A DIMENSION INEQUALITY FOR EXCELLENT, COHEN-MACAULAY RINGS RELATED TO THE POSITIVITY OF SERRE'S INTERSECTION MULTIPLICITY

by

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ABSTRACT

Assume that $(R, \mathfrak{m})$ is a Noetherian local ring. Kurano and Roberts have made the following conjecture related to the positivity of Serre's intersection multiplicity. Assume that R is regular and contains prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$; then

$$\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \quad \text{for all } n \geq 1.$$

We consider this conjecture and the following question, which is a generalization of the conjecture. Assume that R is quasi-unmixed with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$. Does the inequality

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$$

hold? We answer this question in the affirmative in the following cases:

1. R is excellent and contains a field.
2. $\mathrm{ht}\,(\mathfrak{p}) = 0$.
3. R is Nagata and $\mathrm{ht}\,(\mathfrak{q}) = 0$.
4. $\dim(R/\mathfrak{q}) = 1$.
5. R is Nagata and $\dim(R/\mathfrak{p}) = 1$.
6. R is Nagata and $R/\mathfrak{p}$ is regular.

We also verify the original conjecture of Kurano and Roberts in a number of cases (with no excellence restriction), most notably when

1. R contains a field.

2. $\mathfrak{p}$ is generated by a regular sequence.

3. $\mathfrak{q}$ is generated by part of a regular system of parameters.

We also present a number of examples that demonstrate the necessity of each of the assumptions of the conjectures as well as the limitations of some of our results.

To Joy

CONTENTS

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CHAPTER 1

INTRODUCTION

Throughout this work, all rings are assumed to be commutative and Noetherian with identity, and all modules are assumed to be unitary.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension d, and let M and N be finitely generated R-modules such that $M \otimes_R N$ is a module of finite length. Serre defined the *intersection multiplicity* of M and N to be

$$\chi(M, N) = \sum_{i=0}^{n} (-1)^i \text{length}(\text{Tor}_i^R(M, N))$$

and conjectured that $\chi(M, N)$ satisfies the following properties:

1. $\dim(M) + \dim(N) \leq \dim(R)$.

2. (Nonnegativity) $\chi(M, N) \geq 0$.

3. $\chi(M, N) > 0$ if and only if $\dim(M) + \dim(N) = \dim(R)$.

or, equivalently,

1. $\dim(M) + \dim(N) \leq \dim(R)$.

2. (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, then $\chi(M, N) = 0$.

3. (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, then $\chi(M, N) > 0$.

Serre was able to verify the first statement for any regular local ring, and to verify the others in the case when R is unramified. Since $\chi(M, N)$ has many of the characteristics we desire from an intersection multiplicity (for example, Bézout's Theorem holds), it was not unreasonable to suppose that these further properties are satisfied for an arbitrary regular local ring. The results were left unproved for ramified rings, and Serre also asked whether a proof existed in the equicharacteristic case which did not use reduction to the diagonal.

The vanishing conjecture was proved about ten years ago by Gillet-Soulé [6] and Roberts [18] using K-theoretic methods. The proof in [6] uses the theory of Adams operations on Grothendieck groups of complexes, while that in [18] uses the theory of local Chern characters. O. Gabber proved the nonnegativity conjecture very recently[1] using a theorem of de Jong [5]. Kurano and Roberts have proved the following theorem using methods introduced by Gabber.

Theorem 1.1 ([12] Theorem 3.2) *Assume that* $(R, \mathfrak{m})$ *is a regular local ring that either contains a field or is ramified. Also, assume that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$*. If* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ *then*

$$\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \qquad \textit{for all } n \geq 1. \tag{1.1}$$

(For the definition of symbolic powers of prime ideals, see Definition 2.9.) As a result, they conjectured that (1.1) should hold for all regular local rings.

Conjecture 1.2 *Assume that* $(R, \mathfrak{m})$ *is a regular local ring and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$*. Then* $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ *for all* $n \geq 1$*.*

Furthermore, Kurano and Roberts asked whether there exists an elementary proof of the conjecture in the equicharacteristic case.

We study Conjecture 1.2, as a verification of this conjecture could introduce new tools to apply to the positivity conjecture.

For any local ring $(A, \mathfrak{n})$ let $e(A)$ denote the Hilbert-Samuel multiplicity of A with respect to $\mathfrak{n}$. (For the definition of Hilbert-Samuel multiplicity, see Definition 2.5.) It is a straightforward exercise to verify that, if R is a regular local ring with prime

[1] I have no direct reference to Gabber's work as it remains unpublished. However, detailed treatments of Gabber's work on this problem can be found in Berthelot [2], Hochster [10] and Roberts [19].

ideal $\mathfrak{p}$ and $0 \neq f \in \mathfrak{p}$, then $e(R_{\mathfrak{p}}/(f)) = e$ if and only if $f \in \mathfrak{p}^{(e)} \smallsetminus \mathfrak{p}^{(e+1)}$. Thus, Conjecture 1.2 may be rephrased as the following.

Conjecture 1.2′ *Assume that* $(R, \mathfrak{m})$ *is a regular local ring and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *If there exists* $0 \neq f \in \mathfrak{p} \cap \mathfrak{q}$ *such that* $e(R_{\mathfrak{p}}/(f)) = e(R/f)$, *then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - 1$.

To see that this is a restatement of Conjecture 1.2, we first recall a classical result on the behavior of symbolic powers of prime ideals in regular local rings.

Theorem 1.3 (Nagata [16] Theorem 38.3) *Assume that* $(R, \mathfrak{m})$ *is a regular local ring with prime ideal* $\mathfrak{p}$. *Then* $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$ *for all* $n \geq 1$.

Now, let $R, \mathfrak{m}, \mathfrak{p}, \mathfrak{q}$ be as in Conjecture 1.2 and suppose that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \not\subseteq \mathfrak{m}^{n+1}$. Then there exists $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$ such that $f \notin \mathfrak{m}^{n+1}$. By Theorem 1.3, $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$ so that $f \in \mathfrak{m}^n$. If $f \in \mathfrak{p}^{(n+1)}$ then $f \in \mathfrak{m}^{n+1}$, a contradiction, so that $e(R_{\mathfrak{p}}/(f)) = n = e(R/(f))$. If Conjecture 1.2′ holds, then this implies that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - 1$, contradicting the assumption that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

Conjecture 1.2′ motivates the following generalization.

Conjecture 1.4 *Assume that* $(R, \mathfrak{m})$ *is a quasi-unmixed local ring of dimension* d *with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $e(R_{\mathfrak{p}}) = e(R)$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq d$.

As we noted above, Serre proved this conjecture in the case where R is regular where the condition $e(R_{\mathfrak{p}}) = e(R)$ holds automatically. The following is a famous conjecture whose statement is very similar to Conjecture 1.4.

Conjecture 1.5 (Peskine and Szpiro [17]) *Assume that* $(R, \mathfrak{m})$ *is a local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\mathfrak{p}$ *has finite projective dimension and* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$.

We shall discuss connections between Conjectures 1.4 and 1.5 below.

In the main results of this work we verify Conjectures 1.2 and 1.4 for a number of cases. Most notably, we verify Conjecture 1.4 for excellent rings containing a field.

In Chapter 2 we prove a generalization of Theorem 1.3, which motivates the restriction of our attention to a certain class of rings where the Hilbert-Samuel multiplicity is well-behaved with respect to localization. We then verify Conjecture 1.4 for excellent rings containing a field, certain low dimensional cases and the case where $R/\mathfrak{p}$ is regular. In Chapter 3 we establish Conjecture 1.2 for regular local rings containing a field, certain low-dimensional cases, and the the cases where $\mathfrak{p}$ is generated by a regular sequence and where $\mathfrak{q}$ is generated by part of a regular system of parameters. In Chapter 4 we present a number of examples that demonstrate the necessity of each of the assumptions of the conjectures as well as the limitations of some of our results.

CHAPTER 2

THE DIMENSION INEQUALITY

In this chapter, we establish Conjecture 1.4 in a number of cases, most notably in the case where R contains a field. We recall the conjecture here.

Conjecture 1.4 *Assume that* $(R, \mathfrak{m})$ *is a Cohen-Macaulay local ring of dimension* d *with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ *and* $e(R_{\mathfrak{p}}) = e(R)$*. Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq d$*.*

Examples 4.1, 4.2 and 4.3 below show that each requirement in this conjecture is necessary.

We begin by proving a generalization of Theorem 1.3 motivated by a theorem of Lech. We then verify Conjecture 1.4 in the case where R is excellent and contains a field. Finally, we verify Conjecture 1.4 in certain low-dimensional cases and the case where $R/\mathfrak{p}$ is regular.

2.1 An Inequality for Multiplicities

In order to deal with Conjecture 1.4 effectively, we need to know under what conditions the Hilbert-Samuel multiplicity is "well-behaved" with respect to localization, that is, when $e(R_{\mathfrak{p}}) \leq e(R)$ for all prime ideals $\mathfrak{p}$. This condition is a generalization of the containment $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$ of Theorem 1.3. One class of rings for which this is true is the class of quasi-unmixed, Nagata rings (see Theorem 2.20 below). We recall a few definitions and background results here.

Definition 2.1 Let $(A, \mathfrak{m})$ be a Noetherian local ring. We say that A is *equidimensional* if $\dim(A/\mathfrak{p}) = \dim(A)$ for all $\mathfrak{p} \in \min(A)$. We say that A is *quasi-unmixed* if its completion $\hat{A}$ is equidimensional. A (nonlocal) ring R is *quasi-unmixed* if $R_{\mathfrak{n}}$ is quasi-unmixed for every maximal ideal $\mathfrak{n}$.

The class of quasi-unmixed rings is closed under localization and polynomial extensions, as the following lemma shows.

Lemma 2.2 (Hermann, Ikeda and Orbanz [8] Theorems 18.13 and 18.17) *Let A be a local ring. Then A is quasi-unmixed if and only if A is equidimensional and universally catenary. If A is quasi-unmixed, then*

1. *$A_{\mathfrak{p}}$ is quasi-unmixed for every prime ideal $\mathfrak{p}$ of A.*

2. *The polynomial ring $A[X_1, \ldots, X_n]$ is quasi-unmixed.*

Definition 2.3 A ring A is called *Nagata* if, for every prime ideal $\mathfrak{p}$ of A and every finite field extension L of the quotient field of $R/\mathfrak{p}$, the integral closure of $A/\mathfrak{p}$ in L is module finite over $A/\mathfrak{p}$.[1]

The class of Nagata rings is closed under algebras essentially of finite type and contains the class of excellent rings, as the following lemma shows. (For a complete definition of excellent rings, see [15] Section 32. Classical examples of excellent rings are complete rings, rings that are of finite type over a field or the ring of integers, and localizations of excellent rings.)

Lemma 2.4 ([16] (36.1) and (36.5), Matsumura [14] Theorem 78) *Let A be a ring.*

1. *If A is Nagata and B is essentially of finite type over A, then B is Nagata.*

2. *If A is excellent, then A is Nagata.*

Hilbert polynomials and multiplicities shall play a central role in our work. An excellent reference for the results on generalized Hilbert polynomials and multiplicities is [8] Chapter 1.

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[1] As is noted in Matsumura [15] p. 264, "Nagata defined and studied the class of pseudo-geometric rings, which were called "anneaux universellement japonais" by Grothendieck. These are now known as 'Nagata rings'."

Definition 2.5 Assume that $(A, \mathfrak{m})$ is a local ring and let M be a finitely generated, nonzero A-module. Given an ideal $\mathfrak{a}$ of A such that $M/\mathfrak{a}M$ has finite length, the *Hilbert polynomial* of $\mathfrak{a}$ on M, denoted $H[\mathfrak{a}, M](n)$, is the polynomial in n of degree $r = \dim(M)$ with rational coefficients such that for $n \gg 0$

$$H[\mathfrak{a}, M](n) = \text{length}(M/\mathfrak{a}^{n+1}M).$$

If e_r is the leading coefficient of $H[\mathfrak{a}, M](n)$, then the *Samuel multiplicity* of $\mathfrak{a}$ on M is $e(\mathfrak{a}, M) = r!e_r$. We denote $e(\mathfrak{m}, M)$ by $e(M)$. If $y_1, \ldots, y_m$ is a system of parameters for M, then for brevity, we let $e(\mathbf{y}, M) = e((\mathbf{y})A, M)$. If $\mathfrak{b} \subset R$ is any ideal of R and $z_1, \ldots, z_l$ is a system of parameters for $M/\mathfrak{b}M$, then define the *generalized Hilbert polynomial* of $\mathbf{z}$ and $\mathfrak{b}$ on M to be the polynomial $H[\mathbf{z}, \mathfrak{b}, M](n)$ such that for $n \gg 0$

$$H[\mathbf{z}, \mathfrak{b}, M](n) = e(\mathbf{z}, M/\mathfrak{b}^{n+1}M).$$

(c.f., [8] (3.4) for the proof that $e(\mathbf{z}, M/\mathfrak{b}^{n+1}M)$ is a polynomial function.) If the ideal $\text{Ann}(M/\mathfrak{b}M) = \mathfrak{p}$ is prime, then the degree of $H[\mathbf{z}, \mathfrak{b}, M]$ is exactly $s = \dim(M_\mathfrak{p})$. If f_s is the leading coefficient of $H[\mathbf{z}, \mathfrak{b}, M]$ then we define the *generalized multiplicity* of $\mathbf{z}$ and $\mathfrak{b}$ on M as $e(\mathbf{z}, \mathfrak{b}, M) = s!f_s$.

We shall find the following *Associativity Formula* helpful when dealing with the Hilbert-Samuel multiplicity.

Lemma 2.6 (Bruns and Herzog [4] Corollary 4.7.8) *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then*

$$e(A) = \sum \text{length}(A_\mathfrak{p})e(A/\mathfrak{p})$$

where the sum is taken over all prime ideals $\mathfrak{p}$ such that $\dim(A/\mathfrak{p}) = \dim(A)$.

The following lemma demonstrates one aspect of the behavior of the Hilbert-Samuel multiplicity under specialization.

Lemma 2.7 ([15] Theorem 14.9) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with R-regular sequence $x_1, \ldots, x_k \in \mathfrak{m}$. Suppose that $x_i \in \mathfrak{m}^{\nu_i}$ for each i. Then $e(R/(\mathbf{x})) \geq e(R)\nu_1 \cdots \nu_k$.*

We make use of generalized multiplicities in exactly one argument (Proposition 2.19), and they seem crucial for the proof of this result. One particularly important property is the following.

Lemma 2.8 ([8] Proposition 3.11) *With notation as above, assume that $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} \supseteq Ann(M)$ and $\sqrt{\mathfrak{p} + (\mathbf{z})A} = \mathfrak{m}$. Then*

$$e(\mathbf{z}, \mathfrak{p}, M) = e(\mathbf{z}, A/\mathfrak{p})e(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

If, in addition, A is equidimensional, then $e(\mathbf{z}, \mathfrak{p}, A) \leq e(\mathfrak{p} + (\mathbf{z})A, A)$.

To motivate the main result of this section, we rephrase Theorem 1.3 in terms of multiplicities. First, we recall the definition of symbolic powers of prime ideals.

Definition 2.9 Assume that A is a ring with prime ideal $\mathfrak{p}$. For every positive integer n, the nth *symbolic power* of $\mathfrak{p}$ is the ideal

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n A_{\mathfrak{p}} \cap A.$$

Recall that if $\mathfrak{p}$ is a prime ideal in a regular local ring R and x is a nonzero element of $\mathfrak{p}$, then $e(R_{\mathfrak{p}}/(x)) = e$ if and only if $x \in \mathfrak{p}^{(e)} \smallsetminus \mathfrak{p}^{(e+1)}$.

We now give our restatement of Theorem 1.3.

Theorem 1.3$'$ *Assume that $(R, \mathfrak{m})$ is a regular local ring with prime ideal $\mathfrak{p}$. Then for all $x \in \mathfrak{p}$, $e(R_{\mathfrak{p}}/(x)) \leq e(R/(x))$.*

To see that this is a restatement of Theorem 1.3, fix $R, \mathfrak{m}, \mathfrak{p}, x$ as in the Theorem 1.3$'$. Our interpretation of multiplicity implies that $e(R_{\mathfrak{p}}/(x)) \leq e(R/(x))$ if and only if $x \in \mathfrak{p}^{(e)} \smallsetminus \mathfrak{p}^{(e+1)}$ and $x \in \mathfrak{m}^e$ where $e = e(R_{\mathfrak{p}}/(x))$.

Example 4.4 below shows that the regularity assumption in Theorem 1.3 is necessary.

Before we state and prove the generalization (which is essentially due to Lech), we need some preliminaries. The following lemma provides a useful description of maximal ideals in polynomial rings.

Lemma 2.10 *Assume that k is a field and that N is a maximal ideal in the polynomial ring $T = k[X_1, \dots, X_n]$. Then N can be generated by n (irreducible) elements $f_1, \dots, f_n$ such that $f_i \in k[X_1, \dots, X_i]$ and f_i is monic in X_i. Furthermore, if k is the residue field of a Noetherian local ring $(A, \mathfrak{n})$ and F_i is a lift of f_i in $S_i = A[X_1, \dots, X_i] \subseteq A[X_1, \dots, X_n] = S$ which is monic in X_i, then the extension $A \to S/(\mathbf{F})S$ is flat. Furthermore, $\mathbf{F}$ is an S-regular sequence.*

Proof. The existence of the f_i is proved in [15] Theorem 5.1. Now, assume that k is the residue field of a Noetherian local ring $(A, \mathfrak{n})$ and F_i is a lift of f_i in S_i which is monic in X_i. (Such lifts always exist.) Then the extension $A \to A[X_1]/(F_1) = S_1/(F_1)$ is finite and free, as F_1 is monic in X_1. in particular, the extension is flat. Similarly, the extension

$$A[X_1]/(F_1) \to (A[X_1]/(F_1))[X_2]/(F_2) = S_2/(F_1, F_2)$$

is flat. In a similar way, we see that each extension

$$A \to A[X_1]/(F_1) \to \cdots \to S/(F_1, \dots, F_n)$$

is flat. Since the composition of flat maps is flat, the map $A \to S/(\mathbf{F})S$ is flat. To see that the sequence is S-regular, note that since $F_1 \in S_1$ is monic in X_1 it is regular on $S_1 = A[X_1]$. Therefore, F_1 is regular on $S = S_1[X_2, \dots, X_n]$. Also, $S/(F_1)S = (A[X_1]/(F_1))[X_2, \dots, X_n] = (A[X_1]/(F_1))[X_2][X_3, \dots, X_n]$ so that the same argument shows that F_2 is regular on $S/(F_1)S$, and similarly for $F_3, \dots, F_n$. ■

The following lemma provides useful information regarding the behavior of certain numerical data under flat, local extensions. For a finitely generated module M over a local ring $(A, \mathfrak{m})$, let $\mu(M)$ denote the minimal number of generators of M, i.e., $\mu(M) = \dim_{A/\mathfrak{m}}(M/\mathfrak{m}M)$.

Lemma 2.11 *Assume that $A \to B$ is a flat local homomorphism of Noetherian local rings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ and that $\mathfrak{a}$ is an ideal of A.*

1. *$\mu(\mathfrak{a}) = \mu(\mathfrak{a}B)$.*

2. *If, in addition,* $\mathfrak{m}B = \mathfrak{n}$ *then* $\text{length}_A(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \text{length}_B(\mathfrak{n}^n/\mathfrak{n}^{n+1})$. *In particular,* $e(A) = e(B)$.

3. *More generally, if* $\mathfrak{m}B$ *is* $\mathfrak{n}$*-primary and* M *is an* A*-module of finite length, then* $M \otimes_A B$ *has finite length over* B *and*

$$\text{length}_B(M \otimes_A B) = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}B)$$

Proof. Part 1 is proved by Herzog [9] Lemma 2.3. Part 2 follows from part 1 by the following computation

$$\text{length}_A(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \mu(\mathfrak{m}^n) = \mu(\mathfrak{m}^n B) = \mu(\mathfrak{n}^n) = \text{length}_B(\mathfrak{n}^n/\mathfrak{n}^{n+1})$$

and the definitions of $e(A)$ and $e(B)$.

To prove part 3, let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a filtration of M such that each quotient $M_i/M_{i-1} \cong A/\mathfrak{m}$ where $n = \text{length}_A(M)$. Then the fact that B is flat over A implies that $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \cdots \subset M_n \otimes_A B = M \otimes_A B$ is a filtration of $M \otimes_A B$ with quotients

$$(M_i \otimes_A B)/(M_{i-1} \otimes_A B) \cong (M_i/M_{i-1}) \otimes_A B \cong (A/\mathfrak{m}) \otimes_A B \cong B/\mathfrak{m}B$$

so that the only associated prime of $M \otimes_A B$ is $\sqrt{\mathfrak{m}B} = \mathfrak{n}$. In particular, $M \otimes_A B$ has finite length over B. Furthermore,

$$\begin{aligned}\text{length}_B(M \otimes_A B) &= \sum_i \text{length}_B(M_i \otimes_A B)/(M_{i-1} \otimes_A B) = \sum_i \text{length}_B(B/\mathfrak{m}B) \\ &= n\,\text{length}_B(B/\mathfrak{m}B) = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}B)\end{aligned}$$

as desired. ■

The following lemma provides useful information regarding the behavior of regular sequences under (faithfully) flat extensions and the persistence of (faithful) flatness under the taking of quotients.

Lemma 2.12 *Assume that* $A \to B$ *is a (faithfully) flat extension of Noetherian rings and that* A *is local with maximal ideal* $\mathfrak{m}$. *Let* $\mathbf{x} = x_1, \ldots, x_k \in \mathfrak{n}$ *be an* A*-regular sequence and* $\mathfrak{a} \subseteq \mathfrak{m}$ *an ideal. Then*

1. $\mathbf{x}$ *is a* B*-regular sequence.*

2. *The extension* $A/\mathfrak{a} \to B/\mathfrak{a}B$ *is (faithfully) flat.*

Proof. Part 1 follows from [4] Proposition 1.1.2. To prove part 2, it suffices to prove that if $A \to B$ is flat, then $A/\mathfrak{a} \to B/\mathfrak{a}B$ is flat. This is sufficient, as if $A \to B$ is a faithfully flat, then $\mathfrak{m}B \neq (1)B$ so that $\mathfrak{m}B/\mathfrak{a}B \neq (1)B/\mathfrak{a}B$. Since $A/\mathfrak{a} \to B/\mathfrak{a}B$ is flat, faithful flatness follows immediately.

Let M be any $A/\mathfrak{a}$-module. Then

$$B/\mathfrak{a}B \otimes_{A/\mathfrak{a}} M \cong B \otimes_A A/\mathfrak{a} \otimes_{A/\mathfrak{a}} M \cong B \otimes_A M$$

It follows that if $M_1 \to M_2 \to M_3$ is an exact sequence of $A/\mathfrak{a}$-modules, then applying $- \otimes_{A/\mathfrak{a}} B/\mathfrak{a}B$ to the sequence yields

$$M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B$$

which is exact if B is flat over A. Thus, $B/\mathfrak{a}B$ is flat over $A/\mathfrak{a}$. ■

Corollary 2.13 *Assume that* $A \to B$ *is a flat local homomorphism of local rings* $(A, \mathfrak{m})$ *and* $(B, \mathfrak{n})$.

1. *If* $\mathfrak{m}B$ *is* $\mathfrak{n}$*-primary, then*

   (a) $\dim(B) = \dim(A)$.

   (b) If A *is Cohen-Macaulay, then* B *is Cohen-Macaulay.*

   (c) $e(\mathfrak{m}B, B) = e(A)\mathrm{length}_B(B/\mathfrak{m}B)$

   (d) $e(B) \leq e(A)\mathrm{length}_B(B/\mathfrak{m}B)$

2. *If* $\mathfrak{p}$ *is a prime ideal of* A *and* P *is a prime ideal of* B *which is minimal over* $\mathfrak{p}B$*, then* $P \cap A = \mathfrak{p}$ *and* $\mathrm{ht}\,(P) = \mathrm{ht}\,(\mathfrak{p})$.

Proof. 1(a). Let $d = \dim(A)$. By flatness, the extension $A \to B$ satisfies going-down. In particular, $d \leq \dim(B)$. Let $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m}$ be a system of parameters for A

and let $\mathfrak{a} = (\mathbf{x})A$. Then $\sqrt{\mathfrak{a}} = \mathfrak{m}$ so that $\sqrt{\mathfrak{a}B} \supseteq \sqrt{\mathfrak{m}B} = \mathfrak{n}$. Thus, $\mathfrak{a}B$ is $\mathfrak{n}$-primary and generated by d elements, so that $d \geq \dim(B)$.

1(b). The system of parameters $\mathbf{x}$ is a regular sequence on A of length d. By Lemma 2.12, $\mathbf{x}$ is a regular sequence on B of length $\dim(B)$.

1(c). By Lemma 2.11 part 3,

$$\text{length}_B(B/\mathfrak{m}^n B) = \text{length}_B((A/\mathfrak{m}^n) \otimes_A B) = \text{length}_A(A/\mathfrak{m}^n)\text{length}_B(B/\mathfrak{m}B)$$

As the left-hand side and the right hand side are both polynomials of degree d in n by part (a), we have the desired equality.

1(d). As $\mathfrak{m}B \subseteq \mathfrak{n}$, we have $e(B) = e(\mathfrak{n}, B) \leq e(\mathfrak{m}B, B) = e(A)\text{length}_B(B/\mathfrak{m}B)$.

2. Suppose that $P \cap A \neq \mathfrak{p}$. Since $P \cap A$ is a prime ideal of A properly containing $\mathfrak{p}$, going-down implies that P contains a prime ideal Q that contracts to $\mathfrak{p}$. But then $\mathfrak{p}B \subseteq Q \subset P$, contradicting the minimality of P. The fact that $\text{ht}\,(P) = \text{ht}\,(\mathfrak{p})$ is proved in [8] Lemma 18.10. ■

Example 4.5 below shows that the inequality in part 1(d) can be strict and that equality may hold, even for flat, local homomorphisms of regular local rings.

The following lemma gives the first tool we need in order to establish the fact that the Hilbert-Samuel multiplicity is well-behaved under polynomial extensions.

Lemma 2.14 *Let $(A, \mathfrak{n})$ be a local ring, $S = A[X_1, \ldots, X_n]$ a ring of polynomials and M a maximal ideal of S such that $M \cap A = \mathfrak{n}$. Let X be an indeterminate, $A(X) = A[X]_{\mathfrak{n}[X]}$ and $S_M(X) = S_M[X]_{M_M[X]}$. Then, there is a maximal ideal ideal N of $A(X)[X_1, \ldots, X_n]$ such that $S_M(X) = A(X)[X_1, \ldots, X_n]_N$.*

Proof. To start, we have

$$\begin{aligned} S_M(X) &= A[X_1, \ldots, X_n]_M[X]_{M[X]} = A[X_1, \ldots, X_n, X]_{M[X]} \\ &= A[X][X_1, \ldots, X_n]_{M[X]}. \end{aligned}$$

The prime ideal $M[X] \subset A[X][X_1, \ldots, X_n]$ avoids the multiplicative subset $A[X] \setminus \mathfrak{n}[X]$ and therefore corresponds to a prime ideal N in $A[X]_{\mathfrak{n}[X]}[X_1, \ldots, X_n]$. Furthermore,

$$\begin{aligned}\operatorname{ht}(N) &= \operatorname{ht}(M[X]) = \dim(A) + n = \dim(A(X)) + n \\ &= \dim(A[X]_{\mathfrak{n}[X]}[X_1, \ldots, X_n])\end{aligned}$$

so that N is maximal. Finally,

$$\begin{aligned}S_M(X) &= A[X][X_1, \ldots, X_n]_{M[X]} = A[X]_{\mathfrak{n}[X]}[X_1, \ldots, X_n]_{M[X]} \\ &= A(X)[X_1, \ldots, X_n]_N\end{aligned}$$

which shows that N has the desired properties. ■

The following lemma gives the first indication that the Hilbert-Samuel multiplicity is well-behaved under polynomial extensions and provides us with the first tool needed to prove that $e(R_{\mathfrak{p}}) \leq e(R)$ for a large class of rings.

Lemma 2.15 *Assume that $(R, \mathfrak{m})$ is a local Noetherian ring of dimension d and n is a positive integer. Let $S = R[X_1, \ldots, X_n]$. Let $R(\mathbf{X}) = S_{\mathfrak{m}S}$. Then $e(R) = e(R(\mathbf{X}))$.*

Proof. If $J = \mathfrak{m}S$ then, it is straightforward to verify that

$$gr_{J_J}(S_J) \cong gr_{\mathfrak{m}}(R)(Z_1, \ldots, Z_n)$$

where

$$\begin{aligned}gr_{\mathfrak{m}}(R)(Z_1, \ldots, Z_n)_k &= (\mathfrak{m}^k/\mathfrak{m}^{k+1})(Z_1, \ldots, Z_n) \\ &= \left\{ \frac{f}{g} \mid f \in (\mathfrak{m}^k/\mathfrak{m}^{k+1})[Z_1, \ldots, Z_n], g \in (R/\mathfrak{m})[Z_1, \ldots, Z_n] \right\}\end{aligned}$$

It follows that, if $w_1, \ldots, w_t \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a basis of $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ over $R/\mathfrak{m}$, then $w_1, \ldots, w_t \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a basis of J_J^k/J_J^{k+1} over S_J/J_J. That is, the Hilbert functions of $gr_{J_J}(S_J)$ and $gr_{\mathfrak{m}}(R)$ are the same, and the claim follows immediately. ■

Corollary 2.21 below gives a surprising generalization of this lemma.

We are interested in proving that $e(R_{\mathfrak{p}}) \leq e(R)$ for a large class of rings. Before we state and prove the result, we need some background information on reductions of ideals and the analytic spread of an ideal.

Definition 2.16 Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}$ be an ideal. The *Rees algebra* of $\mathfrak{a}$ is the graded ring

$$A[\mathfrak{a}t] = \oplus_{n=0}^{\infty} \mathfrak{a}^n t$$

which we consider as a subring of $A[t]$. The *associated graded ring* of $\mathfrak{a}$ is the graded ring

$$gr_{\mathfrak{a}}(A) = \oplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1} = A[\mathfrak{a}t]/\mathfrak{a}A[\mathfrak{a}t].$$

The *special fibre* of the Rees algebra $R[\mathfrak{a}t]$ is the ring

$$F(\mathfrak{a}) = R[\mathfrak{a}t] \otimes_R R/\mathfrak{m}$$

The Krull dimension of $F(\mathfrak{a})$ is the *analytic spread* of $\mathfrak{a}$ and is denoted $s(\mathfrak{a})$. An ideal $\mathfrak{b}$ contained in $\mathfrak{a}$ is a *reduction* of $\mathfrak{a}$ if there exists a positive integer n such that $\mathfrak{a}^{n+1} = \mathfrak{b}\mathfrak{a}^n$. In this case, $\mathfrak{b}$ is a *minimal reduction* of $\mathfrak{a}$ if it is minimal among all reductions of $\mathfrak{a}$ with respect to inclusion.

We summarize some useful properties of the analytic spread of an ideal and of reductions of an ideal.

Lemma 2.17 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and let $\mathfrak{a}$ be an ideal.*

1. $\operatorname{ht}(\mathfrak{a}) \leq s(\mathfrak{a}) \leq \dim(A)$.

2. $s(\mathfrak{a}) = \dim(gr_{\mathfrak{a}}(A)/\mathfrak{m}gr_{\mathfrak{a}}(A))$.

3. *$\mathfrak{a}$ has a minimal reduction $\mathfrak{b}$.*

4. *if $A/\mathfrak{m}$ is infinite, then we may choose a minimal reduction of $\mathfrak{m}$ generated by d elements which form part of a minimal generating set for $\mathfrak{m}$.*

5. *If $A/\mathfrak{m}$ is infinite and $\mathfrak{b}$ is a minimal reduction of $\mathfrak{a}$, then the natural map $gr_{\mathfrak{b}}(A)/\mathfrak{m}gr_{\mathfrak{b}}(A) \to gr_{\mathfrak{a}}(A)/\mathfrak{m}gr_{\mathfrak{a}}(A)$ is a graded inclusion and an integral extension. In particular, $s(\mathfrak{a}) = s(\mathfrak{b})$. Also, $\mu(\mathfrak{b}) = s(\mathfrak{a})$.*

6. If $\mathfrak{a}$ is an m-primary ideal and $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $e(\mathfrak{b}, A) = e(\mathfrak{a}, A)$.

Proof. For 1,2,3,5 and 6, see Vasconcelos [22] Section 5.1 and Brodmann and Sharp [3] Sections 18.1–18.2. Part 4 follows from the construction, which we outline here. Let $F(\mathfrak{m}) = A[\mathfrak{m}t] \otimes_A A/\mathfrak{m} = gr_{\mathfrak{m}}(A)$, which is a finitely generated algebra over $A/\mathfrak{m}$. By Noether normalization, there are elements $y_1, \ldots, y_d$ of $F(\mathfrak{m})$ such that the natural map $A/\mathfrak{m}[Y_1, \ldots, Y_d] \to F(\mathfrak{m})$ is injective and is an integral extension. If $A/\mathfrak{m}$ is infinite, then the y_i can be chosen in degree 1. The generators of the minimal reduction of $\mathfrak{m}$ are exactly lifts of the y_i to $\mathfrak{m}$, and since the y_i have degree 1 these lifts form part of a minimal set of generators for $\mathfrak{m}$. ■

The following theorem gives a second indication that the Hilbert-Samuel multiplicity is well-behaved under polynomial extensions and provides us with the next tool needed to prove that $e(R_{\mathfrak{p}}) \leq e(R)$ for a large class of rings.

Theorem 2.18 *Assume that $(A, \mathfrak{n}, K)$ is a local ring, and let M be a maximal ideal in the polynomial ring $S = A[X_1, \ldots, X_n]$ such that $M \cap A = \mathfrak{n}$. Then $e(A) = e(S_M)$.*

Proof. First, we reduce to the case where K is infinite. Suppose that K is finite. With the notation of Lemma 2.15 we see that $e(A) = e(A(X))$ and $e(S_M) = e(S_M(X))$. By Lemma 2.14, $S_M(X) = A(X)[X_1, \ldots, X_n]_N$ for some maximal ideal N of $A(X)[X_1, \ldots, X_n]$ so that if $e(A(X)) = e(S_M(X))$ then we are done. Thus, we may assume that K is infinite.

Let $T = K[X_1, \ldots, X_n] = S/\mathfrak{n}S$ and $N = M/\mathfrak{n}S$. Then N is a maximal ideal in T, and has a generating set $f_1, \ldots, f_n$ as in Lemma 2.10. Let $F_i \in S$ be a lift of f_i as in Lemma 2.10, so that the extension $A \to S/(\mathbf{F})S$ is flat. This implies that the composition $A \to S/(\mathbf{F})S \to (S/(\mathbf{F})S)_M = S_M/(\mathbf{F})S_M$ is flat. Furthermore, the maximal ideal of $S_M/(\mathbf{F})S_M$ is exactly $M_M/(\mathbf{F})S_M = (\mathfrak{n}S_M + (\mathbf{F})S_M)/(\mathbf{F})S_M = \mathfrak{n}(S_M/(\mathbf{F})S_M)$. By Lemma 2.11, $e(A) = e(S_M/(\mathbf{F})S_M)$, so it suffices to show that $e(S_M) = e(S_M/(\mathbf{F})S_M)$. The fact that K is infinite implies that $\mathfrak{n}$ has a reduction ideal $\mathfrak{a} \subseteq \mathfrak{n}$, which is generated by $\dim(A)$ elements, by Lemma 2.17, part 5. It is straightforward to verify that the ideal $L_M = \mathfrak{a}S_M + (\mathbf{F})S_M \subseteq M_M = \mathfrak{n} + (\mathbf{F})S_M$ is a reduction of M_M. Furthermore, L_M is generated by $\dim(A)+n$ elements, that is, by a system of parameters. By Lemma 2.10, the F_i form a regular sequence on S_M. Thus,

$$e(S_M) = e(L_M, S_M)$$

(as L_M is a reduction ideal of M_M)

$$= e(L_M/(\mathbf{F})S_M, S_M/(\mathbf{F})S_M)$$

(by repeated application of [15] Theorem 14.11)

$$= e(S_M/(\mathbf{F})S_M)$$

as $L_M/(\mathbf{F})S_M$ is a reduction of $M_M/(\mathbf{F})S_M$. ■

Example 4.6 below shows that the condition $M \cap A = \mathfrak{n}$ does not automatically follow from the fact that M is maximal.

The following proposition is the key tool for proving our generalization of Theorem 1.3. We note that its proof contains the only argument using the generalized multiplicities introduced in Definition 2.5.

Proposition 2.19 *Assume that R is an equidimensional local ring and that $\mathfrak{p}$ is a prime ideal of R such that $R/\mathfrak{p}$ is a regular local ring. Then $e(R_\mathfrak{p}) \leq e(R)$.*

Proof. By assumption, there is a sequence $z_1, \ldots, z_l$ in R such that $\mathbf{z}$ is a regular system of parameters for $R/\mathfrak{p}$. Then applying Lemma 2.8

$$\begin{aligned} e(R_\mathfrak{p}) &= e(\mathfrak{p}_\mathfrak{p}, R_\mathfrak{p})e(\mathbf{z}, R/\mathfrak{p}) = e(\mathbf{z}, \mathfrak{p}, R) \\ &\leq e((\mathbf{z})R + \mathfrak{p}, R) = e(\mathfrak{m}, R) = e(R) \end{aligned}$$

as desired. ■

The following is a generalization of Theorem 1.3 that is motivated by a theorem of Lech [13]. Lech's theorem is stronger in the sense that it gives an inequality for Hilbert polynomials instead of multiplicities. It is slightly weaker, though, as it assumes the ring A is excellent instead of quasi-unmixed and Nagata.

Theorem 2.20 *Assume that* $(R, \mathfrak{m})$ *is a local, quasi-unmixed, Nagata ring with prime ideal* $\mathfrak{p}$. *Then* $e(R_\mathfrak{p}) \leq e(R)$.

Proof. If we can prove the inequality for $\dim(R/\mathfrak{p}) = 1$, then we will be done, as follows. Let $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m}$ be a saturated chain of prime ideals in R. Each $R_{\mathfrak{p}_i}$ is a local, quasi-unmixed, Nagata ring with prime ideal $(\mathfrak{p}_{i-1})_{\mathfrak{p}_i}$. The fact that $\dim(R_{\mathfrak{p}_i}/(\mathfrak{p}_{i-1})_{\mathfrak{p}_i}) = 1$ and the one-dimensional case imply that

$$e(R_\mathfrak{p}) = e((R_{\mathfrak{p}_1})_{(\mathfrak{p}_0)_{\mathfrak{p}_1}}) \leq e(R_{\mathfrak{p}_1}) \leq \cdots \leq e(R_{\mathfrak{p}_n}) = e(R)$$

as desired.

Assume that $\dim(R/\mathfrak{p}) = 1$. Since R is Nagata, the integral closure A of $R/\mathfrak{p}$ is module-finite over $R/\mathfrak{p}$. In particular, $\dim(A) = \dim(R/\mathfrak{p}) = 1$. A is Noetherian and integrally closed, and therefore is a Dedekind domain. Because A is module finite over $R/\mathfrak{p}$, there is a surjection $S = A[X_1, \ldots, X_n] \to B$. Let K denote the kernel of this map so that $S/K = A$. The commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{p} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & A \end{array}$$

shows that $K \cap R = \mathfrak{p}$. Let $\mathfrak{r} \subset A$ be a maximal ideal. Since the extension $R/\mathfrak{p} \to A$ is finite, we know that $(\mathfrak{r} \cap R)/\mathfrak{p}$ is maximal in $R/\mathfrak{p}$, that is, $(\mathfrak{r} \cap R)/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$. Let $M = \phi^{-1}(\mathfrak{r})$ which properly contains K and is therefore maximal in S, as $\dim(S/K) = 1$. Then $S_M/K_M \cong A_{MA} = A_\mathfrak{r}$ is a discrete valuation ring. Again by the commuting diagram, $M \cap R = \mathfrak{m}$. By Theorem 2.18 we see that $e(R) = e(S_M)$. Furthermore, after localizing at $\mathfrak{p}$ we see that $e(R_\mathfrak{p}) = e(S_K)$ since K corresponds to a maximal ideal of $R_\mathfrak{p}[X_1, \ldots, X_n]$ such that $K_\mathfrak{p} \cap R_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$. By replacing R with S_M and $\mathfrak{p}$ with K_M, we may assume that $R/\mathfrak{p}$ is a discrete valuation ring. The result now follows from Proposition 2.19. ■

Example 4.7 below shows that the quasi-unmixedness requirement in Theorem 2.20 is necessary.

The following Corollary is a surprising generalization of Lemma 2.15 and Theorem 2.18.

Corollary 2.21 *Assume that* $(R, \mathfrak{m})$ *is a local, quasi-unmixed, Nagata ring and let* $S = R[X_1, \ldots, X_n]$ *be a polynomial ring over* R. *Assume that* P *is a prime ideal of* S *and let* $\mathfrak{p} = P \cap R$. *Then* $e(R_{\mathfrak{p}}) = e(S_P)$.

Proof. The localization $R_{\mathfrak{p}}$ is a local, equidimensional, Nagata ring and the ideal $PR_{\mathfrak{p}}[X_1, \ldots, X_n]$ is a prime ideal of $R_{\mathfrak{p}}[X_1, \ldots, X_n]$. Thus, we may assume that $\mathfrak{p} = \mathfrak{m}$. Let M denote a maximal ideal of S such that $P \subseteq M$. Then the rings S_M and S_P are both local, equidimensional and Nagata so that by Lemma 2.15 and Theorems 2.20 and 2.18

$$e(R_{\mathfrak{p}}) = e(S_{\mathfrak{p}S}) \leq e(S_P) \leq e(S_M) = e(R_{\mathfrak{p}})$$

giving the desired equality. ■

Due to the fact that the proof of Theorem 2.20 depends heavily on the Nagata condition, one might look for a counterexample to the inequality $e(R_{\mathfrak{p}}) \leq e(R)$ among the classical examples of non-Nagata rings. A number of such examples may be found in [16] Appendix. Each example is of the form $T = R[c]$ where R is a regular local ring and c is integral over R. In the following proposition and corollary we demonstrate that, for every prime ideal P and every maximal ideal M containing P in a ring of this form, we have the inequality $e(T_P) \leq e(T_M)$.

Proposition 2.22 *Assume that* R *is a regular local ring and that* T *is a domain containing* R *which is finite as an* R*-module. Assume also that* $T = R[c]$ *for some element* $c \in T$. *Then for every maximal ideal* M *of* T *and every prime ideal* P *of* T *such that* $P \subseteq M$, $e(T_P) \leq e(T_M)$.

Proof. Consider the surjection $R[X] \to R[c] = T$ given by $X \mapsto c$. Since T is module-finite over R, c satisfies a monic polynomial with coefficients in R. Let $f \in R[X]$ be a monic polynomial of minimal degree such that $f(c) = 0$. If f is not irreducible, say $f = f_1 f_2$ with each f_i monic and not a unit in $R[X]$, then $0 = f(c) = f_1(c)f_2(c)$ and the fact that T is a domain implies that $f_i(c) = 0$ for $i = 1$ or $i = 2$. Neither f_i is a constant and therefore have strictly smaller degree than f, contradicting the minimality of the degree of f. Thus, our map $R[X] \to T$ factors through the projection $R[X] \xrightarrow{\pi} R[X]/(f)$ giving a surjective homomorphism $R[X]/(f) \xrightarrow{\rho} T$. The rings $R[X]/(f)$ and T are both domains with dimension equal to $\dim(R)$. Thus, the kernel of ρ must be zero, so that ρ is an isomorphism.

For any maximal ideal M of T let $M_1 = \pi^{-1}(M)$, and for any prime ideal P contained in M let $P_1 = \pi^{-1}(P)$. Then P_1 is a prime ideal in the regular ring $R[X]$ which is contained in the maximal ideal M_1. Furthermore, since $R[X]_{M_1}$ is a regular local ring, Theorem 1.3′ implies that

$$\begin{aligned} e(T_P) = e(R[X]_{P_1}/(f)) &= m_{R[X]_{P_1}}(f) \\ &\leq m_{R[X]_{M_1}}(f) = e(R[X]_{M_1}/(f)) = e(T_M) \end{aligned}$$

which proves the result. ■

Corollary 2.23 *Assume that R is a regular local ring and that T is any ring (not necessarily a domain) containing R which is finite as an R-module. Assume also that $T = R[c]$ for some element $c \in T$. Then for every maximal ideal M of T and every prime ideal P of T such that $P \subseteq M$, $e(T_P) \leq e(T_M)$.*

Proof. Let $Q_1, \ldots, Q_n$ be the minimal prime ideals of T. By the going-up property for finite extensions, we see that $Q_i \cap R = (0)$. Let M be a maximal ideal of T and let P be any prime ideal contained in M. Assume that $Q_1, \ldots, Q_j \subseteq P$ and that $Q_{j+1}, \ldots, Q_m \nsubseteq P$. Then for $Q = 1, \ldots, j$, R is a subring of T/Q_i, and T/Q_i is a finite R-module which is generated by the residue of c as an R-algebra. Thus the inclusion $R \subseteq T/Q_i$ satisfies the hypotheses of the proposition and so $e(T_P/(Q_i)_P) \leq e(T_M/(Q_i)_M)$. By the Associativity Formula

$$\begin{aligned} e(T_P) &= \sum_{i=1}^{j} \text{length}(T_{Q_i}) e(T_P/(Q_i)_P) \leq \sum_{i=1}^{j} \text{length}(T_{Q_i}) e(T_M/(Q_i)_M) \\ &\leq \sum_{i=1}^{n} \text{length}(T_{Q_i}) e(T_M/(Q_i)_M) = e(T_M) \end{aligned}$$

which is the desired result. ■

The following propositions shows that another case where the Nagata condition of Theorem 2.20 may be omitted is the case where $\mathfrak{p}$ is minimal of the "correct" dimension.

Proposition 2.24 *Assume that $(R, \mathfrak{m})$ is a Noetherian local ring with minimal prime ideal $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = \dim(R)$. Then $e(R_{\mathfrak{p}}) \leq e(R)$.*

Proof. The result is a direct consequence of the Associativity Formula which implies that

$$\begin{aligned} e(R) &= \sum \text{length}(R_{\mathfrak{r}}) e(R/\mathfrak{r}) \\ &\geq \text{length}(R_{\mathfrak{p}}) e(R/\mathfrak{p}) \geq \text{length}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}}) \end{aligned}$$

where the sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. ■

2.2 Reduction to the Completion

One standard technique in commutative algebra is to reduce a given question to a question for complete rings. We accomplish this for Conjecture 1.4, assuming that we start with an excellent ring.

The following theorem allows us to reduce Conjecture 1.4 to the case where the quotient $R/\mathfrak{p}$ is a normal domain. This will be the key step in our reduction to the case where R is complete.

Theorem 2.25 *Let $(R, \mathfrak{m})$ be a Nagata, Cohen-Macaulay local ring, and suppose that , for every ring (S, M) which is a localization at a maximal ideal of a polynomial ring over R, the following holds: for all prime ideals P and Q of S such that $\sqrt{P+Q} = M$, $e(S_P) = e(S)$ and S/P is a normal domain,*

$$\dim(S/P) + \dim(S/Q) \leq \dim(S).$$

Then, for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$,

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Proof. Fix prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$. Let B be the integral closure of $R/\mathfrak{p}$. Since R is Nagata, B is module-finite over $R/\mathfrak{p}$. In particular, $\dim(B) = \dim(A/\mathfrak{p})$. Because B is module finite over $A/\mathfrak{p}$, there is a surjection $T = R[X_1, \ldots, X_n] \to B$. Let K denote the kernel of this map so that $T/K = B$. The commutative diagram

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d] & R/\mathfrak{p} \arrow[d] \\ T \arrow[r, "\phi"] & B \end{tikzcd}$$

shows that $K \cap R = \mathfrak{p}$. Let $\mathfrak{n} \subset B$ be a maximal ideal. Since the extension $R/\mathfrak{p} \to B$ is finite, we know that $\mathfrak{n} \cap R/\mathfrak{p}$ is maximal in $R/\mathfrak{p}$, that is, $\mathfrak{n} \cap R/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$. Also, there are no primes of B which are properly contained in $\mathfrak{n}$ and contract to $\mathfrak{m}/\mathfrak{p}$ in $R/\mathfrak{p}$. It follows that $\sqrt{\mathfrak{q}B_{\mathfrak{n}}} = \sqrt{(\mathfrak{m}/\mathfrak{p})B_{\mathfrak{n}}} = \mathfrak{n}_{\mathfrak{n}}$.

Let $N = \phi^{-1}(\mathfrak{n})$, so that $T/N \cong B/\mathfrak{n}$. Then, $K \subseteq N$ and since $\mathfrak{m} = \beta^{-1}(\mathfrak{n}) = R \cap \phi^{-1}(\mathfrak{n}) = R \cap N$, we see that $L = \mathfrak{q}T \subseteq N$. We claim that $\sqrt{K_N + L_N} = N_N$. Since $K + L \supseteq K = \ker(\phi)$ we see that $K + L = \phi^{-1}(\phi(K + L)) = \phi^{-1}(\mathfrak{q}B)$. Let ϕ denote the map $T_N \to B_{\mathfrak{n}}$. If $x \in N_N$, then $\phi(x) \in \mathfrak{n}$ so that for some n, $\phi(x^n) = \phi(x)^n \in \mathfrak{q}B_{\mathfrak{n}}$. Then $x^n \in \phi^{-1}(\mathfrak{q}B_{\mathfrak{n}}) = (K + L)_N$ so that $N_N \subseteq \sqrt{K_N + L_N}$ as desired.

By Theorem 2.18, we see that $e(T_N) = e(R) = e(R_{\mathfrak{p}}) = e(T_K)$. (The final equality follows from the fact that K determines a maximal ideal of $R_{\mathfrak{p}}[X_1, \ldots, X_n]$.) By construction, T_N/K_N is a normal domain. Furthermore, T_N is a good complete intersection ring of type k. Thus, if $\dim(T_N/K_N) + \dim(T_N/L_N) \leq \dim(T_N)$ then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(T_N/K_N) + \dim(T_N/L_N) - n \leq \dim(T_N) - n = \dim(R)$ as desired. ∎

Note that the same procedure allows us to reduce further to the case where both $R/\mathfrak{p}$ and $R/\mathfrak{q}$ are normal domains.

It is clear from the proof that, in the statement of the theorem, "Cohen-Macaulay" may be replaced by either "quasi-unmixed," "Gorenstein" or "complete intersection (of codimension c)."

Example 4.3 below shows that we must assume that our ring is at least equidimensional, for Conjecture 1.4 to hold.

In the following theorem, "Cohen-Macaulay" may be replaced by either "quasi-unmixed," "Gorenstein" or "complete intersection (of codimension c)." This is the result which will allow us to reduce Conjecture 1.4 to the case where R is complete.

Theorem 2.26 *Let $(R, \mathfrak{m})$ be an excellent, Cohen-Macaulay local ring. Assume the following for every ring (S, M) which is the localization of a polynomial ring over R at a maximal ideal: for all prime ideals $\hat{P}$, $\hat{Q}$ of the completion $\hat{S}$ such that $\sqrt{\hat{P}+\hat{Q}} = \hat{M}$, $e(\hat{S}_{\hat{P}}) = e(\hat{S})$ and $\hat{S}/\hat{P}$ is a normal domain,*

$$\dim(\hat{S}/\hat{P}) + \dim(\hat{S}/\hat{Q}) \leq \dim(\hat{S}).$$

Then, for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$,

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Proof. By Theorem 2.25 it suffices to show that for every ring (S, M) which is a localization at a maximal ideal of a polynomial ring over R, the following holds: for all prime ideals P and Q such that $\sqrt{P+Q} = M$, $e(S_P) = e(S)$ and S/P is a normal domain,

$$\dim(S/P) + \dim(S/Q) \leq \dim(S).$$

Let S, M, P, Q satisfy these hypotheses. S is excellent, which implies that S/P is also excellent (c.f., [15] §32). By [15] Theorem 32.2, $\widehat{S/P} = \hat{S}/P\hat{S}$ is normal. In particular, $\hat{P} = P\hat{S}$ is a prime ideal of $\hat{S}$ such that $\hat{S}/\hat{P}$ is a normal domain. Since the map $S \to \hat{S}$ is faithfully flat, $\hat{P} \cap S = P$. Thus, by Corollary 2.13, $\mathrm{ht}\,(\hat{P}) = \mathrm{ht}\,(P)$. By [8] Theorem 18.13 (d), $\dim(\hat{S}/\hat{P}) = \dim(S/P)$. Let $\hat{Q}$ be any minimal prime ideal of $Q\hat{S}$, for which $\dim(\hat{S}/\hat{Q}) = \dim(S/Q)$ by the same reasoning. Furthermore, the extension $S_P \to \hat{S}_{\hat{P}}$ is faithfully flat and $P_P\hat{S}_{\hat{P}} = \hat{P}_{\hat{P}}$ so that, by Lemma 2.11

$$e(\hat{S}_{\hat{P}}) = e(S_P) = e(S) = e(\hat{S})$$

Thus, by assumption

$$\dim(S/P) + \dim(S/Q) = \dim(\hat{S}/\hat{P}) + \dim(\hat{S}/\hat{Q}) \leq \dim(\hat{S}) = \dim(S)$$

as desired. ■

2.3 The Equicharacteristic Case

In this section, we verify Conjecture 1.4 for excellent rings which contain a field. The main tool is Theorem 2.28. Before we prove the theorem, we recall some basic facts in the following lemma that will allow us to apply Theorem 2.28 to Conjecture 1.4.

Lemma 2.27 ([15] Theorem 29.4, [4] Proposition 2.2.11)

1. *Assume that B is a Noetherian complete local ring containing a field. Then there exists a subring $A \subseteq B$ with the following properties: A is a complete regular local ring with the same residue field as B, and B is finitely generated as an A-module.*
2. *Let B be a Noetherian local ring and A a regular local subring such that B is a finite A-module. Then B is Cohen-Macaulay if and only if it is a free A-module.*

The following theorem is the main tool we need to verify Conjecture 1.4 in the case of an excellent ring that contains a field.

Theorem 2.28 *Assume that B is a Cohen-Macaulay ring and $(A, \mathfrak{m})$ a regular local subring such that B is a finite free A-module. Assume that P is a prime ideal of B with $P \cap A = \mathfrak{p}$ and $e(B_P) = \mathrm{rank}_A(B)$. Then P is the unique prime ideal of B which contracts to $\mathfrak{p}$ in A, and $B_P/P_P \cong A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.*

Proof. First, we reduce to the case where $\mathfrak{p} = \mathfrak{m}$. It suffices to show that the extension $A_\mathfrak{p} \to B_\mathfrak{p}$ with $P_\mathfrak{p} \subset B_\mathfrak{p}$ satisfies the hypotheses of the theorem. Any localization of a Cohen-Macaulay ring is Cohen-Macaulay, so $B_\mathfrak{p}$ is Cohen-Macaulay. The ring $A_\mathfrak{p}$ is regular and by the exactness of $- \otimes_A A_\mathfrak{p}$, $A_\mathfrak{p}$ is a subring of $B_\mathfrak{p}$. If $r = \mathrm{rank}_A(B)$ then $B \cong A^r$ and so $B_\mathfrak{p} \cong A_\mathfrak{p}^r$ is finite and free over $A_\mathfrak{p}$. Furthermore, $P_\mathfrak{p}$ is a prime ideal of $B_\mathfrak{p}$ which contracts to the maximal ideal of $A_\mathfrak{p}$ and $e((B_\mathfrak{p})_P) = e(B_P) = \mathrm{rank}_A(B) = r = \mathrm{rank}_{A_\mathfrak{p}}(B_\mathfrak{p})$ so all the hypotheses are satisfied.

By the finiteness of the extension $A \to B$, P is maximal. The ring $B/\mathfrak{m}B$ is Artinian because any ideal of B which contains $\mathfrak{m}B$ must contract to $\mathfrak{m}$ in A and therefore must be maximal. In particular, a regular system of parameters of A passes to a system of parameters of B (and since B is Cohen-Macaulay, a maximal B-regular sequence). Let $K = A/\mathfrak{m}$, $L = B/P$ and $\hat{P} = P/\mathfrak{m}B$. By the finiteness of the extension $A \to B$, the extension $K \to L$ is finite. Since $B/\mathfrak{m}B$ is a finite-dimensional vector space over K, $B/\mathfrak{m}B$ has finite length as an A-module. By the structure theorem for Artinian rings, $B/\mathfrak{m}B$ has a finite number of maximal ideals, $\hat{P} = \hat{P}_1, \hat{P}_2, \ldots, \hat{P}_k$, and $B/\mathfrak{m}B \cong \prod_i (B/\mathfrak{m}B)_{\hat{P}_i}$. In particular, each $(B/\mathfrak{m}B)_{\hat{P}_i}$ is finitely generated over A and has finite length as an A-module. We compute $l = \operatorname{length}_{B_P}(B_P/\mathfrak{m}B_P)$. Let $B_P/\mathfrak{m}B_P = M_l \supset M_{l-1} \supset \cdots \supset M_0 = 0$ be a filtration of $B_P/\mathfrak{m}B_P$ by B_P submodules such that $M_i/M_{i-1} \cong B/P = L$. Then the additivity of length implies that

$$\operatorname{length}_A(B_P/\mathfrak{m}B_P) = \sum_{i=1}^{l} \operatorname{length}_A(M_i/M_{i-1}) = \sum_{i=1}^{l} \dim_K(L) = l \dim_K(L)$$

so that

$$\operatorname{length}_{B_P}(B_P/\mathfrak{m}B_P) = l = \operatorname{length}_A(B_P/\mathfrak{m}B_P)/\dim_K(L).$$

By Lemma 2.7, $e(B_P) \leq e(B_P/\mathfrak{m}B_P)$, as $\mathfrak{m}$ is generated by a B-regular sequence. Our assumptions imply that

$$\begin{aligned}
\sum_i \dim_K((B/\mathfrak{m}B)_{\hat{P}_i}) &= \dim_K(B/\mathfrak{m}B) = \operatorname{rank}_A(B) = e(B_P) \leq e(B_P/\mathfrak{m}B_P) \\
&= \operatorname{length}_{B_P}(B_P/\mathfrak{m}B_P) = \operatorname{length}_A(B_P/\mathfrak{m}B_P)/\dim_K(L) \\
&= \dim_K(B_P/\mathfrak{m}B_P)/\dim_K(L) \leq \dim_K(B_P/\mathfrak{m}B_P) \\
&\leq \sum_i \dim_K((B/\mathfrak{m}B)_{\hat{P}_i})
\end{aligned}$$

so we must have equality. This can happen only if (i) $\hat{P}$ is the unique prime ideal of $B/\mathfrak{m}B$ and (ii) $\dim_K(L) = 1$. These are the desired results. ■

We are now in the position to verify Conjecture 1.4 in the case of an excellent ring that contains a field.

Theorem 2.29 *Assume that* $(R, \mathfrak{m})$ *is an excellent local Cohen-Macaulay ring which contains a field. Also, assume that* P *and* Q *are prime ideals of* R *such that* $\sqrt{P+Q} = \mathfrak{m}$ *and* $e(R_P) = e(R)$*. Then* $\dim(R/P) + \dim(R/Q) \leq \dim(R)$.

Proof. As in the proof of Proposition 2.32, we may pass to the ring $R(X) = R[X]_{\mathfrak{m}[X]}$ to assume that R has infinite residue field. By Theorems 2.25 and 2.26 we may pass to the completion of the localization of a polynomial ring over R at a maximal ideal. In particular, we may assume that R is a complete local Cohen-Macaulay ring which contains a field.

Lemma 2.17 shows that we may choose a system of parameters $y_1, \ldots, y_n$ of R such that (i) the y_i form part of a minimal generating set of $\mathfrak{m}$ and (ii) the y_i generate a minimal reduction of $\mathfrak{m}$. Fix $z_1, \ldots, z_q \in \mathfrak{m}$ such that $y_1, \ldots, y_n, z_1, \ldots, z_q$ form a minimal generating set for $\mathfrak{m}$. Since R is complete, R has a coefficient field K and the natural map $K[\![Y_1, \ldots, Y_n]\!] \to R$ given by $Y_i \mapsto y_i$ is injective and R is module finite over $A = K[\![Y_1, \ldots, Y_n]\!]$ (c.f., [15] §29). By Lemma 2.27, the fact that R is local Cohen-Macaulay implies that R is free over A of finite rank r. By the definition of a coefficient ring, we know that the residue field of R is K. Furthermore, the natural map $\rho : K[\![Y_1, \ldots, Y_n, Z_1, \ldots, Z_q]\!] \to R$ given by $Y_i \mapsto y_i$ and $Z_j \mapsto z_j$ is surjective. Let $A' = K[\![Y_1, \ldots, Y_n, Z_1, \ldots, Z_q]\!]$ with maximal ideal $\mathfrak{m}'$ and $I = \ker(\rho)$. The constructions show that we have a natural commuting diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\psi"'] & A' \arrow[d, "\rho"] \\ & R \end{tikzcd}$$

Let $\mathfrak{p} = P \cap A$. Since the extension $A \to R$ is finite and free (i.e. integral and flat) both going-up and going-down hold so that $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(P)$ and $\dim(A/\mathfrak{p}) = \dim(R/P)$. If we can show that $\sqrt{\mathfrak{p}A' + \rho^{-1}(Q)} = \mathfrak{m}'$, then it follows that

$$\begin{aligned} \dim(R/P) + \dim(R/Q) &= \dim(A/\mathfrak{p}) + \dim(A'/\phi^{-1}(Q)) \\ &= \dim(A'/\mathfrak{p}A') - q + \dim(A'/\phi^{-1}(Q)) \\ &\leq \dim(A') - q = n = \dim(R) \end{aligned}$$

as desired.

For an ideal $\mathfrak{a}$ of A', let $Z(\mathfrak{a}) \subseteq \operatorname{Spec}(A')$ denote the closed subscheme determined by $\mathfrak{a}$. In order to show that $\sqrt{\mathfrak{p}A' + \rho^{-1}(Q)} = \mathfrak{m}'$, it suffices to show that

$$Z(\mathfrak{p}A' + \rho^{-1}(Q)) = Z(\rho^{-1}(P) + \rho^{-1}(Q))$$

as the facts that $\sqrt{P+Q} = \mathfrak{m}$ and ρ is surjective imply that $\sqrt{\rho^{-1}(P) + \rho^{-1}(Q)} = \mathfrak{m}'$. Since $\mathfrak{p}A' + \rho^{-1}(Q) \subseteq \rho^{-1}(P) + \rho^{-1}(Q)$, the containment $Z(\mathfrak{p}A' + \rho^{-1}(Q)) \supseteq Z(\rho^{-1}(P) + \rho^{-1}(Q))$ is clear. To demonstrate the other inclusion, we note that, since $I \subseteq \rho^{-1}(Q)$,

$$\mathfrak{p}A' + \rho^{-1}(Q) = (\mathfrak{p}A' + I) + \rho^{-1}(Q)$$

so that

$$Z(\mathfrak{p}A' + \rho^{-1}(Q)) = Z((\mathfrak{p}A' + I) + \rho^{-1}(Q)) = Z(\mathfrak{p}A' + I) \cap Z(\rho^{-1}(Q)).$$

Since $Z(\rho^{-1}(P) + \rho^{-1}(Q)) = Z(\rho^{-1}(P)) \cap Z(\rho^{-1}(Q))$ it then suffices to show that $Z(\mathfrak{p}A' + I) \subseteq Z(\rho^{-1}(P))$. It suffices to show that $\rho^{-1}(P)$ is the unique minimal prime ideal of $\mathfrak{p}A' + I$ in A'. By our commuting diagram, the (minimal) primes of $\mathfrak{p}A' + I$ are in 1-1 correspondence with the (minimal) primes of $\mathfrak{p}R = \rho(\mathfrak{p}A' + I)$. Thus, it suffices to show that P is the unique minimal prime of $\mathfrak{p}R$. By assumption $P \cap A = \mathfrak{p}$, and $e(R_P) = e(R)$. So, if we can show that $e(R) = \operatorname{rank}_A(R)$, then Theorem 2.28 implies that P is the unique prime ideal of R which contracts to $\mathfrak{p}$ in A. If P' is any minimal prime of $\mathfrak{p}R$, then Corollary 2.13 implies that $P' \cap A = \mathfrak{p}$, a contradiction.

To compute $e(R)$, we use the fact that $(\mathbf{y})R$ is a minimal reduction of $\mathfrak{m}$ so that

$$\begin{aligned} e(R) &= \operatorname{length}(R/(\mathbf{y})R) = \dim_K(R/(\mathbf{y})R) = \dim_K(R \otimes_A A/(\mathbf{Y})A) \\ &= \dim_K(R \otimes_A K) = \dim_K(A^r \otimes_A K) = \dim_K(K^r) = r = \operatorname{rank}_A(R). \end{aligned}$$

This completes the proof. ■

We note here that the excellence assumption is used only in order to reduce to the completion.

2.4 The Case When $R/\mathfrak{p}$ is Regular

In this section, we prove Conjecture 1.4 in the case where $R/\mathfrak{p}$ is regular. The proof depends on reductions of ideals, in particular on the notion of equimultiplicity. Here we give the definition of equimultiple ideals and quote the property that is of primary interest to us.

Definition 2.30 An ideal $\mathfrak{a}$ of a local Noetherian ring is said to be *equimultiple* if $\operatorname{ht}(\mathfrak{a}) = s(\mathfrak{a})$.

The following lemma supplies an explanation for this definition.

Lemma 2.31 ([8] Theorem 20.9) *Let A be a quasi-unmixed local ring and let $\mathfrak{p}$ be a prime ideal of A for which $A/\mathfrak{p}$ is regular. Then the following conditions are equivalent.*

1. $e(A) = e(A_{\mathfrak{p}})$
2. $\operatorname{ht}(\mathfrak{p}) = s(\mathfrak{p})$

Examples 4.8 and 4.9 below show that if $R/\mathfrak{p}$ is not regular, neither implication of the lemma holds.

The following proposition is the main result of this section where we prove Conjecture 1.4 in the case where $R/\mathfrak{p}$ is regular. Notice that the Cohen-Macaulayness assumption is loosened to "quasi-unmixed" here.

Proposition 2.32 *Assume that $(R, \mathfrak{m})$ is a quasi-unmixed local ring with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, $R/\mathfrak{p}$ is regular and $e(R_{\mathfrak{p}}) = e(R)$. Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$.

Proof. If the residue field of R is finite, let $R(X) = R[X]_{\mathfrak{m}[X]}$. By Lemma 2.2, $R(X)$ is quasi-unmixed. The prime ideals $P = \mathfrak{p}R(X)$ and $Q = \mathfrak{q}R(X)$ satisfy the properties $\sqrt{P+Q} = \mathfrak{m}R(X)$, $R(X)/P = R/\mathfrak{p}(X)$ is regular and (by Lemma 2.15) $e(R(X)_P) = e(R_\mathfrak{p}) = e(R) = e(R(X)$. Thus, if the proposition holds for $R(X)$ then

$$\begin{aligned}\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) &= \dim(R(X)/P) + \dim(R(X)/Q)\\ &\leq \dim(R(X)) = \dim(R)\end{aligned}$$

and the proposition holds for R. Thus, we may assume that the residue field of R is infinite.

By Lemma 2.31, the prime ideal $\mathfrak{p} \subset R$ is equimultiple. Since the residue field of R is infinite, $\mathfrak{p}$ contains a sequence $y_1, \dots, y_i$ which generate a minimal reduction of $\mathfrak{p}$ where $i = \mathrm{ht}\,(\mathfrak{p})$. Since $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, we see that $\mathfrak{q}$ is an ideal of definition for $R/\mathfrak{p}$ and therefore $\mathfrak{q}$ contains a system of parameters $z_1, \dots, z_j$ for $R/\mathfrak{p}$. In particular $j = \dim(R/\mathfrak{p})$. We claim that $y_1, \dots, y_i, z_1, \dots, z_j$ is a system of parameters for R. Since $i + j = \mathrm{ht}\,(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$, the sequence has the correct length and we need only check that the sequence generates an $\mathfrak{m}$-primary ideal. We compute $\sqrt{(\mathbf{y},\mathbf{z})R} = \sqrt{\sqrt{(\mathbf{y})R} + \sqrt{(\mathbf{z})R}} = \sqrt{\mathfrak{p} + \sqrt{(\mathbf{z})R}}$. The fact that $\mathbf{z}$ is a system of parameters for $R/\mathfrak{p}$ implies that the only prime ideal of R containing $\mathfrak{p}$ and $\mathbf{z}$ is $\mathfrak{m}$, as desired.

To prove the result, it suffices to show that $i \geq \dim(R/\mathfrak{q})$, as this will show that $\dim(R) = i + j \geq \dim(R/\mathfrak{q}) + \dim(R/\mathfrak{p})$. In the ring $R/\mathfrak{q}$, the images of $\mathbf{y}$ generate an ideal which is primary to $\mathfrak{m}/\mathfrak{q}$ since $\sqrt{(\mathbf{y})R} = \mathfrak{p}$. Since $\dim(R/\mathfrak{q})$ is the least integer l such that an ideal primary to the maximal ideal of $R/\mathfrak{q}$ can be generated by l elements, we are done. ■

With Proposition 2.32 in mind, one might hope that there is a nice relation between the conditions "$e(R_\mathfrak{p}) = e(R)$" and "$\mathfrak{p}$ has finite projective dimension." Examples 4.10 and 4.11 below show that these conditions are mutually exclusive in a ring which is not regular.

2.4.1 The Use of Regular Alterations

With Gabber's work on nonnegativity (c.f., [19]) and Proposition 2.32 in mind, it would make sense to try to use de Jong's Theorem on regular alterations to reduce

Conjecture 1.4 to the case where (a global version of) $\mathfrak{p}$ has a regular quotient. Details of the results concerning existence of regular alterations may be found in [5]. The main result for our purposes is the following.

Theorem 2.33 *Let A be a local integral domain that is a localization of a ring of finite type over a field or a complete discrete valuation ring with algebraically closed residue field. Then there exists a projective map $\phi : X \to Spec(A)$ such that*

1. *X is an integral regular scheme.*
2. *If K is the quotient field of A, then the extension $k(X)$ of K is finite (we say that X is generically finite over $Spec(A)$.*

Such a morphism ϕ will be called an *regular alteration*. For us, the result says that, given such a ring A, there is a natural number n and ideal $I \in \text{Proj}(A[X_0, \dots, X_n])$ such that $I \cap R = (0)$, the scheme $\text{Proj}(A[X_0, \dots, X_n]/I)$ is regular, and the natural morphism $\phi : \text{Proj}(A[X_0, \dots, X_n]/I) \to \text{Spec}(A)$ is generically finite. It follows automatically, since ϕ is proper (c.f., Hartshorne [7] Theorem II.4.9) it is closed. Furthermore, the generically finite condition implies that the schemes $\text{Proj}(A[X_0, \dots, X_n]/I)$ and $\text{Spec}(A)$ have the same dimension.

For our purposes, let $(R, \mathfrak{m})$ be a local, equidimensional, Nagata ring which is of finite type over a field or a complete discrete valuation ring with algebraically closed residue field. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_\mathfrak{p}) = e(R)$. We let $A = R/\mathfrak{p}$ and let $I \in \text{Proj}(R/\mathfrak{p}[X_0, \dots, X_n])$ be an ideal coming from a regular alteration of $R/\mathfrak{p}$. Let $S = R[X_0, \dots, X_n]$ so that I corresponds to an element $P \in \text{Proj}(S)$. Let $Q = \mathfrak{q}S$. If P and Q were to satisfy the same conditions satisfied by $\mathfrak{p}$ and $\mathfrak{q}$, at least locally, then we might hope that a counterexample in R would pass to a counterexample in A. If M is any prime ideal of S that contains P, then Theorem 2.20 implies that

$$e(R) = e(R_\mathfrak{p}) \leq e(R_{M \cap R}) \leq e(R)$$

so that $e(R_\mathfrak{p}) = e(R_{M \cap R})$. Corollary 2.21 then implies that

$$e(S_P) = e(R_\mathfrak{p}) = e(R_{M \cap R}) = e(S_M)$$

so that the multiplicity condition of Conjecture 1.4 is satisfied. Example 4.15 below shows that P and Q will not in general satisfy the condition $\sqrt{P+Q} = M$, even after localizing at a maximal ideal M which contains P and Q.

We note that in Gabber's proof of the Nonnegativity Conjecture, he circumvents this problem by using the fact that the map $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is proper. The essential point is that, any element of $\mathrm{Proj}(S)$ which contains I and $\mathfrak{q}S$ must contract to the maximal ideal of R. This tells us that the map $\mathrm{Proj}(S/(I+\mathfrak{q}S)) \to \mathrm{Spec}(R)$ factors through the natural map $\mathrm{Spec}(R/\mathfrak{m}^k) \to \mathrm{Spec}(R)$ for some k, so that $\mathrm{Proj}(S/(I+\mathfrak{q}S))$ is a scheme over $R/\mathfrak{m}^k$. It may be that this global finiteness condition can replace the local finiteness condition that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length, but there is much work to be done here.

2.5 Low-Dimensional Cases

In this section, we prove Conjecture 1.4 in certain low-dimensional cases. Although the proofs are relatively straightforward, the results are worth mentioning. In each case, the Cohen-Macaulay assumption of the conjecture may be relaxed to "equidimensional" with the possible extra assumption "Nagata."

In the following proposition we verify Conjecture 1.4 when $\dim(R/\mathfrak{p}) = \dim(R)$.

Proposition 2.34 *Assume that $(R, \mathfrak{m})$ is an equidimensional local ring with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. Assume that $\mathfrak{p}$ is a minimal prime of R and $e(R_\mathfrak{p}) = e(R)$. Then $\mathfrak{q} = \mathfrak{m}$, so that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.*

Proof. Our assumptions imply that

$$\mathrm{length}(R_\mathfrak{p}) = e(R_\mathfrak{p}) = e(R)$$

and the Associativity Formula implies that

$$\mathrm{length}(R_\mathfrak{p}) = e(R) = \sum \mathrm{length}(R_\mathfrak{r}) e(R/\mathfrak{r})$$

where the sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. Since each $e(R/\mathfrak{r}) > 0$, it follows that $\mathrm{length}(R_\mathfrak{r}) = 0$ for all $\mathfrak{r} \neq \mathfrak{p}$ with $\dim(R/\mathfrak{r}) =$

$\dim(R)$. Since R is equidimensional, this implies that $\{\mathfrak{p}\} = \min(R)$. In particular, $\mathfrak{q} \supseteq \sqrt{(0)} = \mathfrak{p}$. The fact that $\mathfrak{q}$ is prime implies that $\mathfrak{m} = \sqrt{\mathfrak{p}+\mathfrak{q}} = \sqrt{\mathfrak{q}} = \mathfrak{q}$, as desired. ■

The proof of Proposition 2.34 shows that, for minimal primes $\mathfrak{p}$, the assumption $e(R_{\mathfrak{p}}) = e(R)$ is quite strong. In fact, we have the following.

Proposition 2.35 *Assume that $(R, \mathfrak{m})$ is an equidimensional local ring with minimal prime ideal $\mathfrak{p}$, and consider the following statements.*

1. $e(R/\mathfrak{p}) = 1$.

2. $e(R_{\mathfrak{p}}) = e(R)$.

3. *$\mathfrak{p}$ is the unique minimal prime of R.*

(a) Any two of these conditions imply the third.

(b) If R is quasi-unmixed then condition 1 may be replaced by "$R/\mathfrak{p}$ is regular".

Proof. (a) If $e(R/\mathfrak{p}) = 1$ and $e(R_{\mathfrak{p}}) = e(R)$ then the above computation shows that

$$\begin{aligned} \operatorname{length}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}}) = e(R) &= \sum \operatorname{length}(R_{\mathfrak{r}}) e(R/\mathfrak{r}) \\ &\geq \operatorname{length}(R_{\mathfrak{p}}) e(R/\mathfrak{p}) = \operatorname{length}(R_{\mathfrak{p}}) \end{aligned}$$

where the sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. the only term which occurs in the sum is the term corresponding to $\mathfrak{p}$. Since R is equidimensional, this implies that $\mathfrak{p}$ is the unique minimal prime of R.

If $e(R/\mathfrak{p}) = 1$ and $\mathfrak{p}$ is the unique minimal prime of R

$$e(R) = \sum \operatorname{length}(R_{\mathfrak{r}}) e(R/\mathfrak{r}) = \operatorname{length}(R_{\mathfrak{p}}) e(R/\mathfrak{p}) = \operatorname{length}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}})$$

where the sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. If $e(R_{\mathfrak{p}}) = e(R)$ and $\mathfrak{p}$ is the unique minimal prime of R, then a similar computation shows that $e(R/\mathfrak{p}) = 1$.

(b) If R is quasi-unmixed, then [8] Theorem 6.8 implies that $e(R/\mathfrak{p}) = 1 \Leftrightarrow R/\mathfrak{p}$ is regular. ■

Notice the similarity between this and Lemma 2.31.

In the following proposition we verify Conjecture 1.4 in the case where $\dim(R/\mathfrak{q}) = \dim(R)$. Note that the Cohen-Macaulayness condition has been loosened to "quasi-unmixed," although we need to assume "Nagata."

Proposition 2.36 *Assume that* $(R, \mathfrak{m})$ *is a quasi-unmixed, Nagata local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *Assume that* $e(R) = e(R_\mathfrak{p})$ *and* $\mathfrak{q}$ *is a minimal prime of* R. *Then* $\mathfrak{p} = \mathfrak{m}$, *so that* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R/\mathfrak{q}) = \dim(R)$.

Proof. The Associativity Formula implies that

$$\sum_{\mathfrak{r}} \mathrm{length}(R_\mathfrak{r}) e(R/\mathfrak{r}) = e(R) = e(R_\mathfrak{p}) = \sum_{\mathfrak{r} \subseteq \mathfrak{p}} \mathrm{length}(R_\mathfrak{r}) e(R_\mathfrak{p}/\mathfrak{r}_\mathfrak{p})$$

where the first sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$ and the second sum is taken over all prime ideals $\mathfrak{r}$ of R such that $\dim(R/\mathfrak{r}) = \dim(R)$ and $\mathfrak{r} \subseteq \mathfrak{p}$. Since each $e(R/\mathfrak{r}) \geq e(R_\mathfrak{p}/\mathfrak{r}_\mathfrak{p})$ by Theorem 2.20, the equality implies that every minimal prime of R is contained in $\mathfrak{p}$ and each $e(R/\mathfrak{r}) = e(R_\mathfrak{p}/\mathfrak{r}_\mathfrak{p})$. In particular, $\mathfrak{q} \subseteq \mathfrak{p}$. As above, this implies that $\mathfrak{p} = \mathfrak{m}$. ■

In the following proposition we verify Conjecture 1.4 in the case where $\dim(R/\mathfrak{q}) = 1$. Note that the Cohen-Macaulayness condition has been loosened to "equidimensional,"

Proposition 2.37 *Assume that* $(R, \mathfrak{m})$ *is an equidimensional local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. *Assume that* $e(R_\mathfrak{p}) = e(R)$ *and* $\dim(R/\mathfrak{q}) = 1$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$. *I*

Proof. Suppose that $\dim(R) < \dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R/\mathfrak{p}) + 1 \leq \dim(R) + 1$. Then $\dim(R/\mathfrak{p}) + 1 = \dim(R) + 1$ so that $\dim(R/\mathfrak{p}) = \dim(R)$. As R is equidimensional, this implies that $\mathfrak{p}$ is a minimal prime of R. By Proposition 2.34, $\mathfrak{q} = \mathfrak{m}$, a contradiction. ■

In the following proposition we verify Conjecture 1.4 when $\dim(R/\mathfrak{p}) = 1$. Note that the assumption "Cohen-Macaulay" has been loosened to "quasi-unmixed," although we need to assume "Nagata."

Proposition 2.38 *Assume that* $(R, \mathfrak{m})$ *is a quasi-unmixed, Nagata local ring with prime ideals* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$. *Assume that* $e(R_{\mathfrak{p}}) = e(R)$ *and* $\dim(R/\mathfrak{p}) = 1$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$. *In particular, if* R *is Cohen-Macaulay and Nagata then Conjecture 1.4 holds for* R *when* $\dim(R/\mathfrak{p}) = 1$.

Proof. The proof is identical to that of Proposition 2.37, using Proposition 2.36 instead of Proposition 2.34. ■

CHAPTER 3

THE CONJECTURE OF KURANO AND ROBERTS

In this chapter, we prove Conjecture 1.2 for a number of cases, most notably when (1) R contains a field, (2) $\mathfrak{p}$ is generated by a regular sequence, and (3) $R/\mathfrak{q}$ is regular. We note that each of these cases is proved without any excellence assumptions and therefore the results are not simple consequences of the results of the previous chapter.

We recall the conjecture here.

Conjecture 1.2 *Assume that $(R, \mathfrak{m})$ is a regular local ring and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$. Then $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$.*

Examples 4.16, 4.17 and 4.18 below show that each of the requirements of the conjecture is necessary.

3.1 The Equicharacteristic Case

The proof of Conjecture 1.2 in the equicharacteristic case differs from the proof of Theorem 2.29 in only one place: the method of reduction to the completion. The following lemma is our first tool in making this reduction.

Lemma 3.1 *Assume that $\phi : R \to S$ is a homomorphism of rings, P is a prime ideal in S and $\mathfrak{p} = \phi^{-1}(P)$. Then $\mathfrak{p}^{(n)} \subseteq \phi^{-1}(P^{(n)})$, for all $n > 0$.*

Proof. We have a natural commuting diagram

$$\begin{tikzcd} R_{\mathfrak{p}} \arrow[r, "\psi"] & S_P \\ R \arrow[u] \arrow[r, "\phi"] & S \arrow[u] \end{tikzcd}$$

and we are considering the behavior of the ideals $\mathfrak{p}^n$ and P^n under the appropriate extensions and contractions. Since $\mathfrak{p}S \subseteq P$, we see that $\mathfrak{p}^n S = (\mathfrak{p}S)^n \subseteq P^n$ and therefore that $(\mathfrak{p}^n S)S_P \subseteq P_P^n$. Thus,

$$\begin{aligned}\phi^{-1}(P^{(n)}) &= \phi^{-1}(P_P^n \cap S)\\ &= (\psi^{-1}(P_P^n)) \cap R\end{aligned}$$

(since the diagram commutes)

$$\begin{aligned}&\supseteq (\psi^{-1}((\mathfrak{p}^n S)S_P)) \cap R\\ &= (\psi^{-1}((\mathfrak{p}^n R_{\mathfrak{p}})S_P)) \cap R\\ &= (\psi^{-1}(\mathfrak{p}_{\mathfrak{p}}^n S_P)) \cap R\\ &\supseteq \mathfrak{p}_{\mathfrak{p}}^n \cap R\\ &= \mathfrak{p}^{(n)}\end{aligned}$$

as desired. ■

The following proposition will allow us to reduce Conjecture 1.2 to the case where R is complete.

Proposition 3.2 *Assume that $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are regular local rings and $R \to S$ is a faithfully flat extension such that $\mathfrak{m}S = \mathfrak{n}$. If Conjecture 1.2 holds for $\hat{R}$, then Conjecture 1.2 holds for R.*

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\mathfrak{p}+\mathfrak{q}$ is $\mathfrak{m}$-primary and $\operatorname{ht}\mathfrak{p}+\operatorname{ht}\mathfrak{q} = \dim R$ and fix $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$. Let P be a prime of $\hat{R}$ that is minimal over $\mathfrak{p}\hat{R}$ and let Q be a prime of $\hat{R}$ that is minimal over $\mathfrak{q}\hat{R}$. By Corollary 2.13, $\operatorname{ht}(P) = \operatorname{ht}(\mathfrak{p})$ and $\operatorname{ht}(Q) = \operatorname{ht}(\mathfrak{q})$, so that $\operatorname{ht}(P) + \operatorname{ht}(Q) = \dim(R) = \dim(\hat{R})$. By assumption

$$\sqrt{P+Q} \supseteq \sqrt{\mathfrak{p}+\mathfrak{q}}\hat{R} = \mathfrak{m}\hat{R} = \mathfrak{n}$$

so that $P+Q$ is $\mathfrak{n}$-primary. By Lemma 3.1, $f \in P^{(n)} \cap Q$. Since Conjecture 1.2 holds for $\hat{R}$, $f \in \mathfrak{n}^{n+1} \cap R = \mathfrak{m}^{n+1}$, as desired. ■

In the following theorem, we verify Conjecture 1.2 for regular local rings that contain a field by showing that it follows from the results of the previous chapter.

Theorem 3.3 *Assume that* $(R, \mathfrak{m})$ *is a regular local ring that contains a field and that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals of* R *such that* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ *and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. *Then* $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ *for all positive integers* n.

Proof. Let $\hat{R}$ denote the $\mathfrak{m}$-adic completion of R. Then the extension $R \to \hat{R}$ is faithfully flat (c.f., [15] Theorem 8.14). Also, $\hat{R}$ is a regular local ring with maximal ideal $\mathfrak{m}\hat{R}$ (c.f., Atiyah and MacDonald [1] Propositions 10.15, 10.16 and 11.24). By Proposition 3.2, it suffices to prove that the theorem holds for $\hat{R}$, so we may assume that R is complete.

Suppose that $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$ and $f \notin \mathfrak{m}^{n+1}$. By Theorem 1.3 $f \in \mathfrak{m}^n$, and $f \notin \mathfrak{p}^{(n+1)}$. Let $R' = R/(f)$ and similarly for $\mathfrak{m}'$, $\mathfrak{p}'$ and $\mathfrak{q}'$. Then R' is a complete Cohen-Macaulay local ring with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}'+\mathfrak{q}'} = \mathfrak{m}'$ and $e(R'_{\mathfrak{p}'}) = n = e(R')$. By Theorem 2.29

$$\dim(R) = \dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R'/\mathfrak{p}') + \dim(R'/\mathfrak{q}') \leq \dim(R') = \dim(R) - 1$$

a contradiction. This establishes the result. ■

3.2 A Note on the Mixed-Characteristic Case

The argument of the previous section depends heavily on the assumption that our rings contain a field. Thus, we can not apply these methods to rings of mixed characteristic. We can, however, make a reduction in this case to assume that our rings have algebraically closed residue fields. The following lemma gives the main tool used for this reduction. Recall that a *p-ring* is a discrete valuation ring whose maximal ideal is generated by the prime integer p (that is, the p-fold sum of 1).

Lemma 3.4 *Let* L *be a complete p-ring with residue field* k. *Let* K *be the algebraic closure of* k. *Then there exists complete p-ring* $\tilde{L}$ *dominating* L *with residue field* K. *Furthermore, the inclusion* $L \subseteq \tilde{L}$ *is a flat extension.*

Proof. By [15] Theorem 29.4, there exists a p-ring L_1 dominating L with residue field K. If we let $\tilde{L}$ be the completion of L_1, then $\tilde{L}$ is a complete p-ring dominating L_1 (and therefore dominating L) with residue field K by [15] note (4) on p. 63. The fact that the extension is flat follows from the fact that a module over a discrete valuation ring is flat if and only if it is torsionfree. ∎

The following proposition allows us to reduce Conjecture 1.2 to the case where the ring has algebraically closed residue field.

Proposition 3.5 *Conjecture 1.2 holds for all regular local rings of mixed characteristic if and only if it holds for all regular local rings of mixed characteristic with algebraically closed residue fields.*

Proof. One implication is trivial. Assume that Conjecture 1.2 holds for all regular local rings of mixed characteristic with algebraically closed residue fields, and let R be a regular local rings of mixed characteristic. By Proposition 3.2, we may assume that R is complete. Let $k = R/\mathfrak{m}$. The Cohen Structure Theorem implies that R has a coefficient ring L which is a complete p-ring and that either $R \cong L[\![X_1, \dots, X_d]\!]$ (unramified) or $R \cong L[\![X_1, \dots, X_d]\!][X]/(f)$ where f is an Eisenstein polynomial (ramified). Let $\tilde{L}$ be as in Lemma 3.4. If R is unramified, then let $\tilde{R} = \tilde{L}[\![X_1, \dots, X_d]\!]$. Otherwise, let $\tilde{R} = \tilde{L}[\![X_1, \dots, X_d]\!][X]/(f)$. In either case, $\tilde{R}$ is a complete, regular local ring dominating R, which has the same dimension $(d+1)$ as R. Furthermore, the inclusion $R \subseteq \tilde{R}$ is a flat extension of regular local rings. Let $\tilde{\mathfrak{m}}$ denote the maximal ideal of $\tilde{R}$. Then for all n, $\mathfrak{m}^n\tilde{R} = \tilde{\mathfrak{m}}^n$ and $\tilde{\mathfrak{m}}^n \cap R = \mathfrak{m}^n$. By Corollary 2.13, if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of R such that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R = d+1$, and $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ are primes of $\tilde{R}$, which are minimal over $\mathfrak{p}\tilde{R}$ and $\mathfrak{q}\tilde{R}$ respectively, then $\dim(\tilde{R}/\tilde{\mathfrak{p}}) = \dim(R/\mathfrak{p})$ and $\dim(\tilde{R}/\tilde{\mathfrak{q}}) = \dim(R/\mathfrak{q})$, so that $\dim(\tilde{R}/\tilde{\mathfrak{p}}) + \dim(\tilde{R}/\tilde{\mathfrak{q}}) = d+1 = \dim \tilde{R}$. If, additionally, $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, then $\mathfrak{m}^t \subseteq \mathfrak{p}+\mathfrak{q}$ so that $\tilde{\mathfrak{m}}^t = \mathfrak{m}^t\tilde{R} \subseteq \mathfrak{p}\tilde{R} + \mathfrak{q}\tilde{R} \subseteq \tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}$, which implies that $\sqrt{\tilde{\mathfrak{p}}+\tilde{\mathfrak{q}}} = \tilde{\mathfrak{m}}$. Finally, suppose that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \not\subseteq \mathfrak{m}^{n+1}$. Then there exists $f \in (\mathfrak{p}^{(n)} \cap \mathfrak{q}) \smallsetminus \mathfrak{m}^{n+1}$, and if we consider f as an element of $\tilde{R}$, then (i) $f \in \tilde{\mathfrak{p}}^{(n)} \cap \tilde{\mathfrak{q}}$ by Lemma 3.1, and (ii) $f \notin \tilde{\mathfrak{m}}^{n+1}$ by our previous observations. Thus, a counterexample in R would pass to a counterexample in $\tilde{R}$, and it suffices to prove the theorem for $\tilde{R}$. This is the desired result. ∎

It should be noted that this construction works equally well if $R = k[\![X_1, \dots, X_d]\!]$.

3.3 Low Dimensional Cases

In this section, we prove Conjecture 1.2 in the following cases: (i) $\dim(R) \leq 3$, (ii) $\mathrm{length}(R/(\mathfrak{p}+\mathfrak{q})) = 1$, and (iii) $n = 1$. The motivation for considering such low-dimensional cases is that they are relatively straightforward and could provide the beginning of any number of induction arguments. The following lemma gives the main tool for dealing with the case $\dim(R) \leq 3$.

Lemma 3.6 *Let R be any commutative ring, $\mathfrak{q}$ a prime ideal of R and $\mathfrak{a}$ a principal ideal in R. Then $\mathfrak{a} \cap \mathfrak{q}^{(n)} = \mathfrak{a}\mathfrak{q}^{(n)}$ for every $n > 0$.*

Proof. Let $\mathfrak{a} = aR$. Since $\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{b} \cap \mathfrak{c}$ for any pair of ideals, we need only show that $aR \cap \mathfrak{q}^{(n)} \subseteq aR\mathfrak{q}^{(n)}$. Fix $ax \in aR \cap \mathfrak{q}^{(n)}$. Since $a \notin \mathfrak{q}$ we see that a is a unit in $R_{\mathfrak{q}}$. Thus, in $R_{\mathfrak{q}}$, $x = a^{-1}ax \in \mathfrak{q}_{\mathfrak{q}}^{n}$ so that $x \in \mathfrak{q}_{\mathfrak{q}}^{n} \cap R = \mathfrak{q}^{(n)}$, as desired. ■

The following proposition supplies a verification of Conjecture 1.2 for regular local rings of dimension at most 3.

Proposition 3.7 *Assume that R is a regular local ring of Krull dimension $d \leq 3$ with maximal ideal $\mathfrak{m}$. Then for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$, $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$, for all $m, n > 0$. In particular, Conjecture 1.2 holds for regular local rings of dimension at most 3.*

Proof. The fact that $d \leq 3$ implies that either $\mathfrak{p}$ or $\mathfrak{q}$ has height 0 or 1, that is, either $\mathfrak{p}$ or $\mathfrak{q}$ is principal. If $\mathfrak{p}$ is principal, say $\mathfrak{p} = aR$, then $\mathfrak{p}^{(m)} = a^m R$ and since $a^n \notin \mathfrak{q}$ (otherwise $\mathfrak{p} \subseteq \mathfrak{q}$) Lemma 3.6 implies that $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} = \mathfrak{p}^{(m)}\mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$. The proof is similar if $\mathfrak{q}$ is principal. ■

The following proposition supplies a verification of Conjecture 1.2 in the case where $\mathfrak{p}+\mathfrak{q} = \mathfrak{m}$, i.e., the case where $\mathrm{length}(R/(\mathfrak{p}+\mathfrak{q})) = 1$.

Proposition 3.8 *Assume that R is a regular local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\mathfrak{p}+\mathfrak{q} = \mathfrak{m}$ and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$. *Then for all $m, n \geq 0$, $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$. In particular, Conjecture 1.2 holds under these hypotheses.*

Proof. We begin by demonstrating that there exists a regular system of parameters $t_1, \dots, t_r, u_1, \dots, u_s$ for R such that $\mathfrak{p}$ is generated by the t_i and $\mathfrak{q}$ is generated by the u_j. Let $M = \mathfrak{m}/\mathfrak{m}^2$, $P = (\mathfrak{p} + \mathfrak{m}^2)/\mathfrak{m}^2 \subseteq M$ and $Q = (\mathfrak{q} + \mathfrak{m}^2)/\mathfrak{m}^2 \subseteq M$. Then $P + Q = M$. Let $t_1, \dots, t_r \in \mathfrak{p}$ such that $\{\overline{t_i}\} \subset P$ forms a basis of P over $R/\mathfrak{m} = k$, and similarly for $u_1, \dots, u_s \in \mathfrak{q}$. Since the $\overline{t_i}$ are linearly independent over k, the ideal $\mathfrak{p}' = (t_1, \dots, t_r)$ is prime of height r contained in $\mathfrak{p}$. Similarly, $\mathfrak{q}' = (u_1, \dots, u_s)$ is prime of height s contained in $\mathfrak{q}$. Since $\{\overline{t_i}, \overline{u_j}\}$ span M, we see that $r+s \geq \dim(M) = d$. If $\mathfrak{p}' \neq \mathfrak{p}$, then $r = \operatorname{ht}\mathfrak{p}' < \operatorname{ht}\mathfrak{p}$ so that $d \leq r+s < \operatorname{ht}\mathfrak{p}+\operatorname{ht}\mathfrak{q} = d$, a contradiction. Thus, $\mathfrak{p} = \mathfrak{p}'$ and similarly $\mathfrak{q} = \mathfrak{q}'$. Then $r + s = \operatorname{ht}\mathfrak{p} + \operatorname{ht}\mathfrak{q} = d$, and since $\{\overline{t_i}, \overline{u_j}\}$ span M, we see that this is a basis for M. Thus, we have a regular system of parameters.

Let $G = gr_{\mathfrak{m}}(R) = k[T_1, \dots, T_r, U_1, \dots, U_r]$ where T_i and U_j are the images of t_i and u_j, respectively, in $\mathfrak{m}/\mathfrak{m}^2$. With the notation of Definition 2.9 $T_i = t_i^G$ and similarly for U_j. Let $P = (T_1, \dots, T_r)$, $Q = (U_1, \dots, U_r)$ and $M = P + Q$. Then $\mathfrak{p}^G = P$, and in fact, $(\mathfrak{p}^m)^G = P^m$ for all m since each is generated by the monomials in the $t_i^G = T_i$ of degree m. Similarly, $(\mathfrak{q}^n)^G = Q^n$ and $(\mathfrak{m}^n)^G = M^n$. Also, by checking monomials, we see that $P^m \cap Q^n = P^m Q^n \subseteq M^{m+n}$. Now, suppose that $f \in (\mathfrak{p}^m \cap \mathfrak{q}^n) \setminus \mathfrak{m}^{m+n}$. Then the degree of f is strictly smaller than $m + n$, so that $f^G \notin M^{m+n}$. But $f^G \in (\mathfrak{p}^m)^G \cap (\mathfrak{q}^n)^G = P^m \cap Q^n \subseteq M^{m+n}$, a contradiction. Since $\mathfrak{p}$ and $\mathfrak{q}$ are generated by regular sequences, $\mathfrak{p}^{(m)} = \mathfrak{p}^m$ and $\mathfrak{q}^{(n)} = \mathfrak{q}^n$, proving our result. ■

Note the stronger containment $\mathfrak{p}^{(m)} \cap \mathfrak{q}^{(n)} \subseteq \mathfrak{m}^{m+n}$ proved in the previous two propositions. We do not conjecture that this containment should hold in the arbitrary case, as we have no evidence like Theorem 1.1 to suggest that it should hold in general. However, at this time, I know of no counterexample.

The following proposition gives a verification of Conjecture 1.2 in the case where $n = 1$.

Proposition 3.9 *Assume that R is a regular local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and*

$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$. *Then* $\mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{m}^2$.

Proof. If $\mathfrak{p} \cap \mathfrak{q} \not\subseteq \mathfrak{m}^2$, then fix $f \in (\mathfrak{p} \cap \mathfrak{q}) \setminus \mathfrak{m}^2$. In particular, $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, so that $R' = R/fR$ is a regular local ring of dimension $d-1$. If $\mathfrak{p}'$, $\mathfrak{q}'$ and $\mathfrak{m}'$ are the images of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{m}$, respectively, in R', then $\mathfrak{p}' + \mathfrak{q}'$ is $\mathfrak{m}'$-primary and $\dim(R'/\mathfrak{p}') + \dim(R'/\mathfrak{q}') = d > \dim(R')$, which contradicts Serre [21] Theorem V.6.3. ■

3.4 Ideals Generated by Regular Sequences

One particularly well-behaved class of ideals include those which can be generated by a regular sequence. When $\mathfrak{p}$ is such an ideal, we prove a slightly stronger version of Conjecture 1.2 and a similar result when $\mathfrak{q}$ is generated by part of a regular system of parameters. The following lemma gives a verification of a version of Conjecture 1.2 for graded rings. We shall use the lemma below to verify Conjecture 1.2 when $\mathfrak{p}$ is generated by a regular sequence.

Lemma 3.10 *Let* $(A, \mathfrak{n})$ *be a local ring of Krull dimension* s, $G = A[X_1, \ldots, X_r]$ *a polynomial ring,* $P = \langle X_1, \ldots, X_r \rangle$, *and* $M = \mathfrak{n}G + P$. *Assume that* Q *is a proper homogeneous ideal of* G *(so that, in particular,* $Q \subseteq M$*) such that* $\operatorname{ht} Q = s$. *Let* $Q_0 = Q \cap A$, *and assume that* Q_0 *is* $\mathfrak{n}$*-primary. Then* $Q \cap P^n \subseteq \mathfrak{n}P^n \subseteq M^{n+1}$ *for all* $n \geq 0$.

Proof. Let Q' be a minimal prime of Q such that $\operatorname{ht} Q' = \operatorname{ht} Q$. Then Q' is homogeneous and $\mathfrak{n} \supseteq Q' \cap A \supseteq Q_0$. Since Q_0 is $\mathfrak{n}$-primary and $Q' \cap A$ is prime, we see that $Q' \cap A = \mathfrak{n}$. Furthermore, if we can show that $Q' \cap P^n \subseteq \mathfrak{n}P^n \subseteq M^{n+1}$, then $Q \cap P^n \subseteq Q' \cap P^n \subseteq \mathfrak{n}P^n \subseteq M^{n+1}$ and we are done. Thus, we may assume that Q is prime and that $Q_0 = \mathfrak{n}$. Then Q and $\mathfrak{n}G = Q_0 G \subseteq Q$ are both prime of height s so that $Q = \mathfrak{n}G$. Then, any element f of $Q \cap P^n$ has coefficients in $\mathfrak{n} \subset M$, implying that $Q \cap P^n \subseteq \mathfrak{n}P^n \subseteq MP^n \subseteq M^{n+1}$, as desired. ■

The following is a technical lemma we shall employ in the proof of Conjecture 1.2 in the case where $\mathfrak{p}$ is generated by a regular sequence.

Lemma 3.11 *Let A be a ring with ideals $\mathfrak{a}$ and $\mathfrak{b}$. Let $G = gr_{\mathfrak{b}}(A)$ and $\mathfrak{a}^G \subset G$ the ideal of initial forms of $\mathfrak{a}$ in G which is the homogeneous ideal $\mathfrak{a}^G = \oplus_i((\mathfrak{a} \cap \mathfrak{b}^i) + \mathfrak{b}^{i+1})/\mathfrak{b}^{i+1}$, and let $\mathfrak{b}' = \mathfrak{b}A/\mathfrak{a}$. Then there is a ring isomorphism $G/\mathfrak{a}^G \cong gr_{\mathfrak{b}'}(A/\mathfrak{a})$.*

Proof. The natural map $G \to gr_{\mathfrak{b}'}(A/\mathfrak{a})$, given in degree i by the projection

$$\mathfrak{b}^i/\mathfrak{b}^{i+1} \to (\mathfrak{b}^i + \mathfrak{a})/(\mathfrak{b}^{i+1} + \mathfrak{a}) = \mathfrak{b}^i/(\mathfrak{b}^i \cap (\mathfrak{b}^{i+1} + \mathfrak{a})) = \mathfrak{b}^i/(\mathfrak{b}^{i+1} + (\mathfrak{b}^i \cap \mathfrak{a}))$$

is a well-defined homomorphism of graded rings that is surjective. The kernel in degree i is exactly $(\mathfrak{b}^{i+1} + (\mathfrak{b}^i \cap \mathfrak{a}))/\mathfrak{b}^{i+1}$ which is exactly the term of $\mathfrak{a}^G$ in degree i, so that $\mathfrak{a}^G$ is the kernel. ■

The following proposition gives the essential argument for Conjecture 1.2 when $\mathfrak{p}$ is generated by a regular sequence (Corollary 3.13). We state the proposition separately here as we shall use it later for symbolic powers of ideals that are not necessarily prime (Theorem 3.24). Note that the regularity requirement of Conjecture 1.2 has been loosened here.

Proposition 3.12 *Assume that R is a Cohen-Macaulay, local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let I and J be ideals of R such that I is generated by a regular sequence, $\sqrt{I+J} = \mathfrak{m}$ and $\dim(R/I) + \dim(R/J) = d$. Then $I^n \cap J \subseteq \mathfrak{m}I^n \subseteq \mathfrak{m}^{n+1}$, for all $n \geq 0$.*

Proof. Let $x_1, \dots, x_r \in \mathfrak{m}$ be a regular sequence that generates I. In particular, I has height r. Let $A = R/I$, $\mathfrak{n} = \mathfrak{m}A$ and $G = gr_I(R)$. Since the x_i form a regular sequence, $G = A[X_1, \dots, X_r]$ is a polynomial ring, where $X_i = x_i^G$. Let $s = \dim(A) = d - r = \operatorname{ht}(J)$, and let $P = I^G$ and $Q = J^G$. Then $P = \langle X_1, \dots, X_r \rangle$ and for $n > 0$, $(\mathfrak{p}^n)^G = P^n$ (to see this we can check generators). Furthermore, Lemma 3.11 implies that $G/Q = G/J^G \cong gr_{I'}(R/J)$ so that $\operatorname{ht} Q = \dim(G) - \dim(G/Q) = d - \dim(gr_{I'}(R/J)) = d - \dim(R/J) = \operatorname{ht} J = s$. Finally, $Q_0 = (J+I)/I$ which is $\mathfrak{n}$-primary since $I+J$ is $\mathfrak{m}$-primary. Thus, Lemma 3.10 implies that $Q \cap P^n \subseteq \mathfrak{n}P^n \subseteq M^{n+1}$.

Now, suppose that $I^n \cap J \not\subseteq \mathfrak{m}I^n$. Then there exists $f \in (I^n \cap J) \setminus \mathfrak{m}I^n$. In particular, $f \in I^n \setminus I^{n+1}$ so that $f^G \in G_n$. By construction, though, $f \in Q \cap$

$P^n \subseteq \mathfrak{n}P^n$ so that $f \in M_n^{n+1} = \mathfrak{n}I^n/I^{n+1} = (\mathfrak{m}I^n + I^{n+1})/I^{n+1}$. This implies that $f \in \mathfrak{m}I^n + I^{n+1} \subseteq \mathfrak{m}^{n+1}$, a contradiction. ■

Corollary 3.13 *Assume that R is a Cohen-Macaulay, local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ and J be ideals of R such that $\mathfrak{p}$ is prime, $\sqrt{\mathfrak{p}+J} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/J) = d$. If $\mathfrak{p}$ is generated by a regular sequence, then $\mathfrak{p}^{(n)} \cap J \subseteq \mathfrak{m}^{n+1}$, for all $n \geq 0$.*

Proof. Since $\mathfrak{p}$ is generated by a regular sequence, $\mathfrak{p}^{(n)} = \mathfrak{p}^n$. Now apply Proposition 3.12. ■

We shall see below (Theorem 3.24) that Conjecture 1.2 holds when $\mathfrak{p}$ is generated by a regular sequence and is not necessarily prime.

It is natural to consider next the case where $\mathfrak{p}$ is equimultiple. A theorem of Huneke [11] tells us that in many cases this follows directly from Corollary 3.13.

Theorem 3.14 *Let R be a Noetherian local ring and $\mathfrak{p}$ a prime ideal such that $R_\mathfrak{p}$ is regular. Suppose either R of $gr_R(\mathfrak{p})$ is Cohen-Macaulay. If $\mathfrak{p}$ is equimultiple then $\mathfrak{p}$ is generated by an R-sequence.*

The following proposition gives another criterion for the containment $\mathfrak{p}^n \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ to hold.

Proposition 3.15 *Assume that R is a regular local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$. Let $G = gr_\mathfrak{m}(R)$ and let $P = \mathfrak{p}^G$, $Q = \mathfrak{q}^G$ and $M = \mathfrak{m}^G$ be the ideals of initial forms in G. If $\sqrt{P+Q} = M$, then $\mathfrak{p}^n \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$.*

Proof. First, we observe that $P^n \cap Q \subseteq M^{n+1}$. Since the localization G_M is a regular local ring which contains a field, the theorem holds in this ring. Since $P_M + Q_M$ is M_M-primary and $\operatorname{ht} P_M + \operatorname{ht} Q_M = \operatorname{ht} P + \operatorname{ht} Q = \operatorname{ht} \mathfrak{p} + \operatorname{ht} \mathfrak{q} = d = \dim G_M$, we consider minimal primes of P and Q of the same heights to see that

$$(P^n \cap Q)_M = (P_M)^n \cap Q_M \subseteq (M_M)^{n+1} = (M^{n+1})_M$$

and we contract to G to see that

$$P^n \cap Q \subseteq (P^n \cap Q)_M \cap G \subseteq (M^{n+1})_M \cap G = M^{(n+1)} = M^{n+1}$$

since M is maximal.

Next, we observe that $((\mathfrak{p}^n)^G)_n = (P^n)_n$. Notice that once we show this, we are done. If $f \in (\mathfrak{p}^n \cap \mathfrak{q}) \smallsetminus \mathfrak{m}^{n+1}$ then

$$f^G \in G_n \cap (\mathfrak{p}^n)^G \cap \mathfrak{q}^G = ((\mathfrak{p}^n)^G)_n \cap (\mathfrak{q}^G)_n = (P^n \cap Q)_n \subseteq (M^{n+1})_n = 0$$

which is a contradiction. To prove the desired equality, we observe that

$$(P^n)_n = (P_1)^n = (\mathfrak{p}^n + \mathfrak{m}^{n+1})/\mathfrak{m}^{n+1} = ((\mathfrak{p}^n \cap \mathfrak{m}^n) + \mathfrak{m}^{n+1})/\mathfrak{m}^{n+1} = ((\mathfrak{p}^n)^G)_n$$

as desired. ■

Example 4.19 below shows that, in general, $\sqrt{\mathfrak{p}^G + \mathfrak{q}^G} \neq M$.

With all this work in mind, one might also ask whether there are reasonable conditions that guarantee that $\mathfrak{p}^{(n)} \cap \mathfrak{q} = \mathfrak{p}^{(n)}\mathfrak{q}$. Certainly, this equality would imply Conjecture 1.2 for these ideals. It is always true that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \supseteq \mathfrak{p}^{(n)}\mathfrak{q}$, but equality does not hold in general: for example if $\mathfrak{p}^{(n)}$ is contained in $\mathfrak{q}$ or vice-versa. Under our usual dimension and spanning restrictions, it seems natural to wonder if this equality holds for sufficiently "nice" ideals, for example, ideals generated by a regular sequence or even by part of a regular system of parameters. By Lemma 3.6, if one of the ideals is principal (i.e., generated by a regular sequence of length 1), the answer is "yes." In general, though, the answer is still "no." To see this, we start with a lemma which shows that the different between $\mathfrak{p}^{(n)} \cap \mathfrak{q}$ and $\mathfrak{p}^{(n)}\mathfrak{q}$ can be computed using Tor.

Lemma 3.16 *Assume that R is a local, Noetherian ring with $x_1, \ldots, x_s \in \mathfrak{m}$ a regular sequence. Let $J = (x_1, \ldots, x_s)$, and let I be a nontrivial ideal of R. Then*

$$Tor_1^R(R/I, R/J) \cong (I \cap J)/(IJ)$$

Proof. Since J is generated by the regular sequence $x_1, \ldots, x_s$ we can use the Koszul complex to compute the Tor module. We recall the relevant definitions. Let $K_0 = R$,

$K_1 = R^s$ and $K_2 = r^{\binom{s}{2}}$. Let a basis of K_2 be given as $\{e_{ij} : 1 \le i < j \le s\}$. Then the differentials are given by the following formulas.

$$\begin{array}{ccccc} K_2 & \xrightarrow{d_2} & K_1 & \xrightarrow{d_1} & K_0 \\ e_{ij} & \longmapsto & x_i e_j - x_j e_i & & \\ & & e_i & \longmapsto & x_i \end{array}$$

The module $\mathrm{Tor}_1^R(R/I, R/J)$ is the homology of this complex after tensoring with R/I. We construct an isomorphism ϕ from this module to $(I \cap J)/(IJ)$. An element of $\mathrm{Tor}_1^R(R/I, R/J)$ is represented by an s-tuple $(\overline{a}_i) = (\overline{a}_1, \ldots, \overline{a}_s) \in \ker(d_1) \subseteq K_1 = R^s$. Since $(\overline{a}_i)$ is in the kernel of d_1, $\sum_i \overline{a}_i \overline{x}_i = \overline{0}$ so that $\sum_i a_i x_i \in I$. Since the $x_i \in J$, $\sum_i a_i x_i \in I \cap J$. Thus, we define $\phi : \ker(d_1) \to (I \cap J)/(IJ)$ as $(\overline{a}_i) \mapsto \sum_i a_i x_i$. To see that this is well-defined, the essential step is to show that if $\overline{a}_i = \overline{0}$, then $\sum_i a_i x_i \in IJ$. This is clear, however, as $\overline{a}_i = \overline{0} \Rightarrow a_i \in I$. This map factors through the quotient $\ker(d_1)/\mathrm{Im}(d_2)$, as $\phi(d_2(e_{i,j})) = \phi(x_i e_j - x_j e_i) = \overline{x}_i \overline{x}_j - \overline{x}_j \overline{x}_i = \overline{0}$. This gives a well-defined map $\phi : \mathrm{Tor}_1^R(R/I, R/J) \to (I \cap J)/(IJ)$. It is surjective, as every element of $(I \cap J)/(IJ)$ is of the form $\sum_i a_i x_i$ and is, by definition, in the image of ϕ. To see that ϕ is injective, suppose that $(\overline{a}_i) \mapsto 0$. This implies that $\sum_i a_i x_i \in IJ$ so that $\sum_i a_i x_i = \sum_i p_i x_i$ for some $p_i \in I$. Then $\sum_i (a_i - p_i) x_i = 0$, so that in the original Koszul complex (before tensoring with R/I) the element $(a_i - p_i) \in K_1$ is in the kernel of d_1. This complex is exact, so $(a_i - p_i) = d_2(b)$ for some $b \in K_2$. Thus, $d_2(\overline{b}) = (\overline{a_i - p_i}) = (\overline{a}_i)$ which implies that $(\overline{a}_i) = 0$ in $\mathrm{Tor}_1^R(R/I, R/J)$, as desired. ■

We note that this lemma holds without the assumption that J is generated by a regular sequence (c.f., Rotman [20] Corollary 11.27).

Now assume that R is a regular local ring with ideals I and J such that $\dim(R/I) + \dim(R/J) = \dim R$, $\sqrt{I + J} = \mathfrak{m}$ and J is generated by a regular sequence $x_1, \ldots, x_s \in \mathfrak{m}$. Then the lemma tells us that $I \cap J = IJ$ if and only if $\mathrm{Tor}_1^R(R/I, R/J) = 0$ if and only if the x_i form a regular sequence on R/I. This condition occurs if and only if R/I is Cohen-Macaulay. To see this, assume that R/I is Cohen-Macaulay. Then $\mathrm{depth}(R/I) = \dim(R/I) = s$, by assumption, and $x_1, \ldots, x_s$ generate an ideal of height s in R/I and therefore form a regular sequence on R/I. Conversely, if R/I is not Cohen-Macaulay then $\mathrm{depth}(R/I) < \dim(R/I) = s$ and R/I can not have a regular sequence of length s. It follows that there are numerous examples where the containment $I^{(n)} \cap J \supsetneq I^{(n)}J$ is nontrivial.

In the following proposition, we verify Conjecture 1.2 for the case where $\mathfrak{q}$ is generated by part of a regular system of parameters.

Proposition 3.17 *Assume that $(R, \mathfrak{m})$ is a regular local ring of Krull dimension d. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = d$. Assume that $\mathfrak{q}$ is generated by part of a regular system of parameters. Then $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^n\mathfrak{q}$, for all $n \geq 0$. In particular, Conjecture 1.2 holds when $R/\mathfrak{q}$ is regular.*

Proof. First, we note that, if we can prove that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$, then the desired result follows. We prove this by induction on $s = \mathrm{ht}\,(\mathfrak{q})$. The case $s = 0$ is trivial, and the case $s = 1$ follows from Lemma 3.6. Fix any $u \in \mathfrak{q} \smallsetminus \mathfrak{m}^2$ and let $\overline{R} = R/uR$, $\overline{\mathfrak{q}} = \mathfrak{q}/uR$ and so on. By Proposition 3.9, $u \notin \mathfrak{p}$, so that $\mathrm{ht}\,(\overline{\mathfrak{p}}) = \mathrm{ht}\,(\mathfrak{p})$. Let $\overline{\mathfrak{r}}$ be a minimal prime ideal of $\overline{\mathfrak{p}}$ which has the same height as $\overline{\mathfrak{p}}$. By induction, $\overline{\mathfrak{r}}^{(n)} \cap \overline{\mathfrak{q}} \subseteq \overline{\mathfrak{m}}^n\overline{\mathfrak{q}}$ so that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^n\mathfrak{q} + (u)$. For $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$, $f = \sum_i a_i x_i + bu$ where the $a_i \in \mathfrak{m}^n$. If $b \notin \mathfrak{m}^n$, then $f \notin \mathfrak{m}^{n+1}$ which contradicts the containment $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$.

Suppose that the residue field of R is finite. As in the proof of Theorem 3.3, let $R(X) = R[X]_{\mathfrak{m}[X]}$ which is a faithfully flat extension of R such that $\mathfrak{m}R(X)$ is the maximal ideal M of the regular local ring $R(X)$. Furthermore, since $\mathfrak{q}$ is generated by part of a regular system of parameters for R, the same is true for the extension $Q = \mathfrak{q}R(X)$. Let P be a minimal prime of $\mathfrak{p}R(X)$ and fix $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$. If Part 1 holds for $R(X)$, then $f \in M^{n+1} \cap R = \mathfrak{m}^{n+1}$ by faithful flatness. Thus, we may assume that R has infinite residue field.

Let $x_1, \ldots, x_s$ be part of a regular system of parameters such that $\mathfrak{q} = (x_1, \ldots, x_s)$. We prove each result by induction on s. Lemma 3.6 implies the case $s = 1$ for both results. Assume that $s \geq 2$ and for some $u \in \mathfrak{q} \smallsetminus \mathfrak{m}^2$, let $\overline{R} = R/(u)$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{R}$, and so on. Then $\mathrm{ht}\,(\overline{\mathfrak{p}}) = \mathrm{ht}\,(\mathfrak{p})$ so let $\overline{\mathfrak{r}}$ be a minimal prime ideal of $\overline{\mathfrak{p}}$ which has the same height as $\overline{\mathfrak{p}}$. Then $\overline{\mathfrak{r}}$ and $\overline{\mathfrak{q}}$ satisfy the hypotheses of the proposition, so by induction, $\overline{\mathfrak{r}}^{(n)} \cap \overline{\mathfrak{q}} \subseteq \overline{\mathfrak{m}}^{n+1}$. By Lemma 3.1, this implies that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} + (u)$ for all such u. Suppose that $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$ such that $f \notin \mathfrak{m}^{n+1}$. Since $s \geq 2$, we see that for every unit v of R, $f \in \mathfrak{m}^{n+1} + (x_1 + vx_2)$. Since $f \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$, the initial form of f in $G = gr_{\mathfrak{m}}(R) = R/\mathfrak{m}[X_1, \ldots, X_d]$ is a multiple of $X_1 + v^G X_2$ for every such v. Since R has infinite residue field, there are an infinite number of distinct elements of G of this form so that f^G has an infinite number of prime factors, a contradiction. ■

3.4.1 The Use of Regular Alterations

With Gabber's work on nonnegativity (c.f., [19]) and Propositions 3.12 and 3.17 in mind, it would make sense to try to use Theorem 2.33 to reduce to the case where a global version of $\mathfrak{p}$ or $\mathfrak{q}$ is locally generated by a regular system of parameters.

For our purposes, let $(R, \mathfrak{m})$ be a regular local ring that is of finite type over a field or a complete discrete valuation ring with algebraically closed residue field. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) = \dim(R)$. We let $A = R/\mathfrak{p}$ and let $I \in \operatorname{Proj}(R/\mathfrak{p}[X_0, \ldots, X_n])$ be an ideal coming from a regular alteration of $R/\mathfrak{p}$. Let $S = R[X_0, \ldots, X_n]$ so that I corresponds to an element $P \in \operatorname{Proj}(S)$. Let $Q = \mathfrak{q}S$. If P and Q were to satisfy the same conditions satisfied by $\mathfrak{p}$ and $\mathfrak{q}$, at least locally, then we might hope that a counterexample in R would pass to a counterexample in A. Unfortunately, P and Q will not satisfy the desired conditions, as noted previously. The same problem occurs if we take a regular alteration of $R/\mathfrak{q}$.

3.5 Ordinary and Symbolic Powers of Ideals

We investigate versions of Conjecture 1.2 where $\mathfrak{p}$ and $\mathfrak{q}$ are not necessarily prime ideals and where the inclusion under consideration is $\mathfrak{p}^n \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$. We also give some indication as to how this may help in the investigation of the original version of the conjecture. The following lemma allows us to replace $\mathfrak{q}$ with any ideal with the same radical as $\mathfrak{q}$ in Conjecture 1.2.

Lemma 3.18 *Assume that R is a regular local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let I and J be ideals of R such that $\sqrt{I+J} = \mathfrak{m}$ and* $\dim(R/I) + \dim(R/J) = d$. *Let J' be an ideal of R with the same radical as J (e.g., J' is a reduction of J or the integral closure of J or the radical of J). Then*

1. *$I^n \cap J' \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 1$ if and only if $I^n \cap J \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$.*

2. *If I is unmixed, $I^{(n)} \cap J' \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$ if and only if $I^{(n)} \cap J \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$.*

Proof. The proof of part 1. is almost identical to that of part 2. so we only prove part 2. The assumption that I is unmixed is only needed for the properties of $I^{(n)}$ which we require. The result is symmetric in J' and J, as $\sqrt{J} = \sqrt{J'} \Rightarrow \dim(R/J) = \dim(R/J')$ and $\sqrt{J' + I} = \mathfrak{m}$. Assume that $I^{(n)} \cap J' \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 1$, and suppose that $f \in (I^{(n)} \cap J) \smallsetminus \mathfrak{m}^{n+1}$. Then, the degree of f (with respect to $\mathfrak{m}$) is exactly n, and the degree of f^t is exactly nt for all positive integers t. By assumption, $J^t \subseteq J'$ for some positive integer t. Thus, $f^t \in J' \cap (I^{(n)})^t \subseteq J' \cap I^{(nt)} \subseteq \mathfrak{m}^{nt+1}$, a contradiction. ■

We consider the containment $I^n \cap J \subseteq \mathfrak{m}^{n+1}$ for the following reasons: (i) it is not known, and (ii) if we can verify this containment, then it may lead us closer to establishing Conjecture 1.2.

The following lemma is the first tool to be used in verifying a version of Conjecture 1.2 for a class of ideals in the unramified, mixed-characteristic case.

Lemma 3.19 *Assume that A is a complete regular local ring, $n > 0$, and $f = X^n + a_1 X^{n-1} + \cdots + a_n$ a polynomial with coefficients in A. If $\mathfrak{p} \subset A[\![X]\!]$ is a prime ideal, $\mathfrak{p}_0 = \mathfrak{p} \cap A$ and $f \in \mathfrak{p}^{(n)}$, then $\mathfrak{p}$ is the unique prime ideal of $A[\![X]\!]$ containing f, which contracts to $\mathfrak{p}_0$ in A.*

Proof. First, we reduce to the case where $\mathfrak{p}_0$ is maximal. Let $S = A \smallsetminus \mathfrak{p}_0$. Weierstrass Preparation implies that in the commuting diagram (where all the maps are canonical)

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A[X] \arrow[r] \arrow[d] & A[X]/(f) \arrow[d] \\ & A[\![X]\!] \arrow[r] & A[\![X]\!]/(f) \end{tikzcd}$$

the right-hand vertical map is an isomorphism. The localized diagram

$$\begin{tikzcd} A_{\mathfrak{p}_0} \arrow[r] \arrow[dr] & A_{\mathfrak{p}_0}[X] \arrow[r] \arrow[d] & A_{\mathfrak{p}_0}[X]/(f) \arrow[d, "\cong"'] \\ & A[\![X]\!]_S \arrow[r] & A[\![X]\!]_S/(f) \end{tikzcd}$$

also has an isomorphism in that spot. The prime ideals of $A[\![X]\!]$ that contain f and contract to $\mathfrak{p}_0$ in A are in bijection with the primes of $A[\![X]\!]_S/(f) \cong A_{\mathfrak{p}_0}[X]/(f)$ which contract to the maximal ideal of the local ring $A_{\mathfrak{p}_0}$. Furthermore, letting $\mathfrak{s} = (\mathfrak{p} \cap A[X])_S$ we have

$$f \in (\mathfrak{p}^{(n)})_S \cap A[X]_S = ((\mathfrak{p} \cap A[X])_S)^{(n)} = \mathfrak{s}^{(n)}.$$

If we can show that $\mathfrak{s}$ is the unique prime in $A_{\mathfrak{p}_0}[X]$ containing f which contracts to $\mathfrak{s}_0 = (\mathfrak{p}_0)_{\mathfrak{p}_0}$ in $A_{\mathfrak{p}_0}$, then $\mathfrak{p}$ is the unique such prime in $A[\![X]\!]$. So we may assume that we are considering the extension $A \to A[X]$ and that $\mathfrak{p} \cap A$ is maximal.

The extension $A \to A[X]/(f)$ is finite, so $\overline{\mathfrak{p}} \subset A[X]/(f)$ is maximal. Thus, $\mathfrak{p}$ is maximal. Since $\mathfrak{p}_0$ is maximal, $A/\mathfrak{p}_0[X]$ is a PID and so $\hat{\mathfrak{p}} \subset A/\mathfrak{p}_0[X] \cong A[X]/\mathfrak{p}_0[X]$ is principal, generated by $\hat{f}_1$, which is a prime factor of the image $\hat{f}$ of f in $\hat{\mathfrak{p}} \subset A/\mathfrak{p}_0[X]$. Since $\mathfrak{p}$ is maximal, $\mathfrak{p}^n$ is $\mathfrak{p}$-primary and $f \in \mathfrak{p}^{(n)} = \mathfrak{p}^n$ so that $\hat{f} \in \hat{\mathfrak{p}}^n = \langle \hat{f}_1^n \rangle$. Thus, $\hat{f} = \hat{f}_1^n \hat{u}$ for some $\hat{u}$. Since $\hat{f}_1$ must have positive degree, we see that $\hat{f}_1$ must be linear, and $\hat{u}$ must be constant, so that modulo $\mathfrak{p}_0$, f has a unique prime factor. Thus, any distinct primes of $A[X]$ containing f and contracting to $\mathfrak{p}_0$ would pass to distinct primes of $A/\mathfrak{p}_0[X]$ containing $\hat{f}$. But the only such prime is $\langle \hat{f}_1 \rangle$, so we have uniqueness. ■

The following lemma is the main tool to be used in verifying a version of Conjecture 1.2 for a class of ideals in the unramified, mixed-characteristic case.

Lemma 3.20 *Assume that A is a complete regular local ring, $n > 0$, and $f = X^n + a_1 X^{n-1} + \cdots + a_n$ a polynomial with coefficients in A. Let $R = A[\![X]\!]$ with maximal ideal $\mathfrak{m}$ and assume that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$. If $\mathfrak{p}_0 = \mathfrak{p} \cap A$, then $\mathfrak{p}_0 R + \mathfrak{q}$ is also $\mathfrak{m}$-primary.*

Proof. For an ideal $\mathfrak{a} \subseteq R$, let $Z(\mathfrak{a}) \subseteq \mathrm{Spec}(R)$ denote the closed subscheme determined by $\mathfrak{a}$. Since $\mathfrak{a}$ is $\mathfrak{m}$-primary iff $Z(\mathfrak{a}) \subseteq \{\mathfrak{m}\}$, we need only show that

$Z(\mathfrak{p}_0R+\mathfrak{q}) = Z(\mathfrak{p}+\mathfrak{q})$. Since $\mathfrak{p}_0R+\mathfrak{q} \subseteq \mathfrak{p}+\mathfrak{q}$, we know that $Z(\mathfrak{p}_0R+\mathfrak{q}) \supseteq Z(\mathfrak{p}+\mathfrak{q})$, so we need only demonstrate the reverse inclusion. The fact that f is in $\mathfrak{q}$ tells us that $\mathfrak{p}_0R+\mathfrak{q} = (\mathfrak{p}_0R+fR)+\mathfrak{q}$, so that $Z(\mathfrak{p}_0R+\mathfrak{q}) = Z((\mathfrak{p}_0R+fR)+\mathfrak{q}) = Z(\mathfrak{p}_0R+fR)\cap Z(\mathfrak{q})$. Since $Z(\mathfrak{p}+\mathfrak{q}) = Z(\mathfrak{p})\cap Z(\mathfrak{q})$, it suffices to show that $Z(\mathfrak{p}_0R+fR) \subseteq Z(\mathfrak{p})$. Furthermore, it suffices to show that if $\overline{R} = R/(\mathfrak{p}_0R+fR)$, then the image $\overline{\mathfrak{p}}$ of $\mathfrak{p}$ in $\overline{R}$ is the unique minimal prime of $\overline{R}$, since then any prime of $\overline{R}$ must also contain $\overline{\mathfrak{p}}$.

Since $A/\mathfrak{p}_0$ is complete and local, and since f is monic we see that the extension $A/\mathfrak{p}_0 \to A/\mathfrak{p}_0[X]/(f) = \overline{R}$ is finite. Furthermore, $\overline{R}$ is a free $A/\mathfrak{p}_0$-module, so both the going-up and going-down theorems hold for this extension. In particular, $\operatorname{ht}\overline{\mathfrak{p}} = \operatorname{ht}(\overline{\mathfrak{p}}\cap A/\mathfrak{p}_0) = \operatorname{ht}(0) = 0$ so that $\overline{\mathfrak{p}}$ is a minimal prime of $\overline{R}$. Now, let $\mathfrak{s}$ be any minimal prime of $\overline{R}$. By Lemma 3.19, it suffices to show that the contraction of $\mathfrak{s}$ in $A/\mathfrak{p}_0$ is (0). Since $A/\mathfrak{p}_0$ is an integral domain, it suffices to show that $\operatorname{ht}(\mathfrak{s}\cap A/\mathfrak{p}_0) = 0$. This follows by going-up and going-down, since $\operatorname{ht}(\mathfrak{s}\cap A/\mathfrak{p}_0) = \operatorname{ht}\mathfrak{s} = 0$. ■

The following proposition gives another class of ideals satisfying the containment $I^n \cap J \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 1$.

Proposition 3.21 *Let L be a complete p-ring with infinite residue field k and let $R = L[\![X_1, \ldots, X_d]\!]$ with maximal ideal $\mathfrak{m}$. Assume that I and J are ideals of R such that $\dim(R/I) + \dim(R/J) = d+1$ and $\sqrt{I+J} = \mathfrak{m}$. Let $\overline{p}$ denote the image of p in $\mathfrak{m}/\mathfrak{m}^2$, and let $\overline{I} = (I+\mathfrak{m}^2)/\mathfrak{m}^2 \subseteq \mathfrak{m}/\mathfrak{m}^2$. If $k\overline{p} \not\subseteq \overline{I}$, then $I^n \cap J \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 1$.*

Proof. If $I \subseteq \mathfrak{m}^2$, then $I^n \subseteq \mathfrak{m}^{2n} \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$. Otherwise, let $d(I) = \dim_k(\overline{I}) > 0$. For $d(I) \geq 2$, the proof of Proposition 3.17 shows that we can reduce to the case $d(I) = 1$. (Let $H = gr_{\mathfrak{m}}(R)$, and for $z \in R$, let $Z \in H$ denote the initial form of z in H. Fixing $x, y \in \mathfrak{p} \smallsetminus \mathfrak{m}^2$ such that $kX \neq kY$, we need to make sure that there is an infinite number of elements $x+uy$ that are relatively prime modulo $\mathfrak{m}^2$ and such that $R/(x+uy)R$ is unramified of mixed characteristic. That is, the $x+uy$ must satisfy $x+uy \notin (p)+\mathfrak{m}^2$. To see that this is possible, we note that, if $P|(X+UY)$, then for every nonzero $V \in k$, $P \nmid (X+V(X+UY))$.) Similarly, we may assume that $d(J) \leq 1$.

Fix $x \in \mathfrak{p} \setminus \mathfrak{m}^2$. Let $G = gr_{\mathfrak{p}}(R)$ and and consider the natural homomorphism of graded rings $\phi : G \to H$. Let $\mathfrak{q}^G$ denote the ideal of initial forms of $\mathfrak{q}$ in G, and for any element f of $\mathfrak{q}$ let f^G denote the initial form of f in G. To show that $\mathfrak{p}^n \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ for all $n \geq 1$, it is equivalent to show that $\phi(\mathfrak{q}^G) = 0$. Since $\mathfrak{p} \subset (x) + \mathfrak{m}^2$, we see that the image of ϕ is exactly $k[X]$. Suppose that $f \in (\mathfrak{p}^n \cap \mathfrak{q}) \setminus \mathfrak{m}^{n+1}$. Then $\phi(f^G) = aX^n$ for some nonzero element $a \in k$, and it follows that $f \in (x^n) + \mathfrak{m}^{n+1}$. If $x \notin (p) + \mathfrak{m}^2$, then we may assume that $x = X_1$, and Lemma 3.20 gives a contradiction. ■

For what follows, we need a notion of symbolic powers for arbitrary ideals.

Definition 3.22 Let I be an ideal of a ring A. The nth *symbolic power* of I is the ideal

$$I^{(n)} = \bigcap (I^n A_{\mathfrak{p}} \cap A)$$

where the intersection is taken over all minimal prime ideals of A/I. If I is a prime ideal, then this definition agrees with the previous definition.

The following lemma shows that symbolic powers of ideals generated by regular sequences are exactly the regular powers. This shall give us the generalization of Corollary 3.13 which was mentioned above.

Lemma 3.23 *Assume that I is generated by a regular sequence (not necessarily prime) of height r in a Cohen-Macaulay local ring A. Then $I^{(n)} = I^n$ for all $n \geq 1$.*

Proof. Let $a_1, \ldots, a_r$ be a regular sequence generating I, and let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_j$ be an irredundant primary decomposition of I with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$. Since I is generated by a regular sequence and A is Cohen-Macaulay, we know that $\operatorname{ht}(\mathfrak{p}_i) = r$ for each i. First, we note that $I^{(1)} = I$. The correspondence of primary ideals under localization implies

$$I^{(1)} = \cap_i (\mathfrak{q}_{\mathfrak{p}_i} \cap A) = \cap_i \mathfrak{q}_i = I$$

as desired.

Now, assume that $n > 1$ and fix $x \in I^{(n)}$. By induction on n, we know that $x \in I^{(n-1)} = I^{n-1}$. Suppose that $x \notin I^n$. By the definition of $I^{(n)}$, there exists $s_i \in R \smallsetminus \mathfrak{p}_i$ such that $s_i x \in I^n$. Since $x \in I^{(n-1)} = I^{n-1}$, the fact that the a_l form a quasi-regular sequence implies that there exists a homogeneous polynomial $F(x_1, \ldots, x_r) \in R[x_1, \ldots, x_r]$ of degree $n-1$ such that $F(a_1, \ldots, a_r)$ and the coefficients of F are not all in I. The polynomial $s_i F$ then satisfies $s_i F(a_1, \ldots, a_r) = s_i x \in I^n$ so that the coefficients of $s_i F$ are all in I. Because such an s_i exists for each $\mathfrak{p}_i$, it follows that the coefficients of F are all in $I^{(1)} = I$, a contradiction. ■

Theorem 3.24 *Assume that R is a Cohen-Macaulay, local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. Let I and J be ideals of R such that $\sqrt{I+J} = \mathfrak{m}$ and $\dim(R/I) + \dim(R/J) = d$. If I is generated by a regular sequence, then $I^{(n)} \cap J \subseteq \mathfrak{m} I^{(n)} \subseteq \mathfrak{m}^{n+1}$, for all $n \geq 0$.*

Proof. In light of the proof of Corollary 3.13, the result follows from Lemma 3.23. ■

3.6 Nonregular Rings

The nature of induction arguments we might try lead us to take quotients of our rings. For example, recall the induction argument for Proposition 3.17. However, without minimal generators of $\mathfrak{m}$ in our ideals, our quotients will no longer be regular. One might ask whether we can loosen the restriction of regularity if we require more from our ideals $\mathfrak{p}$ and $\mathfrak{q}$. For instance, we might require $\mathfrak{p}$ and $\mathfrak{q}$ to have finite projective dimension. As we have shown, we may need to assume that R satisfies some reasonable hypotheses, for instance, quasi-unmixedness or Cohen-Macaulayness. It seems clear that we should work within a class of rings which properly contains the regular local ones and is closed under specialization (i.e., passing to a quotient by a regular sequence). We desire this in order to be able to take more quotients, since our ideals will contain plenty of regular elements, even if they contain no minimal generators of $\mathfrak{m}$.

In light of Example 4.16, we see that the assumption that $\mathfrak{p}$ and $\mathfrak{q}$ have finite projective dimension is quite restrictive. In fact, we immediately have the following.

Lemma 3.25 *Assume that $(R, \mathfrak{m})$ is a local Noetherian ring with prime ideal $\mathfrak{p}$ of finite projective dimension. Then $R_\mathfrak{p}$ is a regular local ring. In particular, $\mathfrak{p}$ is locally generated by a regular sequence.*

Proof. The fact that $\mathfrak{p}$ has finite projective dimension implies that $R/\mathfrak{p}$ has finite projective dimension. Let $F_\bullet \to R/\mathfrak{p}$ be a free resolution of $R/\mathfrak{p}$ over R. Because localization is exact, we see that $(R_\mathfrak{p}) \otimes_R F_\bullet \to (R_\mathfrak{p}) \otimes_R R/\mathfrak{p} = k(\mathfrak{p})$ is a finite free resolution of $k(\mathfrak{p})$ over $R_\mathfrak{p}$. Thus, the residue field of $R_\mathfrak{p}$ has finite projective dimension over $R_\mathfrak{p}$. The theorem of Auslander-Buchsbaum-Serre (c.f., [4] Theorem 2.2.7) implies that $R_\mathfrak{p}$ is regular. ■

Another indication of the strength of the finite projective dimension condition is supplied by the following lemma.

Lemma 3.26 *Assume that $(R, \mathfrak{m})$ is a local Noetherian ring with finitely generated module M of finite projective dimension. If $Ass(R) \cap Supp(M) \neq \emptyset$, then $Supp(M) = Spec(R)$.*

Proof. Fix $\mathfrak{p} \in \mathrm{Ass}(R) \cap \mathrm{Supp}(M)$. The exactness of localization implies that $M_\mathfrak{p} \neq 0$ has finite projective dimension over $R_\mathfrak{p}$. Also, $A_\mathfrak{p}$ has depth 0 since the maximal ideal of $R_\mathfrak{p}$ is $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(R_\mathfrak{p})$ and therefore $\mathfrak{p}_\mathfrak{p}$ consists of zerodivisors and we can not start a regular sequence. By the theorem of Auslander-Buchsbaum (c.f., [4] Theorem 1.3.3) $\mathrm{pdim}(M_\mathfrak{p}) \leq \mathrm{depth}(R_\mathfrak{p})$ so that $\mathrm{pdim}(M_\mathfrak{p}) = 0$. It follows that $M_\mathfrak{p}$ is projective, and since $R_\mathfrak{p}$ is local, that $M_\mathfrak{p}$ is free. Let $F_\bullet \to M$ be a finite free resolution of M over R. For every prime ideal $\mathfrak{q}$ of R, $(F_\bullet)_\mathfrak{q} \to M_\mathfrak{q}$ is a finite free resolution of $M_\mathfrak{q}$ over $R_\mathfrak{q}$, and each $(F_i)_\mathfrak{q}$ has the same rank over $R_\mathfrak{q}$ as F_i has over R. Applying this to $\mathfrak{p} = \mathfrak{q}$, we see that the free module $M_\mathfrak{p} \neq 0$ has rank

$$0 < \mathrm{rank}(M_\mathfrak{p}) = \sum_i (-1)^i \mathrm{rank}(F_i)_\mathfrak{p} = \sum_i (-1)^i \mathrm{rank}(F_i)$$

If $\mathfrak{q}$ is a prime of R which is not in $\mathrm{Supp}(M)$, then $M_\mathfrak{q} = 0$ so that

$$0 = \sum_i (-1)^i \mathrm{rank}(F_i)_\mathfrak{q} = \sum_i (-1)^i \mathrm{rank}(F_i)$$

a contraction. Thus, $\mathrm{Supp}(M) = \mathrm{Spec}(R)$, as desired. ■

The following corollary gives another condition under which the regularity requirement of Conjecture 1.4 may be loosened.

Corollary 3.27 *Assume that* $(R, \mathfrak{m})$ *is a local Cohen-Macaulay ring of dimension* d *with ideals* $\mathfrak{p}$ *and* J *such that* $\mathfrak{p}$ *is prime of finite projective dimension,* $\dim(R/\mathfrak{p}) + \dim(R/J) = d$ *and* $\sqrt{\mathfrak{p} + J} = \mathfrak{m}$*. If* $\mathrm{ht}\,(\mathfrak{p}) = 0$ *or* $\mathrm{ht}\,(J) = 0$ *then* $\mathfrak{p}^{(n)} \cap J \subseteq \mathfrak{m}^{n+1}$*. In particular, this containment holds for local Cohen-Macaulay rings of dimension 0 and 1.*

Proof. First, assume that $\mathrm{ht}\,(J) = 0$. Then $\mathrm{ht}\,(\mathfrak{p}) = d$ so that $\mathfrak{p} = \mathfrak{m}$. Lemma 3.25 implies that $R = R_{\mathfrak{m}} = R_{\mathfrak{p}}$ is regular. In particular, R is an integral domain so that $J = 0$. The result is now trivial in this case.

Next, assume that $\mathrm{ht}\,(\mathfrak{p}) = 0$ so that $\mathrm{ht}\,(J) = d$, from which it follows that J is $\mathfrak{m}$-primary. Since $\mathfrak{p} \in \mathrm{Ass}(R) \cap \mathrm{Supp}(R/\mathfrak{p})$, Lemma 3.26 implies that $V(\mathfrak{p}) = \mathrm{Supp}(R/\mathfrak{p}) = \mathrm{Spec}(R)$. Thus, $\mathfrak{p}$ is the unique minimal prime of R. It follows that $\mathfrak{p} = 0$ (from which the result is trivial), as follows. Since $R_{\mathfrak{p}}$ is a zero-dimensional regular local ring, $\mathfrak{p}_{\mathfrak{p}} = 0$. This implies that there exists $s \in R \smallsetminus \mathfrak{p}$ such that $s\mathfrak{p} = 0$. If $\mathfrak{p} \neq 0$, then s is a zerodivisor on R. However, the fact that R is Cohen-Macaulay implies that the set of zerodivisors is exactly the union of the associated primes of R, i.e., it is $\mathfrak{p}$. It follows that $s \in \mathfrak{p}$, a contradiction. ■

Notice that in the corollary, we do not require J to be prime or to have finite projective dimension. One thing to consider is that, since R is no longer assumed to be regular, there is no guarantee that $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$. (See Example 4.16 below.) It may be, however, that the assumption of finite projective dimension will mend this deficiency, though this is an important open question in the field.

CHAPTER 4

EXAMPLES

In this chapter we give a number of examples that demonstrate the reasons for the assumptions we place on our conjectures, as well as the limitations of some of the results.

The following example shows that the requirement $e(R_{\mathfrak{p}}) = e(R)$ is necessary for Conjecture 1.4.

Example 4.1 Let k be a field, $R = k[\![X, Y, Z, W]\!]/(XY - ZW) = k[\![x, y, z, w]\!]$ with $\mathfrak{p} = (x, z)R$ and $\mathfrak{q} = (y, w)R$. Then $e(R) = 2 > 1 = e(R_{\mathfrak{p}})$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 4 > 3 = \dim(R)$.

The following example shows that the requirement $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ is necessary for Conjecture 1.4.

Example 4.2 Let k be a field, $R = k[\![X]\!]$, $\mathfrak{p} = \mathfrak{q} = (0)$. Then $e(R_{\mathfrak{p}}) = e(R) = 1$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > 1 = \dim(R)$.

The following example shows that, in order for Conjecture 1.4 to hold, we must assume that our ring is at least equidimensional.

Example 4.3 Let k be a field and $R = k[\![X]\!] \times_k k[\![Y, Z]\!]$. That is, in the diagram

$$\begin{array}{ccc} k[\![X]\!] \times k[\![Y, Z]\!] & \longrightarrow & k[\![X]\!] \\ \downarrow & & \downarrow \beta \\ k[\![Y, Z]\!] & \xrightarrow{\alpha} & k \end{array}$$

$R = \{(a, b) \in k[\![X]\!] \times k[\![Y, Z]\!] : \alpha(a) = \beta(b)\}$. It is straightforward to verify the following facts: (i) R is a local ring with maximal ideal $\mathfrak{m} = ((X, 0), (0, Y), (0, Z))$; (ii) $\dim(R) = 2$; (iii) $e(R) = 1$; and if $\mathfrak{p} = ((X, 0))R$ and $\mathfrak{q} = ((0, Y), (0, Z))R$ then

$\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R_{\mathfrak{q}}) = e(R) = 1$. However, $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 3 > 2 = \dim(R)$. he essential problem is that R has components of different dimension.

We should note, as this pertains to Conjecture 1.5, that both $\mathfrak{p}$ and $\mathfrak{q}$ have infinite projective dimension. We demonstrate this in steps. First, $\operatorname{depth}(R) = 1$ so that, in particular, R is not Cohen-Macaulay. It is straightforward to verify that the element (X, Y) is R-regular. The element $(X, 0)$ is not in the ideal $(X, Y)R$: if $(X, 0) = (X, Y)(f, g) = (Xf, Yg)$ then $g = 0$ and $f = 1$ contradicting the fact that $(f, g) \in K[\![X]\!] \times_K K[\![Y, Z]\!]$. Also, every nonunit of $R/(X, Y)R$ annihilates the (nonzero) image of $(X, 0)$: $(f, g)(X, 0) = (Xf, 0) = (X, Y)(f, 0) \in (X, Y)R$. Thus, the longest regular sequence in R has length 1, as desired.

Second, we note the following properties of $\mathfrak{p}$ and $\mathfrak{q}$. $\operatorname{depth}_R(R/\mathfrak{p}) = 2$, as $R/\mathfrak{p} \cong K[\![Y, Z]\!]$ via the natural map $R \to K[\![Y, Z]\!]$ and the sequence $(0, Y), (0, Z)$ is a maximal regular sequence on $R/\mathfrak{p}$. Similarly, $\operatorname{depth}(R/\mathfrak{q}) = 1$. Furthermore, $R/\mathfrak{q}$ is not free, as $\dim(R/\mathfrak{q}) = 1$ and $\dim(R) = 2$.

Now, suppose that $R/\mathfrak{p}$ has finite projective dimension. By the formula of Auslander and Buchsbaum, we have

$$\operatorname{pdim}(R/\mathfrak{p}) + \operatorname{depth}(R/\mathfrak{p}) = \operatorname{depth}(R) = 1$$

But the left-hand side of this equation is $\operatorname{pdim}(R/\mathfrak{p}) + 2 > 2$, a contradiction. If $R/\mathfrak{q}$ had finite projective dimension, then a similar computation would show that $\operatorname{pdim}(R/\mathfrak{q}) = 0$ so that $R/\mathfrak{q}$ is free, a contradiction.

The following example shows that the regularity assumption in Theorem 1.3 is necessary.

Example 4.4 Let k be a field and let $R = k[\![X, Y, Z]\!]/(X^2 - YZ) = k[\![x, y, z]\!]$. Then R is a complete intersection of dimension 2, but is not regular. Let $\mathfrak{p} = (x, z)$ which is a prime ideal in R. It is straightforward to verify that $\mathfrak{p}^{(2)} = (z)$ which is not contained in $\mathfrak{m}^2$. Note that $\mathfrak{p}$ does not have finite projective dimension. This shall be relevant below when we compare Conjectures 1.4 and 1.5.

The following example shows that the bound in Corollary 2.13 part 1(d) can be achieved and can be strict, even for a flat, local homomorphism of regular local rings.

Example 4.5 Let k be a field, $A = k[\![X,Y]\!]$ and $\hat{A} = k[\![X,\sqrt{Y}]\!]$. Then $\hat{A}/\mathfrak{n}\hat{A} = k[\![X,\sqrt{Y}]\!]/(X,Y)$ has length 2. However,

$$\begin{aligned} e(\hat{A}/(X)) &= 1 < 2 = e(A/(X))\mathrm{length}_{\hat{A}}(\hat{A}/\mathfrak{n}\hat{A}) \\ e(\hat{A}/(Y)) &= 2 = e(A/(Y))\mathrm{length}_{\hat{A}}(\hat{A}/\mathfrak{n}\hat{A}) \end{aligned}$$

so, the inequality can be strict or not.

The following example shows that the condition $M \cap A = \mathfrak{n}$ in Theorem 2.18 does not automatically follow from the fact that M is maximal.

Example 4.6 Let A be a discrete valuation ring with uniformizing parameter t. Then the field of fractions of A is

$$K = A[t^{-1}] = A[X]/(1-tX)$$

so that the ideal $(1-tX)A[X]$ is maximal. However, it is straightforward to verify that $(1-tX)A[X] \cap A = (0)$, which is not maximal.

The following example shows that the requirement of equidimensionality in Theorem 2.20 is necessary.

Example 4.7 Let k be a field, $A = k[\![X]\!]$, $B = k[\![Y]\!]/(Y^n)$ for some $n > 1$, and $C = A \times_k B$. That is, in the diagram

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A \arrow[d, "\beta"'] \\ B \arrow[r, "\alpha"] & k \end{tikzcd}$$

$C = \{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$. It is straightforward to verify the following facts: (i) C is a local ring with maximal ideal $\mathfrak{n} = (X)A \times (Y)B$; (ii) $\dim(C) = 1$; (iii) $e(C) = 1$; and (iv) the ideal $\mathfrak{p} = (0)A \times (Y)B$ is a prime such that $e(C_{\mathfrak{p}}) = n > 1 = e(C)$. The essential problem is that C has components of different dimension.

The following two examples show that if $R/\mathfrak{p}$ is not regular, neither implication of Lemma 2.31 holds.

Example 4.8 Let $R = k[\![X, Y, Z, U, V, W]\!]$ and let $\mathfrak{p}$ be the ideal generated by the 2×2 minors of the generic matrix

$$\begin{pmatrix} X & Y & Z \\ U & V & W \end{pmatrix}$$

Since R is regular, $e(R) = 1 = e(R_\mathfrak{p})$. However, $\operatorname{ht}(\mathfrak{p}) = 2$ and $s(\mathfrak{p}) = 3$, since otherwise $\mathfrak{p}$ would be generated by a regular sequence by Theorem 3.14. We note that it can be shown, using local cohomology, that there is no regular sequence in $\mathfrak{p}$ which generates $\mathfrak{p}$ up to radical, i.e., $\mathfrak{p}$ is not a set-theoretic complete intersection ideal. (I am grateful to Anurag Singh for showing me a proof of this fact.)

Example 4.9 Let $R = k[\![X, Y, Z, W]\!]/(XY - ZW)$ with $\mathfrak{p} = (0)R$. Then $\operatorname{ht}(\mathfrak{p}) = 0 = s(\mathfrak{p})$. However, $e(R) = 2$ and $e(R_\mathfrak{p}) = 1$.

The following two examples show that, in a ring that is not regular, neither of the following conditions implies the other: "$e(R_\mathfrak{p}) = e(R)$" and "$\mathfrak{p}$ has finite projective dimension."

Example 4.10 Let $R = k[\![X, Y, Z, W]\!]/(X^2 - YZ) = k[\![x, y, z, w]\!]$ with $\mathfrak{p} = (x, z)$. Then $\mathfrak{p}$ has infinite projective dimension and $e(R) = 2 = e(R_\mathfrak{p})$. Notice that R is a domain and $R/\mathfrak{p}$ is regular.

Example 4.11 Let $R = k[\![X, Y, Z, W]\!]/(X^2 - YZ)$ with $\mathfrak{q} = (0)$. Then $R/\mathfrak{q}$ is free (and therefore has finite projective dimension) and $e(R) = 2 > 1 = e(R_\mathfrak{q})$.

We notice that Example 4.11 is not as satisfying as Example 4.10, in the sense that $R/\mathfrak{q}$ is not regular. We were attempting to find a local domain R with prime ideal $\mathfrak{p}$ satisfying the following conditions: (i) $R/\mathfrak{p}$ is regular, (ii) $\mathfrak{p}$ has finite projective dimension, and (iii) $e(R) > e(R_\mathfrak{p})$. We have already seen that these conditions are quite restrictive. In fact, such an example does not exist, as we see in Theorem 4.13 below. The following lemma supplies the main tool for proving the theorem.

Lemma 4.12 *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Let $F_\bullet \to M$ be an R-free resolution of M and assume that $\mathbf{x} = x_1, \dots, x_i \in R$ is a sequence which is regular on M. Let $K_\bullet = K_\bullet(\mathbf{x})$ denote the Koszul complex of $\mathbf{x}$ on R. Then $F_\bullet \otimes_R K_\bullet$ is a free resolution of $M/\mathbf{x}M$.*

Proof. Since $K_\bullet(\mathbf{x}_1, \dots, x_i) = K_\bullet(x_1) \otimes K_\bullet(x_2, \dots, x_i)$, we may assume without loss of generality that $i = 1$, i.e., that we have a regular sequence of length 1. In this case, $K_\bullet$ is exactly

$$0 \to R \xrightarrow{\cdot x} R \to 0$$

The tensor product of complexes $F_\bullet \otimes_R K_\bullet$ is given as

$$(F_\bullet \otimes_R K_\bullet)_i = F_i \oplus F_{i-1}$$

with differentials $F_i \oplus F_{i-1} \xrightarrow{D_i} F_{i-1} \oplus F_{i-2}$ given by

$$(a, b) \mapsto (d(a) + (-1)^{i-1} x b, d(b))$$

where d represents the differential of the complex $F_\bullet$. In particular, $F_\bullet \otimes_R K_\bullet$ is a finite free complex. The natural split exact sequence

$$0 \to F_i \to F_i \oplus F_{i-1} \to F_{i-1} \to 0$$

gives us a short exact sequence of chain complexes

$$0 \to F_\bullet \to F_\bullet \otimes_R K_\bullet \to F_\bullet[-1] \to 0$$

where $F_\bullet[-1]$ is the twisted complex of $F_\bullet$ so that $F_\bullet[-1]_i = F_{i-1}$. This gives a long exact sequence in homology

$$\cdots \to H_i(F_\bullet) \to H_i(F_\bullet \otimes_R K_\bullet) \to H_{i-1}(F_\bullet) \xrightarrow{\delta} H_{i-1}(F_\bullet) \to \cdots \tag{4.1}$$

It is straightforward to verify that the connecting homomorphism δ is given by multiplication by $(-1)^{i-1} x$. For $i > 1$ the sequence (4.1) is

$$\cdots \to 0 \to H_i(F_\bullet \otimes_R K_\bullet) \to 0 \to 0 \to \cdots$$

so that $H_i(F_\bullet \otimes_R K_\bullet) = 0$. For $i = 1$ we have the sequence

$$\cdots \to 0 \to H_1(F_\bullet \otimes_R K_\bullet) \to M \xrightarrow{\cdot x} M$$

Since x is regular on M, the map $M \to M$ is injective so that $H_1(F_\bullet \otimes_R K_\bullet) = 0$. For $i < 0$, $(F_\bullet \otimes_R K_\bullet)_i = 0$, and it follows that $F_\bullet \otimes_R K_\bullet$ is a free resolution of

$$\ker(D_0) = \mathrm{Coker}(M \xrightarrow{\cdot x} M) = M/xM$$

as desired. ■

The following theorem demonstrates that there does not exist a local domain R with prime ideal $\mathfrak{p}$ satisfying the following conditions: (i) $R/\mathfrak{p}$ is regular, (ii) $\mathfrak{p}$ has finite projective dimension, and (iii) $e(R) > e(R_\mathfrak{p})$.

Theorem 4.13 *Let $(R, \mathfrak{m})$ be a local ring with prime ideal $\mathfrak{p}$ such that $\mathfrak{p}$ has finite projective dimension and let $\hat{R}$ denote the completion of R. Assume that one of the following holds:*

1. *$R/\mathfrak{p}$ is regular.*

2. *R is quasi-unmixed and $e(R_\mathfrak{p}) = e(R)$.*

Then R is regular. (In particular, $e(R_\mathfrak{p}) = 1 = e(R)$ and every prime ideal has finite projective dimension.)

Proof. 1. Assume that $(R, \mathfrak{m})$ is a local ring with prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ is regular and $\mathfrak{p}$ has finite projective dimension. Let $F_\bullet \to R/\mathfrak{p}$ be a minimal, finite R-free resolution of $R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is regular, let $\mathbf{x} = x_1, \ldots, x_i \in R$ be a sequence which forms a regular system of parameters of $R/\mathfrak{p}$. Let $K_\bullet = K_\bullet(\mathbf{x})$ denote the Koszul complex of $\mathbf{x}$ on R, so that, by the lemma, $F_\bullet \otimes_R K_\bullet$ is a finite, R-free resolution of $(R/\mathfrak{p})/(\mathbf{x}R/\mathfrak{p}) = R/\mathfrak{m}$. The existence of such a resolution implies that R is regular.

2. Assume that $e(R_{\mathfrak{p}}) = e(R)$, R is quasi-unmixed. By [8] Theorem 6.8, R is regular if and only if $e(R) = 1$. By Lemma 3.25, $R_{\mathfrak{p}}$ is regular so that $e(R) = e(R_{\mathfrak{p}}) = 1$, as desired. ■

A surprising benefit of prime ideals satisfying conditions 1 and 2 of the theorem is that the multiplicity of R can be computed using any minimal reduction of $\mathfrak{p}$. More specifically we have the following.

Proposition 4.14 *Assume that $(R, \mathfrak{m})$ is a quasi-unmixed, Nagata local ring with infinite residue field and prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ is regular and $e(R_{\mathfrak{p}}) = e(R)$. Then, for any minimal reduction $\mathfrak{a}$ of $\mathfrak{p}$, $e(R) = e(R/\mathfrak{a})$.*

Proof. Let $z_1, \ldots, z_j \in \mathfrak{m}$ be a sequence whose residues in $R/\mathfrak{p}$ form a regular system of parameters and let $y_1, \ldots, y_i \in \mathfrak{p}$ generate a minimal reduction $\mathfrak{a}$ of $\mathfrak{p}$. By Lemma 2.31 $i = \operatorname{ht}(\mathfrak{p})$, and since $j = \dim(R/\mathfrak{p})$ we see that $i+j = \dim(R)$. The ideal $\mathfrak{b} = (\mathbf{z}, \mathbf{y})R$ is a minimal reduction of $\mathfrak{m}$, as it is generated by the correct number of elements and, assuming that $\mathfrak{p}^n\mathfrak{a} = \mathfrak{p}^{n+1}$ we have

$$\begin{aligned} \mathfrak{m}^n\mathfrak{b} &= (\mathfrak{p}, \mathbf{z})^n(\mathbf{y}, \mathbf{z}) \supseteq (\mathfrak{p}^n(\mathbf{y}), \mathfrak{p}^n(\mathbf{z}), \mathfrak{p}^{n-1}(\mathbf{z})^2, \ldots, (\mathbf{z})^{n+1}) \\ &= (\mathfrak{p}^{n+1}, \mathfrak{p}^n(\mathbf{z}), \mathfrak{p}^{n-1}(\mathbf{z})^2, \ldots, (\mathbf{z})^{n+1}) = \mathfrak{m}^{n+1}. \end{aligned}$$

In particular, $\mathfrak{b}/\mathfrak{a}$ is a reduction of $\mathfrak{m}/\mathfrak{a}$ so that

$$e(R) = e(\mathfrak{b}, R) = e(\mathfrak{b}/\mathfrak{a}, R/\mathfrak{a}) = e(R/\mathfrak{a})$$

as desired. ■

The following example shows that ideals P and Q which arise after a regular alteration of the situation of Conjecture 1.4 will not in general satisfy the condition $\sqrt{P+Q} = M$, even after localizing at a maximal ideal M which contains P and Q.

Example 4.15 Let $R = k[\![x, y, z, w]\!]$, $\mathfrak{p} = (x, y)$ and $\mathfrak{q} = (z, w)$. Then $R/\mathfrak{p} = k[\![z, w]\!]$ and if we blow-up along the ideal (z, w) then we get a birational resolution with $P = (x, y, zT - wS) \subset k[\![x, y, z, w]\!][S, T]$. (Even though $R/\mathfrak{p}$ is already regular, there is no way to guarantee that we will pick the "nicest" regular alteration possible.) With the above notation, $Q = (z, w)$. Even though the dimension statement from above shows that, locally, P and Q have the "correct" heights, their sum is not locally primary to the maximal ideal. For example, if we consider the open region $U_{(T)} \subseteq \operatorname{Proj}(A/P)$, which is determined by taking the homogeneous localization $(A/P)_{(T)}$, we have $P_{(T)} + Q_{(T)} = (x, y, z - w\frac{S}{T}) + (z, w) = (x, y, z, w, w\frac{S}{T})$. For this sum to be the right size, we would need (some power of) $\frac{S}{T}$ to be in the sum as well, which is not the case.

The following example shows that the assumption of regularity in Conjecture 1.2 is necessary.

Example 4.16 Let k be a field and let $R = k[\![X, Y, Z]\!]/(X^2 - YZ) = k[\![x, y, z]\!]$. Then R is a complete intersection of dimension 2, but is not regular. Let $\mathfrak{p} = (x, z)$ and $\mathfrak{q} = (x, y)$ which are prime ideals in R such that $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 1 + 1 = \dim R$. However, $x \in \mathfrak{p} \cap \mathfrak{q}$ so that $\mathfrak{p} \cap \mathfrak{q} \not\subseteq \mathfrak{m}^2$.

The following example shows that the assumption $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ in Conjecture 1.2 is necessary.

Example 4.17 Let $R = k[\![X, Y]\!]$ and $\mathfrak{p} = \mathfrak{q} = (X)$. Then R is regular and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 1 + 1 = \dim R$. However, $\mathfrak{p} \cap \mathfrak{q} = (X) \not\subseteq \mathfrak{m}^2$.

The following example shows that the assumption $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$ in Conjecture 1.2 is necessary.

Example 4.18 Let $R = k[\![X]\!]$ and $\mathfrak{p} = \mathfrak{q} = (X)$. Then R is regular and $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$. However, $\mathfrak{p} \cap \mathfrak{q} = (X) \not\subseteq \mathfrak{m}^2$.

The following example shows that the hypothesis $\sqrt{P + Q} = M$ in Proposition 3.6 will not hold in general.

Example 4.19 Let R have dimension 2 with regular system of parameters t, u. Let $\mathfrak{p} = (t)$ and $\mathfrak{q} = (t + u^2)$. Then $P = Q = (t^G)$.

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