Archimedean multiplicity one theorems*

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1 Gelfand pair

We consider a pair of finite groups $G \supset G'$.

- (G, G') is called a Gelfand pair if for any $(\pi, V) \in \widehat{G}$, the dimension of G'-fixed vectors in V is at most one:

$$\dim V^{G'} \leq 1.$$

- Equivalently:

$$\dim \operatorname{Hom}_{G'}(\pi, \mathbb{1}_{G'}) \leq 1,$$

  where $\mathbb{1}_{G'}$ is the trivial representation of G'.

The followings are equivalent:

- (G, G') is a Gelfand pair.
- The G representation $\mathbb{C}[G/G']$ is multiplicity one.
- The algebra $\mathbb{C}[G'\backslash G/G']$ is abelian (with multiplication given by convolution).

Gelfand's observation:

- Suppose that there exists an anti-automorphism σ of G such that $\sigma(g) \in G'gG'$ for all $g \in G$. Then $\mathbb{C}[G'\backslash G/G']$ is abelian and consequently (G, G') is a Gelfand pair.

Proof: for $f_1, f_2 \in \mathbb{C}[G'\backslash G/G']$, we have

$$f_1 \star f_2 = (f_1 \star f_2)^\sigma = f_2^\sigma \star f_1^\sigma = f_2 \star f_1.$$

2 Multiplicity one pair

- (G, G') is called a multiplicity-one pair (or a strong Gelfand pair) if for any $\pi \in \widehat{G}$, and any $\rho \in \widehat{G'}$, we have

$$\dim \operatorname{Hom}_{G'}(\pi, \rho) \leq 1.$$

Remark: (G, G') is a multiplicity one pair if and only if $(G \times G', \Delta G')$ is a Gelfand pair.

The followings are equivalent:

- (G, G') is a multiplicity one pair.
- The G representation $\mathrm{Ind}_{G'}^{G}(\rho)$ is multiplicity one, for any $\rho \in \widehat{G'}$.
- The algebra of functions on G invariant under conjugation by G' is commutative.

3 Gelfand-Kazhdan criteria

- Adaptation and extension of Gelfand's idea to various and more general settings.
- First appeared in I.M. Gelfand and D. Kazhdan, "Representations of the group $\mathrm{GL}(n, K)$, where K is a local field".
- We shall state a version for real reductive groups, in terms of generalized functions.

We consider real reductive groups, and the following class of representations:

- smooth, Fréchet, of moderate growth, admissible and finitely generated.

Suggested terminologies:

- smooth Harish-Chandra representations.
- CW-HC class. CW stands for Casselman-Wallach, HC stands for Harish-Chandra.

Let G be a real reductive group.

- Recall that there is a notion of rapidly decreasing functions (and densities) on G.
- Denote by $C^{-\xi}(G)$ the space of tempered generalized functions on G, which is the continuous dual of the space of rapidly decreasing densities on G.

Let S_1 and S_2 be two closed subgroups of G, with continuous characters

$$\chi_i : S_i \to \mathbb{C}^\times, \quad i = 1, 2.$$

(a) Assume that there is a continuous anti-automorphism τ of G such that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})^G$, the conditions

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1, \ \text{and}$$

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply

$$f(x^\tau) = f(x).$$

Then for any two irreducible smooth Harish-Chandra representations U^∞ and V^∞ of G which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \, \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \leq 1.$$

(b) Assume that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g})^G$, the conditions

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1, \quad \text{and}$$

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply

$$f = 0.$$

Then for any two irreducible smooth Harish-Chandra representations U^∞ and V^∞ of G which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) = 0.$$

4 Main results

Let (G, G') be one of the following five pairs of classical Lie groups:

$$(\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_{n-1}(\mathbb{R})),\ (\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_{n-1}(\mathbb{C})),$$

$$(\mathrm{O}(p,q), \mathrm{O}(p,q-1)),\ (\mathrm{O}_n(\mathbb{C}), \mathrm{O}_{n-1}(\mathbb{C})),\ (\mathrm{U}(p,q), \mathrm{U}(p,q-1)).$$

Theorem A: (G, G') is a multiplicity one pair.

Remarks:

- Theorem A and its p-adic counterpart have been expected (by Bernstein and Rallis) since the 1980's.
- For general linear groups, Theorem A is independently due to Aizenbud and Gourevitch.
- Earlier results on Gelfand pairs by Aizenbud, Gourevitch and Sayag, and by van Dijk.
- The p-adic counterpart of Theorem A was proved in [AGRS, 2007] by Aizenbud, Gourevitch, Rallis and Schiffmann.

Theorem B: Suppose that $f \in C^{-\infty}(G)$ satisfies

$$f(gxg^{-1}) = f(x), \quad \forall\, g \in G'.$$

Then we have

$$f(x^\sigma) = f(x),$$

where σ is the anti-involution of G given by

$$x^\sigma = \begin{cases} x^t, & G \text{ general linear,} \\ x^{-1}, & G \text{ orthogonal,} \\ \bar{x}^{-1}, & G \text{ unitary.} \end{cases}$$

5 A uniform set-up

(A, τ): a (semisimple) commutative involutive algebra over $\mathbb{R}$.

E: a hermitian A-module, with its isometry group $\mathrm{U}(E)$.

Every simple (A, τ) is isomorphic to

$$(\mathbb{R}, 1),\ (\mathbb{C}, 1),\ (\mathbb{C}, ^{-}),\ (\mathbb{R}\times\mathbb{R}, \leftrightarrow),\ (\mathbb{C}\times\mathbb{C}, \leftrightarrow).$$

Its hermitian A-module is isomorphic to

$$(\mathbb{R}^{p+q}, \langle\ ,\ \rangle_{\mathrm{O}(p,q)}), \quad (\mathbb{C}^n, \langle\ ,\ \rangle_{\mathrm{O}(n)}), \quad (\mathbb{C}^{p+q}, \langle\ ,\ \rangle_{\mathrm{U}(p,q)}),$$
$$(\mathbb{R}^n \oplus \mathbb{R}^n, \langle\ ,\ \rangle_{\mathbb{R},n}), \quad (\mathbb{C}^n \oplus \mathbb{C}^n, \langle\ ,\ \rangle_{\mathbb{C},n}).$$

The group $\mathrm{U}(E)$ is isomorphic to

$$\mathrm{O}(p,q),\ \mathrm{O}_n(\mathbb{C}),\ \mathrm{U}(p,q),\ \mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}).$$

Introduce $\check{\mathrm{U}}(E)$ (an extension of $\mathrm{U}(E)$ of index two):

$$\{1\} \to \mathrm{U}(E) \to \check{\mathrm{U}}(E) \xrightarrow{\chi_E} \{\pm 1\} \to \{1\}.$$

Elements of $\check{\mathrm{U}}(E) \subset \mathrm{GL}_{\mathbb{R}}(E) \times \{\pm 1\}$ consist of pairs (g, δ) such that either

$$\delta = 1 \quad \text{and} \quad g \in \mathrm{U}(E),$$

or

$$\delta = -1, \quad g \text{ is conjugate linear and anti-isometry.}$$

Remark: The group $\check{\mathrm{U}}(E)$ appeared implicitly in the work of Moeglin, Vigneras and Waldspurger (which established p-adic Howe correspondence, for $p \neq 2$).

$\check{\mathrm{U}}(E)$ acts on $\mathrm{U}(E)$ by

$$(g,\delta)x := gx^{\delta}g^{-1},$$

and on E by

$$(g,\delta)v := \delta gv.$$

We will show

$$C^{-\infty}_{\chi_E}(\mathrm{U}(E)\times E) = 0.$$

It may be derived from the Lie algebra version:

$$C^{-\infty}_{\chi_E}(\mathfrak{u}(E) \times E) = 0.$$

By a result of Aizenbud and Gourevitch, it suffices to show

$$\boxed{C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E) = 0.}$$

6 Reduction of support to the null set and within the null set

$\mathcal{N}_E \subset \mathfrak{u}(E)$: the cone of nilpotent elements.

The null cone of E:

$$\Gamma_E := \{v \in E \mid \langle v, v\rangle_E = 0\}.$$

Proposition: Assume that A is simple, $\mathrm{sdim}_A(E) \geq 1$, and for all commutative involutive algebra A' and all hermitian A'-module E',

$$\mathrm{sdim}_{A'}(E') < \mathrm{sdim}_A(E) \quad \text{implies} \quad C^{-\xi}_{\chi_{E'}}(\mathfrak{u}(E') \times E') = 0.$$

Then every $f \in C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_E) \times \Gamma_E$.

Idea: Harish-Chandra, and also Jacquet and Rallis. Our set-up and formulation make the argument easier.

Let

$$\mathcal{N}_E = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_k = \{0\} \supset \mathcal{N}_{k+1} = \emptyset$$

be a filtration of $\mathcal{N}_E$ by its closed subsets so that each difference

$$\mathcal{O}_i := \mathcal{N}_i \setminus \mathcal{N}_{i+1}, \quad 0 \leq i \leq k,$$

is a $\mathrm{U}(E)$-adjoint orbit.

Proposition: Assume that A is simple, $\mathrm{sdim}_A(E) \geq 1$, and that every element of $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_i) \times \Gamma_E$, for some fixed $0 \leq i \leq k$. Then every element of $C^{-\xi}_{\chi_E}(\mathfrak{u}(E) \times E)$ is supported in $(\mathfrak{z}(E) + \mathcal{N}_{i+1}) \times \Gamma_E$.

This is complicated, and is at the heart of the matter.

7 In the null cone: the easier part of reduction

Definition: A submanifold Z of a pseudo Riemannian manifold M is said to be **metrically proper** if for all $z \in Z$, the tangent space $\mathrm{T}_z(Z)$ is contained in a proper nondegenerate subspace of $\mathrm{T}_z(M)$.

Lemma: Let Z be a metrically proper submanifold of M, and let Δ be a Laplacian type differential operator on M. Then if $f \in C^{-\infty}(M)$ is supported in Z and is annihilated by some power of Δ, then $f = 0$.

Remarks:

- Notion first appeared in Jiang-Sun-Zhu (Archimedean uniqueness of the Ginzburg-Rallis models).
- Upshot: Laplace type operators are transversal to any metrically proper submanifold.
- Lemma may be viewed as a form of uncertainty principle.

Denote

$$\mathfrak{s} := \mathfrak{su}(E).$$

We shall view $\mathfrak{s}$ as a non-degenerate real quadratic space via the real trace form. Note that the null cone $\mathcal{N}_E$ is contained in the null cone of $\mathfrak{s}$ as a real quadratic space.

Lemma: Let $\mathcal{O} \subset \mathcal{N}_E$ be a nilpotent orbit which is not distinguished, i.e., it commutes with a nonzero semisimple element of $\mathfrak{s}$. Then $\mathcal{O}$ is metrically proper in $\mathfrak{s}$.

Proof of reduction for non-distinguished nilpotent orbits:

- The partial Fourier transform $\mathcal{F}_{\mathfrak{s}}$ (along $\mathfrak{s}$) commutes with the action of $\check{\mathrm{U}}(E)$ on $C^{-\xi}(\mathfrak{s} \times E)$;
- Every element f of $C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$, so is $\mathcal{F}_{\mathfrak{s}}(f)$;
- Some positive power of the partial Laplacian $\Delta_{\mathfrak{s}}$ annihilates f;
- Its restriction $\bar{f}$ to $\mathcal{O}_i \times \Gamma_E$ (open in $\mathcal{N}_i \times \Gamma_E$) must be zero, by the metrical properness of $\mathcal{O}_i$ and the associated lemma.

8 In the null cone: exploiting the "extra" symmetry, a la [AGRS]

For $x \in \mathfrak{s}$, one can associate a closed subset $E(x) \subset E$, which has the following properties.

Fact 1: If every element of $C^{-\xi}_{\chi_E}(\mathfrak{s} \times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$ for some i, then its support must be contained in

$$(\mathcal{N}_{i+1} \times \Gamma_E) \cup \big(\bigsqcup_{\mathbf{e} \in \mathcal{O}_i} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E)\big).$$

Fact 2: For $\mathbf{e}$ nilpotent and $v \in E(\mathbf{e})$, we have

$$\langle e^j v, v \rangle_E = 0, \quad \text{for } j > 0.$$

Remark: this says that for the open part $\mathcal{O}_i$ of $\mathcal{N}_i$, the Lie algebra and vector components of the support must satisfy additional constraints.

Moral: as a guiding principle, one should do reduction to the joint null cone, rather than $\mathcal{N}_E \times \Gamma_E$. The joint null cone satisfies additional conditions such as

$$\langle e^j v, v \rangle_E = 0, \quad \text{for } j > 0.$$

9 In the null cone: the eigenvalue bound on $\mathfrak{s} \times E$

For a nilpotent orbit $\mathcal{O} \subset \mathcal{N}_E$, denote by

$$\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}} \subset C^{-\xi}(\mathfrak{s} \times E; \mathcal{O} \times E)^{\mathrm{U}(E)}$$

the subspace consisting of those f such that the supports of both f and its partial Fourier transform $\mathcal{F}_E(f)$ (along E) are contained in

$$\bigsqcup_{\mathbf{e}\in\mathcal{O}} \{\mathbf{e}\} \times (E(\mathbf{e}) \cap \Gamma_E)$$

This is the piece that we wish to drop!

The key is the following

Proposition: Assume that A is simple, $\mathrm{sdim}_A(E) \geq 1$, and $\mathcal{O}$ is distinguished. Then the Euler vector field $\epsilon_{\mathfrak{s}}$ acts semisimply on $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}}$, and all its eigenvalues are real numbers $< -\frac{1}{2}\dim_{\mathbb{R}} \mathfrak{s}$.

This implies reduction for distinguished nilpotent orbits, as follows.

- $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$ is a module for the standard triple

$$\epsilon_{\mathfrak{s}} + \frac{1}{2}\dim_{\mathbb{R}}\mathfrak{s}, \quad -\frac{1}{2}q_{\mathfrak{s}}, \quad \frac{1}{2}\Delta_{\mathfrak{s}}.$$

- Eigenvalue bound + $\mathfrak{sl}_2$ theory imply the injectivity of

$$\Delta_{\mathfrak{s}} : \mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i} \to \mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}.$$

- If every element f of $C^{-\xi}_{\chi_E}(\mathfrak{s}\times E)$ is supported in $\mathcal{N}_i \times \Gamma_E$, then some positive power of the partial Laplacian $\Delta_{\mathfrak{s}}$ annihilates f.
- Its restriction $\bar{f}$ to $\mathcal{O}_i \times \Gamma_E$ (open in $\mathcal{N}_i \times \Gamma_E$) belongs to $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$.
- $\bar{f} = 0$, by injectivity of $\Delta_{\mathfrak{s}}$ on $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}_i}$.

10 How to achieve the eigenvalue bound of $\epsilon_{\mathfrak{s}}$

We work in a real quadratic space F. For two closed subsets Z_1 and Z_2, denote by $C^{-\xi}(F; Z_1, Z_2)$ the space of all $f \in C^{-\xi}(F)$ such that

- f is supported in Z_1, and
- $\mathcal{F}_F(f)$ is supported in Z_2.

This is a module for the Weyl algebra of F.

For a nilpotent element $\mathbf{e} \in \mathcal{N}_E$, define

$$\mathcal{V}_{E,\mathbf{e}} := C^{-\xi}(E; E(\mathbf{e}) \cap \Gamma_E, E(\mathbf{e}) \cap \Gamma_E)^{\mathrm{Z}(E)}.$$

Let $\mathbf{h}, \mathbf{e}, \mathbf{f}$ be a standard triple, i.e.,

$$[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}, \quad [\mathbf{e}, \mathbf{f}] = \mathbf{h}.$$

The eigenvalue bound on $\epsilon_{\mathfrak{s}} = \epsilon_{\mathfrak{s},\mathrm{I}}$ on $\mathcal{V}_{\mathfrak{s}\times E,\mathcal{O}}$ is implied by

- the following eigenvalue bound of $\epsilon_{E,\mathbf{h}}$ on $\mathcal{V}_{E,\mathbf{e}}$, and
- an (rather easy) eigenvalue bound of $\epsilon_{\mathfrak{s}^{\mathbf{f}},\mathrm{I}-\mathrm{ad}(\mathbf{h}/2)}$ on $C^{-\xi}(\mathfrak{s}^{\mathbf{f}};\{0\})$.

Proposition: Assume that $\mathbf{e}$ is distinguished. Then the vector field $\epsilon_{E,\mathbf{h}}$ acts semisimply on $\mathcal{V}_{E,\mathbf{e}}$, and all its eigenvalues are nonnegative integers.

The proof of the final proposition depends on

- classification of distinguished nilpotent orbits;
- rigidity properties of certain Weyl algebra modules $C^{-\xi}(F; Z_1, Z_2)$; in particular,
- an explicit basis of $C^{-\xi}(F; F', F')$, where $\dim_{\mathbb{R}} F = 2r$ and F' is a totally isotropic subspace of dimension r.

11 Summary of approach

- Reduction to the null sets, analogous to Harish-Chandra's method of descent.
- Reduction for nondistiguished nilpotent orbits, by tranversality reasons.
- Reduction for distinguished nilpotent orbits, by rigidity and representation theoretical reasons (use of $\mathfrak{sl}_2$ triple).

Similar (in spirit) to

- Harish-Chandra's proof of regularity of invariant eigendistributions.

  (C.f. work of Atiyah-Schmid, Varadarajan, Wallach)

- AGRS's proof of p-adic multiplicity one theorems.

Thank you !