Solving Pell's Equation with Fibonacci's Rabbits

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Pell's Equation was a recurring theme in the development of number theory, predating the British mathematician John Pell (1611-1685) by several thousand years (and anyway Pell was wrongly credited with solving it, so the attribution is a total mess). The equation is:

$$y^2 = dx^2 + 1 \tag{1}$$

when $d > 1$ is a (fixed) natural number that is *not* a perfect square.

The graph of this equation in the plane is a hyperbola, of course, but the challenge is to find pairs of **positive integers** (x, y) that solve (1), or, in other words, to find points on the hyperbola that have integer coordinates (and lie in the first quadrant).

Examples:

(a) When $d = 2$, the pair $(x, y) = (2, 3)$ is a solution.

$$9 = 8 + 1; \quad 3^2 = 2(2^2) + 1$$

(b) Devise a trial-and-error method to search for solutions and use it to find a solution when $d = 3, 5, 6, 7$.

(c) Find a different solution to the one above when $d = 2$.

(d) When d is a perfect square, there are no integer solutions. Why?

Fact: Any one solution can be used to generate infinitely many others.

Think of $y^2 - dx^2$ as a difference of squares and factor:

$$y^2 - dx^2 = y^2 - (x\sqrt{d})^2 = (y - x\sqrt{d})(y + x\sqrt{d})$$

(giving the hyperbola's asymptotes). Pell's equation becomes:

$$(y - x\sqrt{d})(y + x\sqrt{d}) = 1$$

If (x_1, y_1) is one solution, then we can raise both sides of:

$$(y_1 - x_1\sqrt{d})(y_1 + x_1\sqrt{d}) = 1$$

to any power n, and get:

$$(y_1 - x_1\sqrt{d})^n(y_1 + x_1\sqrt{d})^n = 1^n = 1$$

These give new solutions, because there is a pair (x_n, y_n) so that:

$$(y - x\sqrt{d})^n = y_n - x_n\sqrt{d} \text{ and } (y + x\sqrt{d})^n = y_n + x_n\sqrt{d}$$

Example: (a) Start from the solution $(2,3)$ to Pell's equation:

$$3^2 - 2(2^2) = (3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$$

Square both sides:

$$(3 - 2\sqrt{2})^2 = (9 - 12\sqrt{2} + 8) = (17 - 12\sqrt{2})$$
$$(3 + 2\sqrt{2})^2 = (9 + 12\sqrt{2} + 8) = (17 + 12\sqrt{2})$$

so $(12, 17)$ is another solution! (Check: $17^2 = 289 = 2(144) + 1$)

Or cube both sides:

$$(3 - 2\sqrt{2})^3 = (27 - 54\sqrt{2} + 72 - 16\sqrt{2}) = 99 - 70\sqrt{2}$$
$$(3 + 2\sqrt{2})^3 = (27 + 54\sqrt{2} + 72 - 16\sqrt{2}) = 99 + 70\sqrt{2}$$

and $(70, 99)$ is another solution! (Check: $99^2 = 9801 = 2(4900) + 1$)

(Do the kids' calculators make this easy to do?)

(b) Find second (and third) solutions when $d = 3, 5, 6, 7$.

An "Application:"

Triangle vs Square Numbers.

We teach about square numbers:

$$1, 4, 9, 16, 25, 36, 49, ..., n^2, ...$$

and about triangle numbers:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, ..., m(m+1)/2, ...$$

and you may have noticed that:

1 and 36 are simultaneously square and triangle numbers

Question: Are there any other square/triangle numbers?

Some Algebra:

$$m(m+1)/2 = n^2$$
$$4m^2 + 4m = 8n^2$$
$$(2m+1)^2 - 1 = 2(2n)^2$$
$$(2m+1)^2 = 2(2n)^2 + 1$$
$$y^2 = 2x^2 + 1 \ (x = 2n, y = 2m + 1)$$

Pell's equation! Let's look at the solutions we've found so far:

$$(x, y) = (2, 3) \Rightarrow (m, n) = (1, 1); 1^2 = 1$$
$$(x, y) = (12, 17) \Rightarrow (m, n) = (8, 6); 6^2 = 36$$
$$(x, y) = (70, 99) \Rightarrow (m, n) = (48, 35); 35^2 = 1225$$

and of course we can now find as many others as we want!

Variation on the Application. Instead, suppose we ask for:

Triangle numbers that are 3 times a perfect square.

The algebra:

$$m(m+1)/2 = 3n^2$$
$$4m^2 + 4m = 24n^2$$
$$(2m+1)^2 - 1 = 6(2n)^2$$
$$(2m+1)^2 = 6(2n)^2 + 1$$
$$y^2 = 6x^2 + 1 \ (x = 2n, y = 2m+1)$$

This is Pell's equation for $d = 6$.

Of course, you can replace 3 by (almost any) number k, and then:

Triangle numbers that are k times a perfect square.

come from solutions to Pell's equation:

$$y^2 = 2kx^2 + 1$$

The Archimedes Cattle Problem was a *very* famous example of this (see the Appendix). After the dust clears, the second part of the cattle problem, which is required if "thou shalt be numbered among the wise" requires finding a triangle number that is:

$$k = 51285802909803$$

times a perfect square. The smallest such number has $206,545$ *digits*! Although a method was devised in 1880 by a German mathematician named Amthor for finding the answer "in principle," it was assumed that it would never be found in practice. In fact, a letter to the New York Times in 1931 stated:

Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution.

My how times have changed! You can see the complete answer at: www.cs.drexel.edu/∼ crorres/Archimedes/Cattle/computer2/computer_output.html

So What About the Rabbits?

In the late 1657, Fermat issued a challenge to the mathematicians across the channel to solve the "simple" equations:

$$y^2 = 61x^2 + 1 \quad \text{and} \quad y^2 = 109x^2 + 1$$

which were chosen "so that they wouldn't cause too much trouble". Quite on the contrary, they were chosen with care to have solutions that were far too large to be found by trial-and-error.

Turns out there is a much more efficient way to solve Pell's equation.

Continued Fractions. Here is a rather nonstandard introduction to the subject of continued fractions. Let's start with Fibonacci's rabbits. Recall that the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

is defined *inductively* by:

$$F_{-1} = 1, \quad F_0 = 1 \text{ and } \; F_{n+1} = F_n + F_{n-1}$$

Alternatively, we can start with one pair of two-year-old rabbits, and another pair of one-year-old rabbits. Each year, each pair of rabbits has a pair of offspring (of opposite sexes), and, alas, the two-year-olds die. Thus in year one, the number of pairs of newborn rabbits is:

$$F_1 = 1 + 1 = 2$$

and, alas, the two-year olds die off. In the year two, the number of newborn rabbits is:

$$F_2 = 1 + 2 = 3$$

and, alas, the two-year-olds die off again.

Now suppose instead that we start with only one pair of one-year-old rabbits (and no two-year-olds). We'll call these the Montague rabbits. They will reproduce according to:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

On the other hand, the Capulet rabbits will start as one pair of two-year-olds (and no one-year-olds). They will reproduce according to:

$$1, 0, 1, 1, 2, 3, 5, 8, 13, 21, ...$$

(one step behind the Montagues). Then the *ratio* of Montagues to Capulets in each year starting with year one will be:

$$1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21, ...$$

Getting out your calculators, you will see that these ratios approach the **golden ratio**:

$$\frac{1+\sqrt{5}}{2} \approx 1.618034...$$

This may have been familiar to you already. The next wrinkle may not be. Suppose that we "reprogram" our Montague and Capulet rabbits as follows. The two-year-old rabbits will have one pair of offspring as before (and then die, as before). But the one-year-olds will have babies according to a rule that we set forth. This rule will look like an infinite string of integer "instructions" to the rabbits. The golden ratio instructions were:

$$1, 1, 1, 1, 1, ...$$

telling each pair of one-year-olds that they need to have one pair of offspring. But suppose the instructions are instead:

$$1, 2, 2, 2, 2, ...$$

telling the one-year-olds in year one to have only one pair of offspring, but all the subsequent one-year-olds to have two. Then the Montagues and Capulets will multiply quite differently. First the Montagues:

$$0, 1, 1, 3, 7, 17, 41, 99, ...$$

and then the Capulets:

$$1, 0, 1, 2, 5, 12, 29, 70, ...$$

Now look at the ratios of these reprogrammed rabbits:

$$1/1, 3/2, 7/5, 17/12, 41/29, 99/70, ...$$

Notice two things. First, these ratios are approaching the square root of two:

$$\sqrt{2} \approx 1.414214$$

and second, **every other** one of these ratios solves Pell's equation!

Note: If you are tired of the rabbits, you can say that the set of instructions is a sequence of whole numbers:

$$a_1, a_2, a_3, a_4, ...$$

and that the corresponding inductive rules are:

$$F_{n+1} = a_n F_n + F_{n-1}$$

Examples: Try out the following "rabbit" instructions:

(a) $1, 1, 2, 1, 2, 1, 2, 1, 2, ...$

(b) $2, 4, 4, 4, 4, 4, 4, ...$

(c) $2, 1, 1, 1, 4, 1, 1, 1, 4, ...$

Turns out that:

Fact 1. Every (positive) real number α has an instruction set with the property that the rabbit ratios approach α. The instruction set may be calculated inductively from α as follows:

$$\alpha_1 = \alpha, a_1 = \lfloor \alpha_1 \rfloor$$
$$\alpha_2 = 1/(\alpha_2 - \alpha_1), a_2 = \lfloor \alpha_2 \rfloor$$
$$\alpha_3 = 1/(\alpha_3 - \alpha_2), a_3 = \lfloor \alpha_3 \rfloor$$

etc. (this is easy to implement on your calculator!)

Fact 2. The instruction sets for rational numbers are finite.

Fact 3. The instruction sets for any $\sqrt{d}$ is infinite repeating, and moreover looks like:

$$\lfloor \sqrt{d} \rfloor, \text{palindrome}, 2\lfloor \sqrt{d} \rfloor, \text{palindrome}, 2\lfloor \sqrt{d} \rfloor, ...$$

where "palindrome" is, literally, a palindrome of instructions.
Examples:

$\sqrt{19}$ instructions: $4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, ...$

$\sqrt{21}$ instructions: $4, 1, 1, 2, 1, 1, 8, 1, 1, 2, 1, 1, 8, ...$

$\sqrt{61}$ instructions: $7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, ...$

These can all be calculated **by hand** using the rules for radicals. And finally, the punchline.

Theorem: The rabbit ratio for the instruction set associated to $\sqrt{d}$ solves Pell's equation:

(a) Just before each "big" $2\lfloor \sqrt{d} \rfloor$ instruction if the palindrome has **odd length**,

(b) Just before every second "big" instruction if the palindrome has **even length**.

Thus, to solve Archimedes' cattle problem, all you have to do(!) is figure out the palindrome for:

$$d = 2(51285802909803)$$

and then let the rabbits multiply!

Appendix. Archimedes' Cattle Problem.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, **equal in depth and breadth**, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a **triangular figure**, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

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