Fractional calculus of Weyl algebra and its applications

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§ **Introduction** global property of special fucntions

Character : Weyl's dimension formula

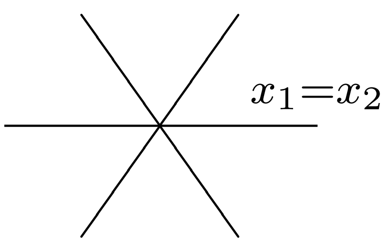
Zonal spherical functions : Gindikin-Karpelevic formula (c-function)

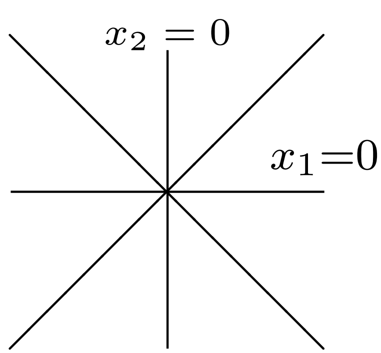
Heckman-Opdam's HG : Gauss' summation formula (by Opdam)

Commuting family of differential operators

$$L(k)_{A_{n-1}} := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{1 \le i < j \le n} 2k \coth(x_i - x_j)\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big)$$

$$L(k)_{B_n} := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{k=1}^{n}(2k_2 \coth x_k + 4k_3 \coth 2x_k)\frac{\partial}{\partial x_k} + \sum_{1 \le i < j \le n} k_1 \left(\coth(x_i - x_j)\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big) + \coth(x_i + x_j)\Big(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\Big)\right)$$





$(A_2)_{x_1=x_2}$: ${}_3F_2(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2; x)$

$(BC_2)_{x_1=0}$: Even family (by Simpson)

$(BC_2)_{x_1=x_2}$: $\rightsquigarrow$ Dotsenko-Fateev eq.

§ Fuchsian differential equations

$$Pu = 0, \quad P := a_n(x)\tfrac{d^n}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\tfrac{d}{dx} + a_0(x)$$

$x = 0$: singularity of P

Normalized at 0: $a_n(0) = \cdots = a_n^{(n-1)}(0) = 0$, $a_n^{(n)}(0) \neq 0$

$x = 0$ is regular singularity $\Leftrightarrow$ (order of zero of $a_j(x)$ at 0) $\geq j$

$$P = \sum_{j=0}^{\infty} x^j p_j(\vartheta), \quad \vartheta := x\partial,\ \partial := \frac{d}{dx}$$

$p_0(s) = 0$: indicial equation

the roots λ_j $(j = 1, \ldots, n)$: characteristic exponents

$\lambda_i - \lambda_j \notin \mathbb{Z} \quad (i \neq j) \ \Rightarrow$

$\exists 1$ solution $u_j(x) = x_j^{\lambda}\phi_j(x)$, $\phi_j(x)$ is analytic at 0 and $\phi_j(0) = 1$

${}_nF_{n-1}(\alpha, \beta; z)$: $\{$exponents at $1\} = \{0, 1, \ldots, n-1, -\beta_n\}$

the local monodromy is generically $(\Leftarrow\ \beta_n \notin \mathbb{Z})$ semisimple

$\Rightarrow$ generalize "characteristic exponents"

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix}, \quad m = 0, 1, \ldots$$

$$n = m_1 + \cdots + m_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{C}$$

Def. P has generalized exponents $\{[\lambda_1]_{(m_1)}, \ldots, [\lambda_k]_{(m_k)}\}$ at $0 \overset{\text{def}}{\Longleftrightarrow}$

$\Lambda := \{\lambda_j + \nu\,;\, 0 \le \nu < m_j,\ j = 1, \ldots, k\}$: char. exponents at 0

- $\lambda_i - \lambda_j \notin \mathbb{Z}$ $(i \ne j)$ $\Rightarrow$

(Def. $\Leftrightarrow$ char. exp. are Λ and local monodromy is semisimple)

- $\lambda_1 = \cdots = \lambda_k$ $\Rightarrow$

(Def. $\Leftrightarrow$ char. exp. are Λ and Jordan normal form of the local monodromy type corresponds to the dual partition of $n = m_1 + \cdots + m_k$)

- $k = 1,\ \lambda_1 = 0$ $\Rightarrow$ (Def. $\Leftrightarrow$ $x = 0$: regular point)
- In general

$$\prod_{j=1}^{k} \prod_{0 \le \nu < m_j - \ell} (s - \lambda_j - \nu) \mid p_\ell(s) \quad (\ell = 0, \ldots, \max\{m_1, \ldots, m_k\} - 1)$$

Def. P has the generalized Riemann scheme (GRS)

$$P\left\{\begin{matrix} x=c_0=\infty & c_1 & \cdots & c_p & \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} & \\ \vdots & \vdots & \vdots & \vdots & ;\ x \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & \end{matrix}\right\}$$

$$\mathbf{m}=(\mathbf{m}_0,\dots,\mathbf{m}_p)=\big((m_{0,1},\dots,m_{0,n_0}),\dots,(m_{p,1},\dots,m_{p,n_p})\big)$$

: $(p+1)$-tuples of partitions of $n=\operatorname{ord}\mathbf{m}$

Fuchs condition (FC): $|\{\lambda_{\mathbf{m}}\}|:=\sum m_{j,\nu}\lambda_{j,\nu}-\operatorname{ord}\mathbf{m}+\frac{1}{2}\operatorname{idx}\mathbf{m}=0$

$\operatorname{idx}\mathbf{m}:=\sum_{j,\nu}m_{j,\nu}^2-(p-1)(\operatorname{ord}\mathbf{m})^2$ (index of rigidity, Katz)

Normal form of P: $\partial=\frac{d}{dx}$

$$P=\Big(\prod_{j=1}^{p}(x-c_j)^n\Big)\partial^n+a_{n-1}(x)\partial^{n-1}+\cdots+a_1(x)\partial+a_0(x)$$

(order of zeros of $a_\nu(x)$ at c_j) $\geq\nu$ and $\deg a_\nu(x)\leq n(p-1)+\nu$

$\mathbf{m}$: realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\, P$ with (GRS) for generic $\lambda_{j,\nu}$ under (FC)

$\mathbf{m}$: irreducibly realizable $\overset{\text{def}}{\Leftrightarrow}$ $\exists\, Pu = 0$ is irreducible for generic $\lambda_{j,\nu}$

Problem. Classify such $\mathbf{m}$! (Deligne-Simpson problem)

$\mathbf{m}$: monotone $\overset{\text{def}}{\Leftrightarrow}$ $m_{j,1} \ge m_{j,2} \ge m_{j,3} \ge \cdots$

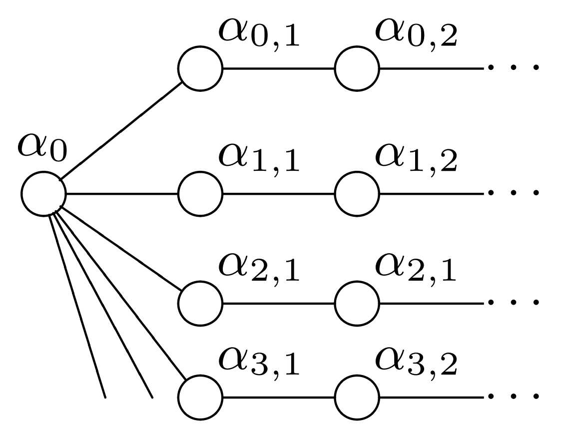
$\mathbf{m}$: indivisible $\overset{\text{def}}{\Leftrightarrow}$ $\gcd \mathbf{m} := \gcd\{m_{j,\nu}\} = 1$

$\mathbf{m}$: basic $\overset{\text{def}}{\Leftrightarrow}$ indivisible, monotone and $m_{j,1}+\cdots+m_{j,p} \le (p-1)\operatorname{ord}\mathbf{m}$

A Kac-Moody root system (Π, W)

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \ne j \quad \text{or} \quad |\mu-\nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu-\nu| = 1) \end{cases}$$



Δ_+^{re}: positive real roots $\qquad \Delta_+ = \Delta_+^{re} \cup \Delta_+^{im} \qquad (W\Delta_+^{re} = \Delta_+^{re} \cup \Delta_-^{re})$

Δ_+^{im}: positive imaginary roots $(k\Delta_+^{im} \subset \Delta_+^{im} = W\Delta_+^{im},\, k = 2, 3, \ldots)$

$\mathbf{m} \leftrightarrow \alpha_{\mathbf{m}} = (\operatorname{ord}\mathbf{m})\alpha_0 + \sum_{j\ge 0,\, k\ge 1}\sum_{\nu>k} m_{j,\nu}\alpha_{j,k}$ (Crawley-Boevey)

Fact. $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

$$\Delta_+^{im} = \{kw\alpha_{\mathbf{m}}\,;\, w\in W,\ k=1,2,\ldots\ \mathbf{m}:\text{basic}\}$$

Thm. $\{\mathbf{m}:\text{realizable}\} \leftrightarrow \{k\alpha;\ \alpha\in\Delta_+\ \ \operatorname{supp}\alpha\ni\alpha_0,\ k=1,2,\ldots\}$

Suppose $\mathbf{m}$ is realizable.

★ $\mathbf{m}$: irreducibly realizable $\Leftrightarrow$ $\mathbf{m}$ is indivisible or $\operatorname{idx}\mathbf{m}<0$

★ $\exists\, P_{\mathbf{m}}$: a universal model with (GRS) $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$

★ $P_{\mathbf{m}}$ is of the normal form with the coefficients $a_\nu(x)\in\mathbb{C}[x,\lambda_{j,\nu},g_i]$

★ $\forall\lambda_{j,\nu}$ under (FC), $\forall P$ with $\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$ are $P_{\mathbf{m}}$

★ $g_1,\ldots,g_N$: accessory parameters $N=\begin{cases}0 & (\operatorname{idx}\mathbf{m}>0)\\ \gcd\mathbf{m} & (\operatorname{idx}\mathbf{m}=0)\\ 1-\frac12\operatorname{idx}\mathbf{m} & (\operatorname{idx}\mathbf{m}<0)\end{cases}$

$$\frac{\partial^2 P_{\mathbf{m}}}{\partial^2 g_i}=0,\quad \operatorname{Top}(P_{\mathbf{m}})=x^{L_i}\partial^{K_i}\operatorname{Top}\Big(\frac{\partial P_{\mathbf{m}}}{\partial g_i}\Big)$$

$\{(L_i,K_i)\,;\, i=1,\ldots,N\}$ are explicitly given

$$Q=(c_kx_k+\cdots+c_0)\partial^m+a_{m-1}(x)\partial^{m-1}+\cdots+a_0(x),\ c_k\neq 0$$

$$\Rightarrow \operatorname{Top}Q=c_kx^k\partial^m$$

Def. $\mathbf{m}$ is rigid $\overset{\text{def}}{\Leftrightarrow}$ irreducibly realizable and $\operatorname{idx}\mathbf{m} = 2 \quad (\Rightarrow N = 0)$

(corresponds to $\alpha \in \Delta_+^{re}$ with $\operatorname{supp}\alpha \ni \alpha_0$)

Rigid tuples : 9 (ord ≤ 4), 306 (ord $= 10$), 19286 (ord $= 20$)

$\mathrm{ord} = 2$ $\quad 11, 11, 11$ (${}_2F_1$; Gauss)

$\mathrm{ord} = 3$ $\quad 111, 111, 21$ (${}_3F_2$) $\qquad 21, 21, 21, 21$(Pochhammer)

$\mathrm{ord} = 4$ $\quad 1^4, 1^4, 31$ (${}_4F_3$) $\qquad 1^4, 211, 22$ (Even family) $\quad 211, 211, 211$

$31, 31, 31, 31, 31$ (Pochhammer) $\qquad 211, 22, 31, 31 \qquad 22, 22, 22, 31$

Simpson's list 1991: $\quad 1^n, 1^n, n-11 \quad 1^n, [\frac{n}{2}][\frac{n-1}{2}]1, [\frac{n+1}{2}][\frac{n}{2}] \quad 1^6, 42, 2^3$

Remark. The existence of $P_{\mathbf{m}}$ for fixed rigid $\mathbf{m}$ and $\{\lambda_{j,\nu}\}$ was an open problem by N. Katz (Rigid Local Systems, 1995).

Reduction by "fractional calculus" $\Leftarrow W$ (Katz's middle convolution)

$\mathbf{m} \to$ trivial ($\Leftarrow$ $\mathbf{m}$: rigid) or basic

$\operatorname{idx}\mathbf{m} = 0 \ \to \ \tilde{D}_4$ ($\to$ Painlevé VI), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (4 types)

$\operatorname{idx}\mathbf{m} = -2 \ \to$ 13 types, etc...

§ Fractional calculus of Weyl algebra

Unified and computable interpretation ($\Rightarrow$ a computer program) of

Construction of equations

Integral representation of solutions

Congruences

Series expansion of solutions

Contiguity relations

Monodromy

Connection problem

Several variables (PDE)

$$W[x] := \langle x, \partial, \xi \rangle \otimes \mathbb{C}(\xi) \subset \overline{W}[x] := W[x] \otimes \mathbb{C}(x, \xi)$$

$$\simeq \overline{W}_L[x] := W[x] \otimes \mathbb{C}(\partial, \xi)$$

$\mathrm{R} : \overline{W}[x],\ \overline{W}_L[x] \to W[x]$ (reduced representative)

$\mathrm{L} : \partial_j \mapsto x_j,\ x_j \mapsto -\partial_j$

$$\mathrm{Ad}(f) \in \mathrm{Aut}(\overline{W}[x]),\ \partial_i \mapsto f(x,\xi) \circ \partial_i \circ f(x,\xi)^{-1} = \partial_i - \frac{f_i}{f}, \quad h_i = \frac{f_i}{f} \in \mathbb{C}(x,\xi)$$

$$\overline{\Delta}_+ := \{k\alpha\,;\ k = 1, 2, \ldots,\ \alpha \in \Delta_+\}$$

$$\begin{array}{ccc} \{P_{\mathbf{m}} : \text{Fuchsian differential operators}\} & \leftrightarrow & \overline{\Delta}_+ = \{\alpha_{\mathbf{m}}\} \\ \downarrow \text{Fractional operations} & & \downarrow W\text{-action} \\ \{P_{\mathbf{m}} : \text{Fuchsian differential operators}\} & \leftrightarrow & \overline{\Delta}_+ = \{\alpha_{\mathbf{m}}\} \end{array}$$

"W-action" for operators, series expansions and integral representations of solutions, contiguity relations, connection coefficients ,... are concretely determined.

Remark. On Fuchsian systems of Schlesinger canonical form

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - c_j} u$$

the W-action is given by Katz + Detteweiler-Reiter + Crawley-Boevey.

Example: Jordan-Pochhammer Eq. ($p = 2 \Rightarrow$ Gauss)

$p-11, p-11, \ldots, p-11$: $(p+1)$-tuple of partitions of p

$$P := \mathrm{RAd}\big(\partial^{-\mu}\big) \circ \mathrm{RAd}\Big(x^{\lambda_0}\prod_{j=1}^{p-1}(1-c_j x)^{\lambda_j}\Big)\partial$$

$$= \mathrm{RAd}(\partial^{-\mu}) \circ \mathrm{R}\Big(\partial - \frac{\lambda_0}{x} + \sum_{j=2}^{p-1}\frac{c_j\lambda_j}{1-c_j x}\Big)$$

$$= \partial^{-\mu+p-1}\Big(p_0(x)\partial + q(x)\Big)\partial^{\mu} = \sum_{k=0}^{p} p_k(x)\partial^{p-k}$$

$$p_0(x) = x\prod_{j=2}^{p-1}(1-c_j x) \qquad q(x) = p_0(x)\Big(-\frac{\lambda_0}{x} + \sum_{j=2}^{p-1}\frac{c_j\lambda_j}{1-c_j x}\Big)$$

$$p_k(x) = \binom{-\mu+p-1}{k}p_0^{(k)}(x) + \binom{-\mu+p-1}{k-1}q^{(k-1)}(x)$$

$$u(x) = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\lambda_0+1)\Gamma(\mu)} \int_0^x \left(t^{\lambda_0} \prod_{j=2}^{p-1} (1-c_j t)^{\lambda_j} \right) (x-t)^{\mu-1} dt$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_{p-1}=0}^{\infty} \frac{(\lambda_0+1)_{m_1+\cdots+m_{p-1}} (-\lambda_1)_{m_1} \cdots (-\lambda_{p-1})_{m_{p-1}}}{(\lambda_0+\mu+1)_{m_1+\cdots+m_{p-1}} m_1! \cdots m_{p-1}!}$$

$$c_2^{m_2} \cdots c_{p-1}^{m_{p-1}} x^{\lambda_0+\mu+m_1+\cdots+m_{p-1}}$$

$$P \left\{ \begin{matrix} x=0 & 1=\frac{1}{c_1} & \cdots & \frac{1}{c_{p-1}} & \infty \\ [0]_{(p-1)} & {\color{blue}[0]}_{(p-1)} & \cdots & [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ {\color{red}\lambda_0+\mu} & {\color{blue}\lambda_1+\mu} & \cdots & \lambda_{p-1}+\mu & -\lambda_1-\cdots-\lambda_{p-1}-\mu \end{matrix} \right\}$$

$$c({\color{red}\lambda_0+\mu} \rightsquigarrow {\color{blue}\lambda_1+\mu}) = \frac{\Gamma(\lambda_0+\mu+1)\Gamma(-\lambda_1-\mu)}{\Gamma(\lambda_0+1)\Gamma(-\lambda_1)} \prod_{j=2}^{p-1} (1-c_j)^{\lambda_j}$$

$$c({\color{red}\lambda_0+\mu} \rightsquigarrow {\color{blue}0}) = \frac{\Gamma(\lambda_0+\mu+1)}{\Gamma(\mu)\Gamma(\lambda_0+1)} \int_0^1 t^{\lambda_0}(1-t)^{\lambda_1+\mu-1} \prod_{j=2}^{p-1} (1-c_j t)^{\lambda_j} dt$$

Versal Pochhammer operator

$$p_0(x)=\prod_{j=1}^{p}(1-c_jx),\quad q(x)=\sum_{k=1}^{p}\lambda_k x^{k-1}\prod_{j=k+1}^{p}(1-c_jx)$$

$$P\left\{\begin{matrix} x=\frac{1}{c_j}\ (j=1,\dots,p) & \infty \\ [0]_{(p-1)} & [1-\mu]_{(p-1)} \\ \sum_{k=j}^{p}\frac{\lambda_k}{c_j\prod_{\substack{1\le\nu\le k\\ \nu\neq j}}(c_j-c_\nu)}+\mu & \sum_{k=1}^{p}\frac{(-1)^k\lambda_k}{c_1\dots c_k}-\mu \end{matrix}\right\}$$

$$u_C(x)=\int_C\left(\exp\int_0^t\sum_{j=1}^{p}\frac{-\lambda_j s^{j-1}}{\prod_{1\le\nu\le j}(1-c_\nu s)}ds\right)(x-t)^{\mu-1}dt$$

$p=2\Rightarrow$ Unifying Gauss + Kummer + Hermite-Weber

$$c_1=\cdots=c_p=0\Rightarrow u_C(x)=\int_{\infty}^{x}\exp\Big(-\sum_{j=1}^{p}\frac{\lambda_j t^j}{j!}\Big)(x-t)^{\mu-1}dt$$

Thm. $\mathbf{m}$: rigid monotone with $m_{0,n_0} = m_{1,n_1} = 1$, $\frac{1}{c_0} = 0$, $\frac{1}{c_1} = 1$

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\prod_{\nu=1}^{n_0-1} \Gamma\big(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\big) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)}{\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m} \\ m'_{0,n_0}=m''_{1,n_1}=1}} \Gamma\big(|\{\lambda_{\mathbf{m}'}\}|\big) \cdot \prod_{j=2}^{p-1} (1-c_j)^{L_j}}$$

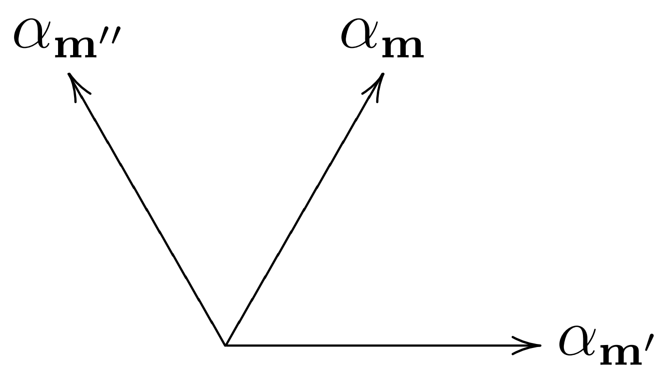
$$|\{\lambda_{\mathbf{m}'}\}| = \sum m'_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m}' + 1$$

$\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}'' \overset{\text{def}}{\Longleftrightarrow} \mathbf{m},\ \mathbf{m}'$ realizable and $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$

$$\text{Gauss}: \left\{\begin{matrix} x = \frac{1}{c_0} = 0 & \frac{1}{c_1} = 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \overline{\lambda_{0,2}} & \underline{\lambda_{1,2}} & \lambda_{2,2} \end{matrix}\right\} = \left\{\begin{matrix} x = 0 & 1 & \infty \\ 1-\gamma & \gamma-\alpha-\beta & \alpha \\ 0 & 0 & \beta \end{matrix}\right\} \quad \begin{matrix} 1\bar{1}, 1\underline{1}, 11 \\ = 0\bar{1}, 10, 10 \\ \oplus 10, 0\bar{1}, 01 \end{matrix}$$

$$c(\lambda_{0,2} \rightsquigarrow \lambda_{1,2}) = \frac{\Gamma(\lambda_{0,2} - \lambda_{0,1} + 1)\,\Gamma(\lambda_{1,1} - \lambda_{1,2})}{\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1})\,\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2})} \quad \updownarrow \ \Leftrightarrow$$

$$P\left\{\begin{matrix} x=\frac{1}{c_0}=0 & \frac{1}{c_1}=1 & \cdots & \frac{1}{c_p}=\infty \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}$$



$\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$: rigid $\iff$ $\alpha_{\mathbf{m}}=\alpha_{\mathbf{m}'}+\alpha_{\mathbf{m}''}$: positive real roots

$\mathrm{ord}\le 40,\ p=2\Rightarrow$ 4,111,704 independent cases by a computer

$1^n,1^n,n-11$: ${}_nF_{n-1}\longrightarrow c$-function of type A_n

$1^{2n},nn-11,nn$: Even family of order $2n\longrightarrow c$-function of type B_n

Thank you!   End!