1. If \( a + b = 4 \) and \( a^2 + b^2 = 9 \), then \( a^3 + b^3 = \)

16

\( a = 4 - b \)
\((4-b)^2 + b^2 = 9\)
\(16 - 8b + b^2 = 9\)
\(2b^2 - 8b + 7 = 0\)
\(b = \frac{8 \pm \sqrt{64 - 4(2)(7)}}{4}\)
\(b = \frac{4 \pm \sqrt{2}}{2} \rightarrow a = 4 - \left(2 \pm \frac{\sqrt{2}}{2}\right) = 2 \pm \frac{\sqrt{2}}{2}\)

\(a + b^3 = \left(2 - \frac{\sqrt{2}}{2}\right)^3 + \left(2 + \frac{\sqrt{2}}{2}\right)^3\)
\[= 2^3 + 3(2^2)\left(-\frac{\sqrt{2}}{2}\right) + 3(2)\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^3\]
\[+ 2^3 + 3(2^2)\left(\frac{\sqrt{2}}{2}\right) + 3(2)\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^3\]
\[= 8 + 8 + 3 + 3 = 22\]

2. Points \( A, B, C \) lie on a circle with radius \( r \) centered at \( O \). Segment \( DE \) has length \( r \).

If \( m(< BOC) = 30^\circ \), then \( M(< BEC) = \)
1. \(x = ?\)

\(\triangle EDO\) is isosceles \(\Rightarrow\) base angles \(x\) are same

\(2x + \theta = 180^\circ\) but also \(\theta + x = 180^\circ\)

\(\Rightarrow\) \(x = 2x\)

\(\triangle DOC\) is isosceles \(\Rightarrow\) \(\angle ODC \cong \angle OCD\)

\(\omega + 2x = 180^\circ\) and \(x + \omega + 30^\circ = 180^\circ\)

\(\omega = 180^\circ - 2x \Rightarrow x + 180^\circ - 4x + 30^\circ = 180^\circ\)

\(\omega = 180^\circ - 4x\)

\(-3x = -30^\circ\)

\(x = 10^\circ\)

3. The circle to the right has diameter 3 m. Find the area of the shaded region if its boundary consists of semicircles:

\[(a)\ 3\pi\ m^2\quad (b)\ \pi\ m^2\quad (c)\ \frac{3}{2}\pi\ m^2\quad (d)\ \frac{3}{4}\pi\ m^2\]

\[A_{shaded} = 2\left(\frac{\pi \left(\frac{1}{2}\right)^2 - \pi \left(\frac{1}{4}\right)^2}{2}\right)\]

\[= \pi - \frac{1}{4} \pi = \frac{3}{4} \pi \quad m^2\]

4. If \(f\) is a function such that \(f(x - 1) = x^2 - 3x + 5\) then \(f(x + 1) = ?\)
5. Chords are drawn in a circle as shown. The value of $x$ is 

(a) 44  (b) 48  (c) 52  (d) 34  (e) 84
\[ \angle AEC \cong \angle BEC \] \text{vertical angles} \\
\Rightarrow \angle AEB = m\angle CED = 84^\circ \] \text{(supplementary angles)} \\
\[ m\angle ACD = 180^\circ - 44^\circ - 84^\circ \]
\[ = 52^\circ \]
and \[ \angle ACD \] and \[ \angle ABD \] cut the same arc \\
\Rightarrow \angle ACD \cong \angle ABD \\
\Rightarrow m\angle ABD = x = 52^\circ \\

6. A 3 \times 3 magic square uses integers 1, 2, \ldots, 9 once each in such a way that each column, each row, and each diagonal sums to 15. Find the value of \( N \) for the magic square, a portion of which is shown below:

\[
\begin{array}{ccc}
8 & & \\
 & N & 7 \\
\end{array}
\]

(a) 1  \hspace{1cm} (b) 2  \hspace{1cm} (c) 3  \hspace{1cm} (d) 4  \hspace{1cm} (e) 5
7. If \( i = \sqrt{-1} \), then

\[
\frac{8 - 4i}{4 + 2i} = \]

(a) \( 2 - \frac{8}{3}i \) \hspace{1cm} (b) \( 2 \) \hspace{1cm} (c) \( 2 - 2i \) \hspace{1cm} (d) \( \frac{6}{5} - \frac{8}{5}i \) \hspace{1cm} (e) \( \frac{10}{3} - \frac{8}{3}i \)

\[
\left( \frac{8-4i}{4+2i} \right) \left( \frac{4-2i}{4-2i} \right) = \frac{32-16i-16i+8i^2}{16-4i^2} = \frac{32-32i-8}{16+4} = \frac{24-32i}{20} = \frac{24}{20} - \frac{32}{20}i = \frac{6}{5} - \frac{8}{5}i.
\]

8. The ratio of the circumference of a circle to the perimeter of an inscribed square is:
9. Which of the following conditions imply that the real number \( x \) is rational?

I. \( \sqrt{x} \) is rational
II. \( x^2 \) and \( x^3 \) are rational
III. \( x^2 \) and \( x^4 \) are rational

(a) I only
(b) I and II only
(c) I and III only
(d) II and III only
(e) I, II, and III
10. If the roots of \( x^2 + bx + c = 0 \) are \( \pi \) and \( \sqrt{2} \), then \( b = \)

(a) \( \pi \sqrt{2} \)
(b) \( 2\pi \sqrt{2} \)
(c) \( 2(\pi + \sqrt{2}) \)
(d) \( \pi + \sqrt{2} \)
(e) \( -(\pi + \sqrt{2}) \)

\[ x^2 + bx + c = 0 \]

By Vieta's formulas, the product of the roots is \( c \) and the sum of the roots is \( -b \).

Roots: \( \pi \) and \( \sqrt{2} \)  
Factors are:  
\((x - \pi) \) and \((x - \sqrt{2})\)  

\[ \Rightarrow (x - \pi)(x - \sqrt{2}) = 0 \]

\[ x^2 - \pi x - \sqrt{2} x + \pi \sqrt{2} = 0 \]

\[ x^2 + (\pi - \sqrt{2} \pi) x + (\sqrt{2} \pi) = 0 \]

\[ \Rightarrow b = -(\pi - \sqrt{2} \pi) = -(-\pi + \sqrt{2} \pi) \]

11. One year, there were exactly four Fridays and four Mondays in July. What day of the week was July 20?
12. What is the smallest number of circles that must be removed from the figure so that no three remaining circles form an equilateral triangle?

(a) 1  (b) 2  (c) 3  (d) 4  (e) 5
13. How many four-digit numbers are there with all odd digits?

\[
\text{(a) 625} \quad \text{(b) 5!} \quad \text{(c) 5000} \quad \text{(d) 5001} \quad \text{(e) 20}
\]

```
can use digits 1, 3, 5, 7 or 9
so we have 5 choices for each of 4 digits

\Rightarrow 5(5)(5)(5) = 25(25) = 625
```

14. Which of the following numbers is a factor of 68,574,961?

\[
\text{(a) 3} \quad \text{(b) 9} \quad \text{(c) 7} \quad \text{(d) 2} \quad \text{(e) 11}
\]
15. An equilateral triangle is divided into more than one smaller equilateral triangles. What is the smallest possible number of such triangles?

(a) 3  (b) 4  (c) 5  (d) 6  (e) 9

Create more equilateral 
by connecting middle points.

16. Suppose 50 cities are connected by roads in such a way that three roads lead in and out of each city. How many roads are there total?
(a) 50  (b) 75  (c) 100  (d) 125  (e) 150

We can only have 3 roads in and out of every city if there is an even # of cities.

<table>
<thead>
<tr>
<th>n</th>
<th>cities</th>
<th># roads</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>n^a</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
<td>75</td>
</tr>
</tbody>
</table>

You can notice this pattern.

OR you can notice that #
roads = # cities
multiplied by 3
roads each, but
we have to divide
that in half
since we counted
each road twice

\[
\frac{3(50)}{2} = 75
\]

17. Solve \(\log(5x) + \log(x - 1) = 2\).

(a) 3  (b) 5  (c) -4  (d) 3 and 5  (e) 5 and -4

\[
\log[(5x)(x-1)] = 2
\]

\[
\log [5x(x-1)] = 2
\]

\[
\frac{2}{10} = 10
\]

5x(x-1) = 100
5x^2 - 5x - 100 = 0
5(x^2 - x - 20) = 0
5(x-5)(x+4) = 0

x = 5 or x = -4

\[\text{domain:} \]
\[5x > 0 \quad \text{or} \quad x > 0 \quad \text{and} \quad x-1 > 0 \quad x > 1 \quad \Rightarrow \quad x > 1 \]

\[\text{throw away}\]
18. How many three-digit whole numbers have the property that doubling them results in reversing their digits? (For instance 125 does not have this property since $2(125) = 250$ which does not equal 521, the number obtained by reversing the digits of 125. Also, 025 is not considered a three-digit number, but rather the two-digit number 25.)

(a) 0  (b) 1  (c) 2  (d) 6  (e) none of the above

Let original # be abc by value $100a + 10b + c$. Then, we need $200a + 20b + 2c = 100c + 10b + a$

There are 4 cases:
1. $2a = c$, $2b = b$, $2c = a$ $\Rightarrow a = b = c = 0$
2. $2a = c$, $b = 2b + 1$, $a = 2c - 10$
   $\Rightarrow b = -\frac{1}{2}$
3. $c = 2a + 1$, $b = 2b - 10$, $a = 2c$
   $\Rightarrow c = -\frac{1}{3}$
4. $c = 2a + 1$, $b = 2b - 10 + 1$, $a = 2c - 10$
   $\Rightarrow c = 18/4$

\( \Rightarrow 3 \) no positive integer solutions.

19. A boy and a girl are sitting on the porch. "I'm a boy," says the child with black hair. "I'm a girl," says the child with red hair. At least one of them is lying. What is the maximum number of statements below that can be true?

(I) The person with red hair is a boy.
(II) The person with red hair is a girl.
(III) The person with black hair is a girl.

(a) 0  (b) 1  (c) 2  (d) 3  (e) There is not enough information to determine the answer.
20. Mary had a coin purse with fifty coins (which are either pennies, nickels, dimes or quarters) totaling exactly $1.00. Unfortunately, while counting her change, she dropped one coin. What is the probability that it was a penny?

(a) 50%
(b) 75%
(c) 85%
(d) 90%
(e) There is not enough information to determine the answer.
21. Given **STATE** **MATH** how many arrangements are there of these blocks?

(a) 10!  \hspace{1cm} (b) \frac{10!}{5!} \hspace{1cm} (c) \frac{10!}{3!} \hspace{1cm} (d) \frac{10!}{12} \hspace{1cm} (e) \frac{10!}{6}

If all the blocks were different, it would be 10!. But since some are the same, it's

\[
\frac{10!}{3! \cdot 2!} = \frac{10!}{(3! \cdot 2!)} = \frac{10!}{12}
\]

because 2 \[4\] and 3 \[1\] and we can't tell difference

we can't tell difference

22. In triangle ABC, the measure of the angle at vertex A is three times
the measure of the angle at vertex B and half the measure of the angle at vertex C. What is the measure of the angle at vertex A?

(a) 30°  (b) 36°  (c) 54°  (d) 60°  (e) 72°

\[ 3b = \frac{1}{2}C \]
\[ 6b = C \]
\[ \text{and } 3b + b + C = 180° \]
\[ 3b + b + 6b = 180° \]
\[ 10b = 180° \]
\[ b = 18° \]

\[ \Rightarrow A : \hspace{1cm} 3b = 3(18°) = 54° \]

23. A park has the shape of a regular hexagon of sides 2 km each. Alice walks a distance of 5km around the perimeter. What is the direct distance between the start point and the end point?

(a) \( \sqrt{13} \)  (b) \( \sqrt{14} \)  (c) \( \sqrt{15} \)  (d) \( \sqrt{16} \)  (e) \( \sqrt{17} \)
24. A nanobillion is

(a) .01  (b) 0.1  (c) 1.0  (d) 10  (e) 100

"nano" means one-billionth

\[ \frac{1}{1,000,000,000} = 1 \text{ nanobillion} \]
25. Which of the following is not equal to $\pi$?

(a) area of this circle

(b) surface area of the label for this cylinder

(c) volume of this cone

(d) shaded area under this curve

(e) all of the above are numerically equal to $\pi$
26. I have three cakes, each divided into 5 equal pieces. A serving is $\frac{2}{5}$ of a cake. How many servings do I have altogether?

(a) $\frac{2}{15}$  (b) $1\frac{1}{5}$  (c) $3\frac{2}{5}$  (d) $7\frac{1}{5}$  (e) $7\frac{1}{2}$

OR

$3 \div \frac{2}{5} = 3 \cdot \frac{5}{2} = \frac{15}{2} = 7\frac{1}{2}$

27. Given an equilateral triangular piece of cardboard, create an open box (i.e., without a lid) by cutting the same shape from each corner.
and folding up the flaps. What is the height of the box of maximal volume? (Assume length of the leg of original cardboard piece is \( x \)).

\[
(a) \quad \frac{x}{6} \quad (b) \quad \frac{\sqrt{3}x}{18} \quad (c) \quad \frac{x}{\sqrt{3}} \quad (d) \quad \frac{\sqrt{3}}{6}x \quad (e) \quad \frac{x}{3}
\]

28. Amaliea is putting her stack of pennies into rolls, keeping out the shiny ones. She notices that every other penny she picks up is dull and every third one is discolored and every fourth one is nicked or bent. How many pennies will she have to roll up if she ends up with fifty shiny pennies?

\[
(a) \quad 50 \quad (b) \quad 100 \quad (c) \quad 120 \quad (d) \quad 150 \quad (e) \quad 160
\]
$x = \text{every other one dull} \quad d = \text{every third one discolored}$

$$
\begin{array}{c}
\overbrace{x \quad d \quad x \quad \text{--} \quad x \quad d} \\
\overbrace{x \quad d \quad x \quad \text{--} \quad x \quad d} \\
\overbrace{x \quad d \quad x \quad \text{--} \quad x \quad d} \\
\overbrace{x \quad d \quad x \quad \text{--} \quad x \quad d}
\end{array}
$$

The last fact that every 4th penny is nickel doesn't rule out any more pennies.

$\Rightarrow$ Basically, for every 6 pennies, we only keep out 2 shiny pennies, and we roll up 4 pennies.

$\Rightarrow$ For 50 shiny pennies, we have 25 groups of 6, so $25(4) = 100$ rolled up

29. In this expression $ax + by + c = d$, which constants and coefficients determine the $y$-intercept?

(a) a, b and c
(b) b, c and d
(c) a, b and d
(d) a, c and d
(e) a, b, c and d

$$
\begin{align*}
ax + by + c &= d \\
by &= -ax - c + d \\
y &= -\frac{a}{b}x + \frac{d-c}{b}
\end{align*}
$$

$\Rightarrow$ $y$-intercept is $(0, \frac{d-c}{b})$
30. Solve for $x$.

$$8^{2x} = 2^x \left( \frac{64^6}{2} \right)$$

(a) 5  (b) 6  (c) 7  (d) 10  (e) 11

$$8^{2x} = 2^x \left( (2^{12})^6 \right)$$

$$2^{2x} = 2^x 2^{36} \frac{2}{2^{x+36-1}}$$

$$2^x = 2$$

$$6x = x + 35$$

$$5x = 35$$

$$x = 7$$