = \frac{24}{12} = 2,$$

and it is the minimum ratio that can be achieved on $C \times C$.

Note that in this case there is another curve (from the other g_3^1) that give the minimum ratio. The reason is in this case $C_0^2 = 0$ while in case of $g = 3$ we have $C_0^2 < 0$ (thus unique).

(3) $g \geq 5$: Assume C has a g_d^1 ($d \geq 3$), call it L . As in (2), consider the curve $C_0 = \{(p, q) | |L - p - q| > 0\} \subset C_2$. Then $C_0 \cdot F = d - 1$, and $C_0 \cdot \Delta = 2d + 2g - 2$ (degree of ramification divisor of L). Thus $C_0 = dF - \frac{1}{2}\Delta$. Lift to $C \times C$ we get $C_0'' = d(F_1 + F_2) - \Delta$. Now first we have

$$\frac{C_0'' \cdot \lambda^* \Theta}{C_0'' \cdot \Delta} = \frac{dg}{d + g - 1} > 2.$$

Secondly for any irreducible $C'' = aF_1 + bF_2 + c\Delta \subset C \times C$ not contracted by λ , either $C'' \cdot C_0'' < 0$, or

$$\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} \geq \frac{d + g - 1}{d} \geq 2 \text{ if } d \leq g - 1.$$

Since the Brill-Noether number for g_d^1 is non-negative if $d \geq \frac{g+2}{2}$, both ratios above are at least 2. If the minimum ratio $\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} < 2$, then all the curves C_0 must be reducible and contain an irreducible component C_1 whose lift to $C \times C$ gives small ratio. It is easy to see that $C_1^2 < 0$, thus unique in C_2 . This is certainly impossible. (For example, if there is two different g_d^1 for some d , then to have a common component for corresponding $C_0 \subset C_2$, one coordinate has to be a base point of g_d^1 , thus it is linear combination of fibers, and since the component is irreducible, it is a fiber. On the other hand, the corresponding ratio for a fiber is $g > 2$.)

Note that the minimum ratio exists and can be achieved if $\frac{dg}{d+g-1} \leq \frac{d+g-1}{d}$, which is equivalent to $\frac{dg}{d+g-1} \leq \sqrt{g}$, or $d \leq \sqrt{g} + 1$. ■

2.4 Other Problems of Seshadri Constants

For nonhyperelliptic cases when $g \geq 5$, to find the Seshadri constants, the first step is to look at the curves in C_2 . It is related to the problem whether the cone of effective curves of C_2 is closed. If it is, the curve from the boundary will give a better upper bound of $\epsilon(\Theta)$ which is less than $\sqrt{g}$. In all special cases we have

discussed (hyperelliptic, small genus, curves with g_d^1 for small d), the cone is closed. For general case there is some indication that the curve from one boundary (the other one being the diagonal), if closed, will give the ratio $\frac{gq}{p}$ where (p, q) is the primitive solution of Pell's equation $x^2 - gy^2 = 1$. The following example gives some indication that it could be true.

Example 14 If C is a plane quintic (i.e., genus is 6), consider the curve $C_0 = \{(p, q) \mid |\mathcal{O}_C(1) - p - q - 2r| > 0\} \subset C_2$. Then C_0 is irreducible and $C_0 = 50F - 7\Delta$. For any curve $C' \subset C_2$ satisfies $C_0 \cdot C' \geq 0$, calculation shows on $C \times C$, $\frac{C'' \cdot \lambda^* \Theta}{C'' \cdot \Delta} \geq \frac{12}{5}$ holds for all irreducible curves (except Δ). Note that the bound $\frac{12}{5}$ is what the conjecture gives. Also note one expects small values for plane curves which are special in the moduli of curves, so general curves of genus 6 must also satisfy that bound.

For $g \geq 5$, if $\epsilon < 2$ it follows that C is hyperelliptic. In all cases hyperelliptic curves gives us the smallest Seshadri constants. From the known result ([16]) of $Bs(|2\Theta|_{00})$ in dimension 4, it is easy to see that:

If A is an indecomposable principally polarized abelian variety of dimension 4 and $\epsilon(\Theta) < 2$, then A is the Jacobian of a hyperelliptic curve C of genus 4.

It is very reasonable to ask same question for any genus and seems like it could be true.

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