Complementary Series of Split Real Groups

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joint with Annegret Paul and Susana Salamanca-Riba

(some of the techniques used are joint work with D. Barbasch)

\[\text{CS}(\text{Mp}(6), \delta^{2,1}) = \text{CS}(\text{SO}(4,3), \delta^{2,1}) \]

\[\text{CS}(\text{SO}_0(3,2),1) \times \text{CS}(\text{SO}_0(2,1),1) = \]

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Introduction

Aim

Discuss the unitarity of minimal principal series of $Mp(2n)$ and $SO(n + 1, n)$.

- Genuine complementary series of $Mp(2n)$
- Complementary series of $SO(n+1,n)$
- Union of spherical complementary series of certain orthogonal groups
PART 1

Genuine Complementary Series of $Mp(2n)$

- Genuine complementary series of $Mp(2n)$
- Complementary series of $SO(n+1,n)$
- Union of spherical complementary series of certain orthogonal groups

FIRST
• $G := Mp(2n)$ the connected double cover of $Sp(2n, \mathbb{R})$

• $K := \tilde{U}(n)$ the maximal compact subgroup of $G$
  $= \{ [g, z] \in U(n) \times U(1): \det(g) = z^2 \}$

• $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$

• $a_0 :=$ maximal abelian subspace of $\mathfrak{p}_0$

• $M := Z_K(a_0)$

• $\Delta(g_0, a_0) = \{ \pm \epsilon_k \pm \epsilon_l \}_{k,l=1...n} \cup \{ \pm 2\epsilon_k \}_{k=1...n} \text{ type } C_n$

• $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \text{ all permutations and sign changes}$
The group $M$ and its genuine representations

$M = Z_K(a_0)$ subgroup of $K$ generated by the elements $m_k = \begin{bmatrix} \text{diag}(1, \ldots, 1, -1, 1, \ldots, 1), i \end{bmatrix}$, $k = 1 \ldots n$ (of order 4)

Genuine $M$-types Irreducible repr.s $\delta$ of $M$ s.t. $\delta([I, -1]) \neq +1$.

$\uparrow$

Subsets $S \subset \{1 \ldots n\}$ $m_k^2 = [I, -1] \rightarrow$ each generator $m_k$ acts by $\pm i$

$S$ keeps track of which generators act by $-i$

$\delta_S(m_k) = \begin{cases} -i & \text{if } k \in S \\ +i & \text{otherwise} \end{cases}$

$\begin{array}{c|c|c|c|c} Mp(6) & m_1 & m_2 & m_3 \\ \hline \delta_{\{2,3\}} & +i & -i & -i \end{array}$
An action of the Weyl group on genuine $M$-types

$W$ acts on $\hat{M} \leftarrow (s_\alpha \cdot \delta)(m) := \delta(\sigma_\alpha^{-1}m\sigma_\alpha) \quad \forall m \in M, \forall \alpha \in \Delta$

The stabilizer of $\delta$ in $W$ is \[ W^\delta := \{w \in W : w \cdot \delta \simeq \delta \}. \]

For all $S \subset \{1, \ldots, n\}$, set $q = |S|$, $p = |S^c|$.

- $W^\delta_S \simeq W(C_p) \times W(C_q) \leftarrow s_{2\epsilon_k} \& s_{\epsilon_{k \pm \epsilon_l}}, k, l \in S$ or $S^C$
- $W \cdot \delta_S = \{\delta_T : |T| = q, |T^c| = p\}$

$W$-orbits of genuine $M$-types $\leftrightarrow$ pairs $(p, q)$: $p, q \in \mathbb{N}, p + q = n$

Pick representatives $\delta^{p,q} := \delta_{\{p+1, \ldots, n\}}$. $\delta^{p,q}(m_k) = \begin{cases} +i & \text{if } k \leq p \\ -i & \text{if } k > p. \end{cases}$
The group $K$ and its genuine representations

Maximal compact subgroup of $G$: \[ K = \tilde{U}(n) \]

Genuine $K$-types parameterized by highest weight $(a_1, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_j \in \mathbb{Z} + \frac{1}{2}, \forall j$

<table>
<thead>
<tr>
<th>fine $K$-types</th>
<th>highest weight</th>
<th>restriction to $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^p(\mathbb{C}^n) \otimes \det^{-1/2}$</td>
<td>$\left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}\right)$</td>
<td>$W \cdot \delta^{p,q}$</td>
</tr>
</tbody>
</table>

- If we restrict a fine $K$-type to $M$, we get one full $W$-orbit in $\widehat{M}$
- Each genuine $M$-type $\delta$ is contained in a unique fine $K$-type $\mu_{\delta}$. 

Genuine Complementary Series of $Mp(2n)$

- $MA :=$ Levi factor of a minimal parabolic
- $\delta :=$ genuine irreducible representation of $M$
- $\nu :=$ real character of $A$
- $P = MAN :=$ a minimal parabolic making $\nu$ weakly dominant.

**Minimal Principal Series**

$I_P(\delta, \nu) := \text{Ind}^G_P (\delta \otimes \nu \otimes 1)$

**Langlands Quotient**

$J(\delta, \nu) :=$ composition factor of $I_P(\delta, \nu) \supseteq \mu_\delta$

**$\delta$-Complementary Series**

$CS(G, \delta) := \{ \nu \in a^*_R \mid J(\delta, \nu) \text{ is unitary} \}$

**Problem:** Find $CS(Mp(2n), \delta^{p,q})$
**Theorem 1:** For all $\nu \in a_*^R$, write $\nu := (\nu^p|\nu^q)$. The map:

$$CS(Mp(2n), \delta^{p,q}) \rightarrow CS(SO(p+1,p)_0, 1) \times CS(SO(q+1,q)_0, 1)$$

$$\nu \mapsto (\nu^p, \nu^q)$$

is a well defined injection. (1 denotes the trivial $M$-type)

Spherical complementary series of real split orthogonal groups are known (Barbasch). Hence this theorem provides explicit necessary conditions for the unitarity of genuine principal series of $Mp(2n)$.
Example: $CS(Mp(6), \delta^{2,1}) \rightarrow CS(SO(3, 2)_0, 1) \times CS(SO(2, 1)_0, 1)$

$⇒ CS(Mp(6), \delta^{2,1})$ embeds into:
A reformulation of THEOREM 1

For all $p, q \in \mathbb{N}$ s.t. $p + q = n$, set:

$$G_{\delta}^{p,q} \equiv SO(p + 1, p)_0 \times SO(q + 1, q)_0$$

and note that $W(G_{\delta}^{p,q}) = W_{\delta}^{p,q}$.

$G_{\delta}^{p,q} :=$ connected real split group whose root system is dual to the system of good roots for $\delta_{p,q}$.

**Theorem 1:** The $\delta_{p,q}$-complementary series of $Mp(2n)$ embeds into the spherical complementary series of $G_{\delta}^{p,q}$.

**Proof:** based on Barbasch’s idea to use calculations on petite K-types to compare unitary parameters for different groups.
Comparing unitary parameters for $Mp(2n)$ and $G^{\delta_{p,q}}$

$J(\delta_{p,q}, \nu)$ unitary for $Mp(2n)$

$\uparrow$

$T(\mu, \delta_{p,q}, \nu)$

pos. semidefinite

$\forall \mu \in \hat{K}$

$J(1, \nu)$ unitary for $G^{\delta_{p,q}}$

$\uparrow$

$A(\psi, 1, \nu)$

pos. semidefinite

$\forall \psi \in \hat{W}^{\delta_{p,q}}$

relevant
A matching of operators

Key Proposition:

\[ \forall \text{ relevant } W^{\delta_{p,q}}\text{-type } \psi, \exists \text{ a “petite” } K\text{-type } \mu \text{ s.t.} \]
\[ T(\mu, \delta_{p,q}, \nu) = A(\psi, 1, \nu) \]

operator for \( M_p(2n) \) \hspace{1cm} operator for \( G^{\delta_{p,q}} \)

Sketch of the proof:

- \( T(\mu, \delta_{p,q}, \nu) \) is defined on \( \text{Hom}_M(\mu, \delta_{p,q}) \)
- This space carries a representation \( \psi_\mu \) of \( W^{\delta_{p,q}} \leftrightarrow W(G^{\delta_{p,q}}) \)
- Attached to \( \psi_\mu \), \( \exists \) a spherical operator \( A(\psi_\mu, 1, \nu) \) for \( G^{\delta_{p,q}} \)
- If \( \mu \) is petite, \( T(\mu, \delta_{p,q}, \nu) = A(\psi_\mu, 1, \nu) \)
- For all \( \psi \in W^{\delta_{p,q}} \) relevant, \( \exists \mu \in \hat{K} \) petite such that \( \psi = \psi_\mu. \square \)
A matching of relevant $W^\delta_{p,q}$-types with petite $K$-types

<p>| | |</p>
<table>
<thead>
<tr>
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</table>
| $((p - s) \times (s)) \otimes \text{triv}$ | \[
\begin{pmatrix}
\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2}, \ldots, -\frac{3}{2}
\end{pmatrix}
\]
|                           | \[
\begin{pmatrix}
p-s & q & s
\end{pmatrix}
\]
| $\quad (p - s, s) \otimes \text{triv}$ | \[
\begin{pmatrix}
\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}
\end{pmatrix}
\]
|                           | \[
\begin{pmatrix}
s & p-2s & q+s
\end{pmatrix}
\]
| $\quad \text{triv} \otimes ((q - r) \times (r))$ | \[
\begin{pmatrix}
\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}
\end{pmatrix}
\]
|                           | \[
\begin{pmatrix}
r & p & q-r
\end{pmatrix}
\]
| $\quad \text{triv} \otimes (q - r, r)$ | \[
\begin{pmatrix}
\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2}, \ldots, -\frac{3}{2}
\end{pmatrix}
\]
|                           | \[
\begin{pmatrix}
p+r & q-2r & r
\end{pmatrix}
\]
\( J(\delta^{p,q}, \nu) \) unitary for \( Mp(2n) \)

\[ T(\mu, \delta^{p,q}, \nu) \]

pos. semidefinite

\[ \forall \mu \in \hat{K} \]

\[ \downarrow \]

\[ T(\mu, \delta^{p,q}, \nu) \]

pos. semidefinite

\[ \forall \mu \in \hat{K} \text{ petite} \]

\( J(1, \nu) \) unitary for \( G'\delta^{p,q} \)

\[ A(\psi, 1, \nu) \]

pos. semidefinite

\[ \forall \psi \in \hat{W}^{\delta^{p,q}} \]

\[ \downarrow \]

\[ A(\psi, 1, \nu) \]

pos. semidefinite

\[ \forall \psi \in \hat{W}^{\delta^{p,q}} \text{ relevant} \]

\( \forall \psi \in \hat{W}^{\delta^{p,q}} \text{ relevant}, \exists \mu \in \hat{K} \text{ petite s.t. } A(\psi, 1, \nu) = T(\mu, \delta^{p,q}, \nu) \)
Let $G^{\delta_{p,q}} = SO(p + 1, p)_0 \times SO(q + q, q)_0$. For all $\nu = (\nu^p|\nu^q)$:

$$J(\delta_{p,q}, \nu) \text{ unitary for } Mp(2n) \implies J(1, \nu) \text{ unitary for } G^{\delta_{p,q}}.$$  

The spherical unitary dual of split orthogonal groups is known. So we get **non-unitarity certificates** for genuine L.Q.s of $Mp(2n)$.

**Theorem 1’**: If

- the spherical L.Q. $J(1, \nu^p)$ of $SO(p + 1, p)_0$ is **not unitary**, or
- the spherical L.Q. $J(1, \nu^q)$ of $SO(q + 1, q)_0$ is **not unitary**

then the genuine L.Q. $J(\delta_{p,q}, (\nu^p|\nu^q))$ of $Mp(2n)$ is also **not unitary**.
Let $\nu = (\nu_1, \ldots, \nu_n)$. We may assume:

$$\nu_1 \geq \cdots \geq \nu_p \geq 0 \quad \text{and} \quad \nu_{p+1} \geq \cdots \geq \nu_n \geq 0,$$

by $W^{\delta_p,q}$-invariance. (Recall $W^{\delta_p,q} = W(C_p) \times W(C_q)$.)

If any of the following conditions holds:

- $\nu_p > 1/2$
- $\nu_n > 1/2$
- $\nu_a - \nu_{a+1} > 1$, for some $a$ with $1 \leq a \leq p - 1$, or
- $\nu_a - \nu_{a+1} > 1$, for some $a$ with $p + 1 \leq a \leq n - 1$

then the genuine Langlands quotient $J(\delta_p,q, \nu)$ of $Mp(2n)$ is not unitary.
This non-unitarity certificate is a key ingredient in the classification of the $\omega$-regular unitary dual of $Mp(2n)$.

**Definition:** A representation of $Mp(2n)$ is called $\omega$-regular if its infinitesimal character is at least as regular as the one of the oscillator representation.

**Corollary:** The only $\omega$-regular complementary series repr.s of $Mp(2n)$ are the two even oscillator representations:

$$J \left( \delta_{0,n}, (n - \frac{1}{2}, \ldots, \frac{3}{2}, \frac{1}{2}) \right) \quad \text{and} \quad J \left( \delta_{n,0}, (n - \frac{1}{2}, \ldots, \frac{3}{2}, \frac{1}{2}) \right).$$
Complementary Series of $SO(n + 1, n)$

- Genuine complementary series of $Mp(2n)$
  \[ \subseteq \]
  Theor. 1

- Complementary series of $SO(n+1,n)$
  \[ \Leftarrow \]

- Union of spherical complementary series of certain orthogonal groups
  \[ \Rightarrow \]
  NEXT
NOTATION

- \(G := SO(n + 1, n)\)
- \(K := S(O(n + 1) \times O(n))\) maximal compact
- \(\Delta(g_0, a_0) = \{\pm \epsilon_k \pm \epsilon_l\} \cup \{\pm \epsilon_k\}\) type \(B_n\) ← dual to previous case
- \(W \cong S_n \times (\mathbb{Z}/2\mathbb{Z})^n\) ← same Weyl group as before
- \(M := Z_K(a_0) = \{\text{diag}(1, t_n, \ldots, t_1, t_1, \ldots, t_n) : t_j = \pm 1, \forall j\}\)
$M$-types

$M$ is generated by the elements

$$m_k = \text{diag}(1, \ldots, 1, \ -1 \ , 1, \ldots, 1, \ -1 \ , 1, \ldots, 1)$$

$k = 1 \ldots n$ (of order 2).

$M$-types $\Leftrightarrow$ Subsets $S \subset \{1 \ldots n\}$

same parametrization as before

The set $S$ keeps track of which generators act by $-1$:

$$\delta_S(m_k) = \begin{cases} 
-1 & \text{if } k \in S \\
+1 & \text{otherwise} 
\end{cases}$$

<table>
<thead>
<tr>
<th>$SO(4, 3)$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{{2,3}}$</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Just like before, we look at the action of $W$ on $\hat{M}$. Then

- $W^{\delta_S} \simeq W(B_p) \times W(B_q)$, for $q = |S|$, $p = |S^c|$ \textcolor{green}{\leftarrow same as before}

- $W \cdot \delta_S = \{\delta_T : |T| = q, |T^c| = p\}$

- $W$-orbits of $M$-types \iff pairs $(p, q) : p, q \in \mathbb{N}, p + q = n$

\[ \uparrow \textcolor{green}{\text{same parametrization as before}} \]

Pick representatives $\delta^{p,q} := \delta_{\{p+1, \ldots, n\}}$. $\delta^{p,q}(m_k) = \begin{cases} +1 & \text{if } k \leq p \\ -1 & \text{if } k > p. \end{cases}$
**$K$-types ($n$ even)**

\[
K = S(O(n+1) \times O(n)), \ n \text{ even}
\]

\[
(a_1, \ldots, a_{n \over 2}; b_1, \ldots, b_{n \over 2}) \text{ with } a_j, b_j \in \mathbb{Z}, \ \forall j \text{ and }
\]

\[
K\text{-types} \hspace{1cm} a_1 \geq \cdots \geq a_{n \over 2} \geq 0; \ b_1 \geq \cdots \geq b_{n \over 2} \geq 0.
\]

If $b_{n \over 2} = 0$, there is also a sign $\epsilon = \pm 1$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>Fine $K$-types</th>
<th>realization</th>
<th>res. to $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q &lt; \frac{n}{2}$</td>
<td>$(0, \ldots, 0; 1, \ldots, 1, 0, \ldots, 0;+)$</td>
<td>$\text{triv} \otimes \Lambda^q \mathbb{C}^n$</td>
<td>$W \cdot \delta^{p,q}$</td>
</tr>
<tr>
<td>$q = \frac{n}{2}$</td>
<td>$(0, \ldots, 0; 1, \ldots, 1)$</td>
<td>$\text{triv} \otimes \Lambda^{n \over 2} \mathbb{C}^n$</td>
<td>$W \cdot \delta^{p,q}$</td>
</tr>
<tr>
<td>$q &gt; \frac{n}{2}$</td>
<td>$(0, \ldots, 0; 1, \ldots, 1, 0, \ldots, 0;-)_{n-q}$</td>
<td>$\text{triv} \otimes \Lambda^q \mathbb{C}^n$</td>
<td>$W \cdot \delta^{p,q}$</td>
</tr>
</tbody>
</table>
**K-types \((n \text{ odd})\)**

\[
K = S(O(n + 1) \times O(n)), \quad n \text{ odd}
\]

\[
(a_1, \ldots, a_{\frac{n+1}{2}}; b_1, \ldots, b_{\frac{n-1}{2}}) \text{ with } a_j, b_j \in \mathbb{Z}, \forall j \text{ and}
\]

\[
a_1 \geq \cdots \geq a_{\frac{n+1}{2}} \geq 0; \quad b_1 \geq \cdots \geq b_{\frac{n-1}{2}} \geq 0.
\]

*If \(a_{\frac{n+1}{2}} = 0\), there is also a sign \(\epsilon = \pm 1\).*

<table>
<thead>
<tr>
<th>(q &lt; \frac{n}{2})</th>
<th>(q &gt; \frac{n}{2})</th>
<th>realization</th>
<th>res. to (M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \ldots, 0; 1, \ldots, 1, 0, \ldots, 0; +))</td>
<td>((0, \ldots, 0; 1, \ldots, 1, 0, \ldots, 0; -))</td>
<td>(\text{triv} \otimes \Lambda^q \mathbb{C}^n)</td>
<td>(W \cdot \delta^{p,q})</td>
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Complementary Series of $SO(n + 1, n)$

- $MA$: Levi factor of a minimal parabolic
- $\delta \in \widehat{M}$
- $\nu \in a^*_R$
- $P = MAN$: a minimal parabolic making $\nu$ weakly dominant.

**Minimal Principal Series** $I_P(\delta, \nu)$

**Langlands Quotient** $J(\delta, \nu)$

**$\delta$-Complementary Series** $CS(SO(n + 1, n), \delta) = \{ \nu | J(\delta, \nu) \text{ unitary} \}$

**Problem**: Find $CS(SO(n + 1, n), \delta^{p,q})$
Theorem 2: For all $\nu \in \mathfrak{a}_R^*$, write $\nu := (\nu^p|\nu^q)$. The map:

$$CS(SO(n + 1, n), \delta^{p,q}) \to CS(SO(p + 1, p)_0, 1) \times CS(SO(q + 1, q)_0, 1)$$

$$\nu \mapsto (\nu^p, \nu^q)$$

is a well defined injection. (1 denotes the trivial M-type.)
A reformulation of THEOREM 2

Set: \( G^{\delta_{p,q}} \equiv SO(p + 1, p)_0 \times SO(q + 1, q)_0 \)

and note that \( W(G^{\delta_{p,q}}) = W^{\delta_{p,q}} \).

\( G^{\delta_{p,q}} \) := connected real split group whose root system is equal to the system of good roots for \( \delta_{p,q} \).

Theorem 2: The \( \delta_{p,q} \)-complementary series of \( SO(n + 1, n) \) embeds into the spherical complementary series of \( G^{\delta_{p,q}} \).

Proof: based on a matching of relevant \( W \)-types for \( G^{\delta_{p,q}} \) with petite \( K \)-types for \( SO(n + 1, n) \).
A matching of relevant $W^{\delta_{p,q}}$-types with petite $K$-types

Recall that $W^{\delta_{p,q}} = W(B_p) \times W(B_q)$ and $K = S(O(n+1) \times O(n))$.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$((p - s) \times (s)) \otimes \text{triv}$</td>
<td>$\Lambda^s(\mathbb{C}^{n+1}) \otimes \Lambda^{q+s}(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$(p - s, s) \otimes \text{triv}$</td>
<td>an irreducible submodule of $\text{triv} \otimes [\Lambda^s(\mathbb{C}^n) \otimes \Lambda^{q+s}(\mathbb{C}^n)]$</td>
</tr>
<tr>
<td>$\text{triv} \otimes ((q - r) \times (r))$</td>
<td>$\Lambda^r(\mathbb{C}^{n+1}) \otimes \Lambda^{q-r}(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$\text{triv} \otimes (q - r, r)$</td>
<td>an irreducible submodule of $\text{triv} \otimes [\Lambda^r(\mathbb{C}^n) \otimes \Lambda^{q-r}(\mathbb{C}^n)]$</td>
</tr>
</tbody>
</table>
PART 3

An example: $n = 3$

Genuine complementary series of $Mp(2n)$ ⊆ Theor. 1

Complementary series of $SO(n+1,n)$ ⊇ Theor. 2

Union of spherical complementary series of certain orthogonal groups
Are these “proper containments” or “equalities”?

Are the L.Q.s $J_{Mp(6)}(δ^{2,1}, ν)$ and $J_{SO(4,3)}(δ^{2,1}, ν)$ unitary for all points $ν$ of the unit cube and all points $ν$ of the 8 line segments?
Unitarity of $J_{Mp(6)}(\delta^{2,1}, \nu)$ for $\nu$ in the unit cube

**Theorem.** The Langlands quotient $J(\delta, \nu)$ of $Mp(2n)$ is unitary for all $\nu$ in the unit cube $\{x \in a_\mathbb{R}^* | 0 \leq |x_j| \leq 1/2, \ \forall \ j\}$.

**Proof.** Note that:

- For $\nu = 0$, all the operators $T(\mu, \delta, \nu)$ are positive definite.
- The signature of $T(\mu, \delta, \nu)$ can only change along the reducibility hyperplanes:
  \[
  \begin{cases}
  \langle \nu, \beta \rangle \in 2\mathbb{Z} + 1 & \text{for some root } \beta \text{ which is good for } \delta \\
  \langle \nu, \beta \rangle \in 2\mathbb{Z} \setminus \{0\} & \text{for a root } \beta \text{ which is bad for } \delta.
  \end{cases}
  \]
- Away from these hyperplanes, $I(\delta, \nu)$ is irreducible ($= J(\delta, \nu)$), and the operators $T(\mu, \delta, \nu)$ have constant signature. In particular, $J(\delta, \nu)$ is unitary throughout the unit cube. $\square$
Theorem. The repr. \( J(\delta^{p,q}, \nu) \) of \( Mp(2n) \) is unitary \( \forall \nu=(\nu^p | \nu^q) \) s.t.

- \( \nu^p \in CS(SO(p+1, p)_0, 1) \), with \( 0 \leq |a_j| \leq 3/2 \) or \( a_j \in \mathbb{Z} + \frac{1}{2} \)
- \( \nu^q \in CS(SO(q+1, q)_0, 1) \), with \( 0 \leq |a_j| \leq \frac{1}{2} \).

Proof. Let \( P_1 \) be a parabolic with \( M_1 A_1 := Mp(2p) \times (\widetilde{GL}(1, \mathbb{R}))^q \).

By double induction, \( J(\delta^{p,q}, \nu) \) is the Langlands quotient of

\[
I(\nu^q) := \text{Ind}_{M_1 A_1 N_1}^{Mp(2n)} \left( (J(\delta^{p,0}, \nu^p) \otimes \delta^{0,q}) \otimes \nu^q \otimes 1 \right).
\]

Here \( J(\delta^{p,0}, \nu^p) \) is a pseudopshерical repr. of \( Mp(2p) \). By results of ABPTV, \( J(\delta^{p,0}, \nu^p) \) is unitary \( \forall \nu^p \in CS(SO(p+1p)_0, 1) \). Then the repr. \( I(\nu^q) \) of \( Mp(2n) \) is unitary at \( \nu^q=0 \) (unitarily induced). For all \( \nu \) of interest, \( I(\nu^q) \) is irreducible, hence it stays unitary by the principle of unitary deformation. \( \square \)
Corollary

\[ CS(\text{Mp}(6), \delta^{2,1}) = CS(\text{SO}(4,3), \delta^{2,1}) = CS(\text{SO}_0(3,2),1) \times CS(\text{SO}_0(2,1),1) \]
More generally...

For all $n \leq 4$ and for all $\delta = \delta^{p,q}$, the following equalities hold:

$$\text{CS}(\text{Mp}(2n), \delta^{p,q}) = \text{CS}(\text{SO}(n+1,n), \delta^{p,q})$$

$$= \text{CS}(\text{SO}_0(p+1,p),1) \times \text{CS}(\text{SO}_0(q+1,q),1)$$
A natural conjecture

Equalities hold for all $n$ and all choices of $\delta^{p,q}$

$$\text{CS}(\text{Mp}(2n), \delta^{p,q}) = \text{CS}(\text{SO}(n+1,n), \delta^{p,q})$$

$$\text{CS}(\text{SO}_o(p+1,p),1) \times \text{CS}(\text{SO}_o(q+1,q),1)$$

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Remark. We may assume $p \geq q$, because

- $J_{Mp(2n)}(\delta^{p,q}, (\nu^p|\nu^q)) = J_{Mp(2n)}(\delta^{q,p}, (\nu^p|\nu^q))^*$
- $J_{SO(n+1,n)}(\delta^{p,q}, (\nu^p|\nu^q)) = J_{SO(n+1,n)}(\delta^{q,p}, (\nu^q|\nu^p)) \otimes \chi$
  ($\chi$ = a unitary character).
(More) evidence for these conjectures

- **The case \((p, q) = (n, 0)\)**

  If \((p, q) = (n, 0)\), the conjectures hold for all \(n\). This is the pseudospherical case for \(Mp(2n)\) and the spherical case for \(SO(n + 1, n)\). (For \(Mp(2n)\), the result is due to ABPTV; for \(SO(n + 1, n)\), it is an empty statement.)

- **A large family of examples**

  Assume \(p > q\). The conjectures hold for all \(\nu = (\rho^p | \nu^q)\) with

  \[ \star \quad \rho^p = \left( p - \frac{1}{2}, p - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \right) = \text{the infinitesimal character of the trivial representation of } SO(p + 1, p)_0, \]

  \[ \star \quad \nu^q \in CS(SO(q + 1, q)_0, 1). \]
PART 5

\[
\text{CS}(\text{Mp}(2n), \delta^{p,q}) \subseteq \text{Theor. 1} \subseteq \text{Conj. 1} \subseteq \text{CS}(\text{SO}(n+1,n), \delta^{p,q})
\]

\[
\text{CS}(\text{SO}(p+1,p),1) \times \text{CS}(\text{SO}(q+1,q),1)
\]
Conjecture 3

For all $n$ and all choices of $\delta^{p,q}$:

$$CS(M_p(2n), \delta^{p,q}) = CS(SO(n + 1, n), \delta^{p,q}).$$

- Conjecture 3 is true for $n = 2, 3, \text{ and } 4$.

- Conjecture 3 is independent of Conjectures 1 and 2.
Consider $G = Sp(2n, \mathbb{R})$, $G' = O(m + 1, m) \subset Sp(2n(2m + 1), \mathbb{R})$. Let $\tilde{G}$ and $\tilde{G}'$ be their preimages in $Mp(2n(2m + 1))$:

$$\tilde{G} = Mp(2n) \quad \tilde{G}' = \tilde{O}(m + 1, m) \text{ linear cover.}$$

- $(G, G')$ is a dual pair in $Sp(2n(2m + 1), \mathbb{R})$ (mutual centralizers)
- The $\theta$-correspondence gives a bijection between certain genuine irreducible representations of $\tilde{G}$ and $\tilde{G}'$.

We can re-interpret this correspondence as a map:

$$\pi \in \widehat{Mp(2n)}_{gen} \leftrightarrow \pi' \in \widehat{SO(m + 1, m)}.$$
Some results of Adams, Barbasch and Li

For all \( k \geq 0 \), let \( \rho_k = (k - \frac{1}{2}, \ldots, \frac{1}{2}) \). The \( \theta \)-correspondence maps:

\[
J_{Mp(2n)}(\delta^{p,q}, \nu) \to J_{SO(n+k+1,n+k)}(\delta^{p+k,q}, (\rho_k | \nu))
\]

\[
J_{Mp(2n+2k)}(\delta^{p+k,q}, (\rho_k | \nu)) \leftarrow J_{SO(n+1,n)}(\delta^{p,q}, \nu)
\]

for all \( p \geq q \).

If \( k \geq n + 1 \), both arrows preserve unitarity. (Stable Range)

**Remark:** If \( k = 0 \), the correspondence

\[
J_{Mp(2n)}(\delta^{p,q}, \nu) \leftrightarrow J_{SO(n+1,n)}(\delta^{p,q}, \nu)
\]

is not known to preserve unitarity.

**Conj.3** \( J_{Mp(2n)}(\delta^{p,q}, \nu) \) unitary \( \iff \) \( J_{SO(n+1,n)}(\delta^{p,q}, \nu) \) unitary
Theorem 3: Conjecture 3 holds in each of the following cases:

(i) Conj.s $A_1$ & $A_2$ hold  
(ii) Conj.s $A_1$ & $B_1$ hold  
(iii) Conj.s $A_2$ & $B_2$ hold  
(iv) Conj.s $B_1$ & $B_2$ hold.

\[
\begin{align*}
  (\rho_{n+2}|\nu) &\in CS(Mp(4n + 4), \delta^{p+n+2,q}) \\
  \nu &\in CS(Mp(2n), \delta^{p,q}) \\
  (\rho_{n+2}|\nu) &\in CS(SO(2n + 3, 2n + 2), \delta^{p+n+2,q}) \\
  \nu &\in CS(SO(n + 1, n), \delta^{p,q})
\end{align*}
\]
THEOREM 3 (a sketch of the proof)

The idea of the proof is similar to the one in ABPTV. We show that:

\[ J_{Mp(2n)}(\delta^{p,q},\nu) \text{ unitary} \]

Conj. 3

\[ J_{SO(n+1,n)}(\delta^{p,q},\nu) \text{ unitary} \]

Conj. B2 or A1

Conj. B1 or A2

Key ingredients:

- Results on \( \theta \)-correspondence (Adams, Barbasch, Li, Przebinda).
- Non-unitarity certificates for both \( Mp(2n) \) and \( SO(n + 1, n) \).
$J_{Mp(2n)}(\delta^{p,q},\nu)$ unitary $\Rightarrow$ Conj. B2 $J_{SO(n+1,n)}(\delta^{p,q},\nu)$ unitary

$J_{SO(2n+3,2n+2)}(\delta^{p+n+2,q},(\rho^{n+2}|\nu))$ unit.

Stable range

Conj. B2

$J_{Mp(2n)}(\delta^{p,q},\nu)$ unit.

$J_{SO(n+1,n)}(\delta^{p,q},\nu)$ unit.
\( J_{cMp(2n)}(\delta^{p,q}, \nu) \) unitary $\iff$ Conj. B1

\( J_{SO(n+1,n)}(\delta^{p,q}, \nu) \) unitary

\( J_{Mp(2n)}(\delta^{p,q}, \nu) \) unit.

\( J_{SO(2n+3,2n+2)}(\delta^{p+n+2,q}, (\rho^{n+2}|\nu)) \) unit.

by non-unitary certificates for Mp(2n)

by non-unitary certificates for SO(n+1,n)

by non-unitary certificates for SO(n+1,n)

Low Rank (Howe)
Conclusions

- Conj. 1 ⇒ Conj. A.
- Conj. 2 ⇒ Conj. B.

If either Conj. 1 (alone) or Conj. 2 (alone) holds, then the 3 parameter sets are all equal.