Archimedean multiplicity one theorems*

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Representation Theory of Real Reductive Groups
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1 Gelfand pair

We consider a pair of finite groups $G \supset G'$.

- $(G, G')$ is called a Gelfand pair if for any $(\pi, V) \in \hat{G}$, the dimension of $G'$-fixed vectors in $V$ is at most one:
  $$\dim V^{G'} \leq 1.$$  

- Equivalently:
  $$\dim \text{Hom}_{G'}(\pi, 1_{G'}) \leq 1,$$
  where $1_{G'}$ is the trivial representation of $G'$.
The followings are equivalent:

- \((G, G')\) is a Gelfand pair.
- The \(G\) representation \(\mathbb{C}[G/G']\) is multiplicity one.
- The algebra \(\mathbb{C}[G' \backslash G/G']\) is abelian (with multiplication given by convolution).
Gelfand’s observation:
- Suppose that there exists an anti-automorphism $\sigma$ of $G$ such that $\sigma(g) \in G'gG'$ for all $g \in G$. Then $\mathbb{C}[G' \backslash G / G']$ is abelian and consequently $(G, G')$ is a Gelfand pair.

**Proof:** for $f_1, f_2 \in \mathbb{C}[G' \backslash G / G']$, we have

$$f_1 \star f_2 = (f_1 \star f_2)^{\sigma} = f_2^{\sigma} \star f_1^{\sigma} = f_2 \star f_1.$$
2 Multiplicity one pair

- \((G, G')\) is called a multiplicity-one pair (or a strong Gelfand pair) if for any \(\pi \in \hat{G}\), and any \(\rho \in \hat{G}'\), we have

\[
\dim \text{Hom}_{G'}(\pi, \rho) \leq 1.
\]

**Remark:** \((G, G')\) is a multiplicity one pair if and only if \((G \times G', \Delta G')\) is a Gelfand pair.
The followings are equivalent:

- $(G, G')$ is a multiplicity one pair.
- The $G$ representation $\text{Ind}_{G'}^{G}(\rho)$ is multiplicity one, for any $\rho \in \hat{G}'$.
- The algebra of functions on $G$ invariant under conjugation by $G'$ is commutative.
3 Gelfand-Kazhdan criteria

- Adaptation and extension of Gelfand’s idea to various and more general settings.
- First appeared in I.M. Gelfand and D. Kazhdan, “Representations of the group $\text{GL}(n, K)$, where $K$ is a local field”.
- We shall state a version for real reductive groups, in terms of generalized functions.
We consider real reductive groups, and the following class of representations:

- smooth, Fréchet, of moderate growth, admissible and finitely generated.

Suggested terminologies:

- smooth Harish-Chandra representations.

- CW-HC class. CW stands for Casselman-Wallach, HC stands for Harish-Chandra.
Let $G$ be a real reductive group.

- Recall that there is a notion of rapidly decreasing functions (and densities) on $G$.
- Denote by $\mathcal{C}^\xi(G)$ the space of tempered generalized functions on $G$, which is the continuous dual of the space of rapidly decreasing densities on $G$.

Let $S_1$ and $S_2$ be two closed subgroups of $G$, with continuous characters

$$\chi_i : S_i \rightarrow \mathbb{C}^\times, \quad i = 1, 2.$$
(a) Assume that there is a continuous anti-automorphism $\tau$ of $G$ such that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $U(g_\mathbb{C})^G$, the conditions

$$f(sx) = \chi_1(s)f(x), \quad s \in S_1,$$

$$f(xs) = \chi_2(s)^{-1}f(x), \quad s \in S_2$$

imply

$$f(x^{\tau}) = f(x).$$

Then for any two irreducible smooth Harish-Chandra representations $U^\infty$ and $V^\infty$ of $G$ which are contragredient to each other, one has that

$$\dim \text{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \text{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \leq 1.$$
(b) Assume that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $U(\mathfrak{g})^G$, the conditions

$$f(sx) = \chi_1(s)f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_2(s)^{-1}f(x), \quad s \in S_2$$

imply

$$f = 0.$$

Then for any two irreducible smooth Harish-Chandra representations $U^\infty$ and $V^\infty$ of $G$ which are contragredient to each other, one has that

$$\dim \text{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \text{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) = 0.$$
4 Main results

Let \((G, G')\) be one of the following five pairs of classical Lie groups:

\[
(\text{GL}_n(\mathbb{R}), \text{GL}_{n-1}(\mathbb{R})), \ (\text{GL}_n(\mathbb{C}), \text{GL}_{n-1}(\mathbb{C})),
\]
\[
(\text{O}(p, q), \text{O}(p, q-1)), \ (\text{O}_n(\mathbb{C}), \text{O}_{n-1}(\mathbb{C})), \ (\text{U}(p, q), \text{U}(p, q-1)).
\]

**Theorem A:** \((G, G')\) is a multiplicity one pair.
Remarks:

- Theorem A and its p-adic counterpart have been expected (by Bernstein and Rallis) since the 1980’s.

- For general linear groups, Theorem A is independently due to Aizenbud and Gourevitch.

- Earlier results on Gelfand pairs by Aizenbud, Gourevitch and Sayag, and by van Dijk.

- The p-adic counterpart of Theorem A was proved in [AGRS, 2007] by Aizenbud, Gourevitch, Rallis and Schiffmann.
**Theorem B:** Suppose that \( f \in C^{-\infty}(G) \) satisfies

\[
f(gxg^{-1}) = f(x), \quad \forall \ g \in G'.
\]

Then we have

\[
f(x^\sigma) = f(x),
\]

where \( \sigma \) is the anti-involution of \( G \) given by

\[
x^\sigma = \begin{cases} 
x^t, & G \text{ general linear,} \\
x^{-1}, & G \text{ orthogonal,} \\
\bar{x}^{-1}, & G \text{ unitary.} 
\end{cases}
\]
5 A uniform set-up

$(A, \tau)$: a (semisimple) commutative involutive algebra over $\mathbb{R}$. 

$E$: a hermitian $A$-module, with its isometry group $U(E)$.

Every simple $(A, \tau)$ is isomorphic to

$(\mathbb{R}, 1), (\mathbb{C}, 1), (\mathbb{C}, -), (\mathbb{R} \times \mathbb{R}, \leftrightarrow), (\mathbb{C} \times \mathbb{C}, \leftrightarrow)$. 
Its hermitian $A$-module is isomorphic to

\[(\mathbb{R}^{p+q}, \langle , \rangle_{O(p,q)}), \quad (\mathbb{C}^n, \langle , \rangle_{O(n)}), \quad (\mathbb{C}^{p+q}, \langle , \rangle_{U(p,q)}), \]
\[(\mathbb{R}^n \oplus \mathbb{R}^n, \langle , \rangle_{\mathbb{R},n}), \quad (\mathbb{C}^n \oplus \mathbb{C}^n, \langle , \rangle_{\mathbb{C},n}).\]

The group $U(E)$ is isomorphic to

\[O(p, q), \quad O_n(\mathbb{C}), \quad U(p, q), \quad GL_n(\mathbb{R}), \quad GL_n(\mathbb{C}).\]
Introduce $\tilde{U}(E)$ (an extension of $U(E)$ of index two):

$$\{1\} \rightarrow U(E) \rightarrow \tilde{U}(E) \xrightarrow{\chi_E} \{\pm 1\} \rightarrow \{1\}.$$

Elements of $\tilde{U}(E) \subset \text{GL}_R(E) \times \{\pm 1\}$ consist of pairs $(g, \delta)$ such that either

$$\delta = 1 \quad \text{and} \quad g \in U(E),$$

or

$$\delta = -1, \quad g \text{ is conjugate linear and anti-isometry.}$$

**Remark:** The group $\tilde{U}(E)$ appeared implicitly in the work of Moeglin, Vigneras and Waldspurger (which established p-adic Howe correspondence, for $p \neq 2$).
$\tilde{U}(E)$ acts on $U(E)$ by

$$(g, \delta)x := gx^\delta g^{-1},$$

and on $E$ by

$$(g, \delta)v := \delta gv.$$

We will show

$$C^{-\infty}_{\chi_E}(U(E) \times E) = 0.$$
It may be derived from the Lie algebra version:

\[ C_{\chi E}^{-\infty} (\mathfrak{u}(E') \times E) = 0. \]

By a result of Aizenbud and Gourevitch, it suffices to show

\[ C_{\chi E}^{-\xi} (\mathfrak{u}(E) \times E') = 0. \]
6 Reduction of support to the null set and within the null set

$\mathcal{N}_E \subset u(E)$: the cone of nilpotent elements.

The null cone of $E$:

$$\Gamma_E := \{ v \in E \mid \langle v, v \rangle_E = 0 \}.$$
Proposition: Assume that $A$ is simple, $\text{sdim}_A(E) \geq 1$, and for all commutative involutive algebra $A'$ and all hermitian $A'$-module $E'$,

$$\text{sdim}_{A'}(E') < \text{sdim}_A(E) \quad \text{implies} \quad C_{\chi_{E'}}^{-\xi}(u(E') \times E') = 0.$$ 

Then every $f \in C_{\chi_E}^{-\xi}(u(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_E) \times \Gamma_E$.

Idea: Harish-Chandra, and also Jacquet and Rallis. Our set-up and formulation make the argument easier.
Let
\[ \mathcal{N}_E = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_k = \{0\} \supset \mathcal{N}_{k+1} = \emptyset \]
be a filtration of \( \mathcal{N}_E \) by its closed subsets so that each difference
\[ \mathcal{O}_i := \mathcal{N}_i \setminus \mathcal{N}_{i+1}, \quad 0 \leq i \leq k, \]
is a \( \text{U}(E) \)-adjoint orbit.
Proposition: Assume that $A$ is simple, $\text{sdim}_A(E) \geq 1$, and that every element of $\mathcal{C}_{\chi_E}^{-\xi}(u(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_i) \times \Gamma_E$, for some fixed $0 \leq i \leq k$. Then every element of $\mathcal{C}_{\chi_E}^{-\xi}(u(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_{i+1}) \times \Gamma_E$.

This is complicated, and is at the heart of the matter.
7 In the null cone: the easier part of reduction

Definition: A submanifold $Z$ of a pseudo Riemannian manifold $M$ is said to be metrically proper if for all $z \in Z$, the tangent space $T_z(Z)$ is contained in a proper nondegenerate subspace of $T_z(M)$.

Lemma: Let $Z$ be a metrically proper submanifold of $M$, and let $\Delta$ be a Laplacian type differential operator on $M$. Then if $f \in C^{-\infty}(M)$ is supported in $Z$ and is annihilated by some power of $\Delta$, then $f = 0$. 
Remarks:


- Upshot: Laplace type operators are transversal to any metrically proper submanifold.

- Lemma may be viewed as a form of uncertainty principle.
Denote
\[ s := \mathfrak{su}(E). \]
We shall view $s$ as a non-degenerate real quadratic space via the real trace form. Note that the null cone $\mathcal{N}_E$ is contained in the null cone of $s$ as a real quadratic space.

**Lemma:** Let $\mathcal{O} \subset \mathcal{N}_E$ be a nilpotent orbit which is not distinguished, i.e., it commutes with a nonzero semisimple element of $s$. Then $\mathcal{O}$ is metrically proper in $s$. 
Proof of reduction for non-distinguished nilpotent orbits:

• The partial Fourier transform $\mathcal{F}_s$ (along $s$) commutes with the action of $\check{U}(E)$ on $C^{-\xi}(s \times E)$;

• Every element $f$ of $C^{-\xi}_{\chi_E}(s \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$, so is $\mathcal{F}_s(f)$;

• Some positive power of the partial Laplacian $\Delta_s$ annihilates $f$;

• Its restriction $\bar{f}$ to $\mathcal{O}_i \times \Gamma_E$ (open in $\mathcal{N}_i \times \Gamma_E$) must be zero, by the metrical properness of $\mathcal{O}_i$ and the associated lemma.
In the null cone: exploiting the “extra” symmetry, a la [AGRS]

For $x \in \mathfrak{s}$, one can associate a closed subset $E(x) \subset E$, which has the following properties.

**Fact 1:** If every element of $C_{\chi_E}^{-\xi}(\mathfrak{s} \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$ for some $i$, then its support must be contained in

\[(\mathcal{N}_{i+1} \times \Gamma_E) \cup \left( \bigsqcup_{e \in \mathcal{O}_i} \{e\} \times (E(e) \cap \Gamma_E) \right)\].

**Fact 2:** For $e$ nilpotent and $v \in E(e)$, we have

\[\langle e^j v, v \rangle_E = 0, \quad \text{for } j > 0.\]
**Remark**: this says that for the open part $\mathcal{O}_i$ of $\mathcal{N}_i$, the Lie algebra and vector components of the support must satisfy additional constraints.

**Moral**: as a guiding principle, one should do reduction to the joint null cone, rather than $\mathcal{N}_E \times \Gamma_E$. The joint null cone satisfies additional conditions such as

$$\langle e^j v, v \rangle_E = 0, \quad \text{for } j > 0.$$
9 In the null cone: the eigenvalue bound on $\mathfrak{s} \times E$

For a nilpotent orbit $\mathcal{O} \subset \mathcal{N}_E$, denote by

$$\mathcal{V}_{\mathfrak{s} \times E, \mathcal{O}} \subset C^{-\xi}(\mathfrak{s} \times E; \mathcal{O} \times E)^{U(E)}$$

the subspace consisting of those $f$ such that the supports of both $f$ and its partial Fourier transform $\mathcal{F}_E(f)$ (along $E$) are contained in

$$\bigsqcup_{\mathbf{e} \in \mathcal{O}} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E)$$

This is the piece that we wish to drop!
The key is the following

**Proposition:** Assume that $A$ is simple, $\text{sdim}_A(E) \geq 1$, and $\mathcal{O}$ is distinguished. Then the Euler vector field $\epsilon_s$ acts semisimply on $\mathcal{V}_{s \times E, \mathcal{O}}$, and all its eigenvalues are real numbers $< -\frac{1}{2} \dim_{\mathbb{R}} s$.

This implies reduction for distinguished nilpotent orbits, as follows.
• \( \mathcal{V}_s \times E, \mathcal{O}_i \) is a module for the standard triple

\[
\epsilon_s + \frac{1}{2} \dim_{\mathbb{R}} s, \quad -\frac{1}{2} q_s, \quad \frac{1}{2} \Delta_s.
\]

• Eigenvalue bound + \( \mathfrak{sl}_2 \) theory imply the injectivity of

\[
\Delta_s : \mathcal{V}_s \times E, \mathcal{O}_i \to \mathcal{V}_s \times E, \mathcal{O}_i.
\]

• If every element \( f \) of \( C^{-\mathcal{E}}_{\chi_E}(s \times E) \) is supported in \( \mathcal{N}_i \times \Gamma_E \), then some positive power of the partial Laplacian \( \Delta_s \) annihilates \( f \).

• Its restriction \( \bar{f} \to \mathcal{O}_i \times \Gamma_E \) (open in \( \mathcal{N}_i \times \Gamma_E \)) belongs to

\[
\mathcal{V}_s \times E, \mathcal{O}_i.
\]

• \( \bar{f} = 0 \), by injectivity of \( \Delta_s \) on \( \mathcal{V}_s \times E, \mathcal{O}_i \).
10 How to achieve the eigenvalue bound of $\epsilon_5$

We work in a real quadratic space $F$. For two closed subsets $Z_1$ and $Z_2$, denote by $C^{-\xi}(F; Z_1, Z_2)$ the space of all $f \in C^{-\xi}(F)$ such that

- $f$ is supported in $Z_1$, and
- $\mathcal{F}_F(f)$ is supported in $Z_2$.

This is a module for the Weyl algebra of $F$. 
For a nilpotent element $e \in \mathcal{N}_E$, define

$$\mathcal{V}_{E,e} := C^{-\xi}(E; E(e) \cap \Gamma_E, E(e) \cap \Gamma_E)^{Z(E)}.$$ 

Let $h, e, f$ be a standard triple, i.e.,

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$
The eigenvalue bound on $\epsilon_s = \epsilon_{s,I}$ on $\mathcal{V}_{s \times E, O}$ is implied by

- the following eigenvalue bound of $\epsilon_{E,h}$ on $\mathcal{V}_{E,e}$, and

- an (rather easy) eigenvalue bound of $\epsilon_{s^f,I - \text{ad}(h/2)}$ on $C^{-\xi}(s^f; \{0\})$.

**Proposition:** Assume that $e$ is distinguished. Then the vector field $\epsilon_{E,h}$ acts semisimply on $\mathcal{V}_{E,e}$, and all its eigenvalues are nonnegative integers.
The proof of the final proposition depends on

- classification of distinguished nilpotent orbits;
- rigidity properties of certain Weyl algebra modules $C^{-\xi}(F; Z_1, Z_2)$; in particular,
- an explicit basis of $C^{-\xi}(F; F', F')$, where $\dim_{\mathbb{R}} F = 2r$ and $F'$ is a totally isotropic subspace of dimension $r$. 
11 Summary of approach

- Reduction to the null sets, analogous to Harish-Chandra’s method of descent.
- Reduction for nondistinguished nilpotent orbits, by tranversality reasons.
- Reduction for distinguished nilpotent orbits, by rigidity and representation theoretical reasons (use of $\mathfrak{sl}_2$ triple).
Similar (in spirit) to

- Harish-Chandra’s proof of regularity of invariant eigendistributions.
  (C.f. work of Atiyah-Schmid, Varadarajan, Wallach)

- AGRS’s proof of p-adic multiplicity one theorems.
Thank you !