Hairy Graphs and the Homology of $\mathrm{Out}(F_n)$

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   4. cf. $\mathrm{Mod}(S)$!

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n	2	3	4	5	6
$\chi_{\mathrm{orb}}(\mathrm{Out}(F_n))$	$-\frac{1}{24}$	$-\frac{1}{48}$	$-\frac{161}{5760}$	$-\frac{367}{5760}$	$-\frac{120257}{580608}$

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   2. A generating function for these orbifold Euler characteristics is known. (Kontsevich, Smillie-Vogtmann)

$H_k(\mathrm{Aut}(F_n);\mathbb{Q})$

$k \backslash n$	2	3	4	5	6	7	8	9	10	11
2	vcd	0	0	stable	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$	$\vert$
3	_	0	0	0	0	stable	$\vert$	$\vert$	$\vert$	$\vert$
4	_	vcd	μ_1	0	0	0	stable	$\vert$	$\vert$	$\vert$
5	_	_	0	0	0	0	0	0	stable	$\vert$
6	_	_	vcd	0	0	0	0	0	0	stable
7	_	_	_	ϵ_1	0	0	0	0	0	0
8	_	_	_	vcd	μ_2	?	?	?	?	?
9	_	_	_	_	?	?	?	?	?	?
10	_	_	_	_	vcd	?	?	?	?	?
11	_	_	_	_	_	ϵ_2	?	?	?	?
12	_	_	_	_	_	vcd	μ_3	?	?	?

$H_k(\mathrm{Out}(F_n); \mathbb{Q})$

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2	_	0	0	0	0	0	0	stable	|	|
3	_	vcd	0	0	0	0	0	0	0	stable
4	_	_	μ_1	0	0	0	0	0	0	0
5	_	_	vcd	0	0	0	0	0	0	0
6	_	_	_	0	0	0	0	0	0	0
7	_	_	_	vcd	0	0	0	0	0	0
8	_	_	_	_	μ_2	?	?	?	?	?
9	_	_	_	_	vcd	?	?	?	?	?
10	_	_	_	_	_	?	?	?	?	?
11	_	_	_	_	_	vcd	?	?	?	?
12	_	_	_	_	_	_	μ_3	?	?	?

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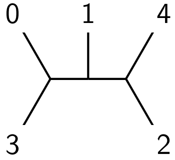
3. Let $\mathcal{S}_{2k}$ be the space of cusp forms for $\mathrm{SL}(2,\mathbb{Z})$ of weight $2k$. (This can be defined as follows. Spaces of all modular forms of weight $2k$, $\mathcal{M}_{2k}$, are defined by $\mathbb{Q}[x_4, x_6] \cong \oplus_k \mathcal{M}_k$. Then $\dim(\mathcal{S}_{2k}) = \dim(\mathcal{M}_{2k}) - 1$ for $k \geq 2$.) There is an embedding

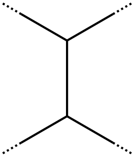
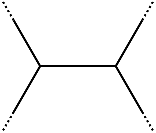
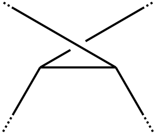
$$\bigwedge^2 \mathcal{S}_{2k}^* \hookrightarrow Z_{4k-2}(\mathrm{Out}(F_{2k+1});\mathbb{Q}).$$

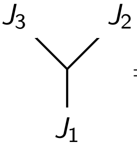
The first potential class lies in $Z_{46}(\mathrm{Out}(F_{25}))$!

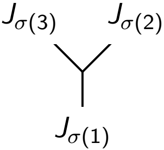
The main goal of this talk is to give an idea how these classes are produced.

The Lie operad

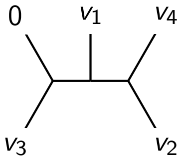
Lie((5)) $= \mathbb{Q}\Bigg\{$  $\Bigg\}$ $/\mathrm{IHX} + \mathrm{AS}$

1. IHX:  $=$  $-$ 

2. AS: 

$= (-1)^{|\sigma|}$ 

1. $\mathsf{L}_n(V) := \mathrm{Lie}((n+1)) \otimes_{\Sigma_n} V^{\otimes n}$.



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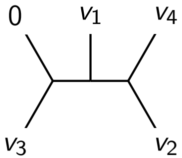

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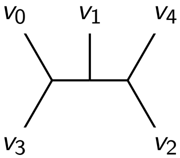
2. $\mathfrak{h}_V[\![d]\!] := \mathrm{Lie}((d+2)) \otimes_{\Sigma_{d+2}} V^{\otimes(d+2)}$.



$$\mathfrak{h}_V := \bigoplus_{d \geq 1} \mathfrak{h}_V[\![d]\!]$$

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    2. If $V = \mathbb{Q}^m$ then $\mathfrak{h}_V$ parameterizes Milnor invariants of m-component links in S^3. (Over $\mathbb{Z}$, $\mathfrak{h}_V$ measures the failure of the Whitney move in 4 dimensions. (C.-Schneiderman-Teichner))

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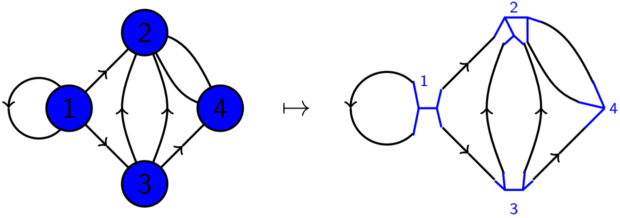
   3. If V is a direct limit of finite-dimensional symplectic vector spaces, then

$$PH^*(\mathfrak{h}_V)^{\mathrm{Sp}} = \bigoplus_{r \geq 2} H_*(\mathrm{Out}(F_r); \mathbb{Q}).\text{(Kontsevich)}$$

$$PH_*(\mathfrak{h}_V; \mathsf{L}(V))_{\mathrm{Sp}} = \bigoplus_{r \geq 2} H^*(\mathrm{Aut}(F_r); \mathbb{Q}).\text{(Gray)}$$

Lie graph homology

1. Informally, one obtains a *Lie graph* by gluing elements of the Lie operad into the vertices of a template graph:



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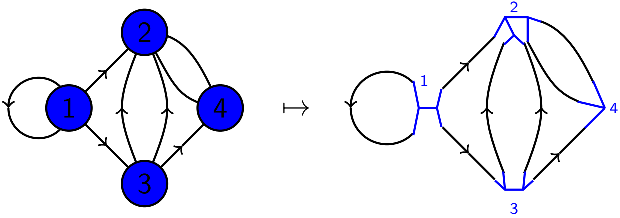

2. More formally, for every template graph G, define

$$\mathcal{G}_G = \left[\bigotimes_{v \in V(G)} \mathsf{Lie}((\mathrm{val}(v))) \right]_{\mathrm{Aut}(G)}$$

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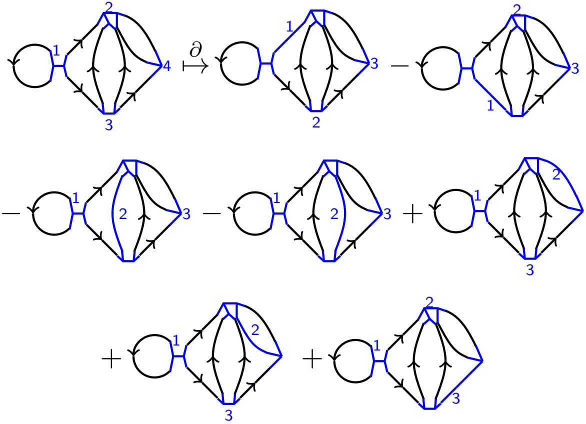

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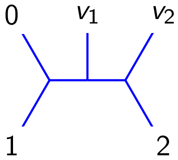
2. This is how the isomorphism to the homology of $\mathfrak{h}_V$ is proven.

The Lie_V operad and Lie_V-graphs

1. $\mathsf{Lie}_V((3)) = \mathbb{Q}\Big\{$  $\Big\}/\mathrm{IHX} + \mathrm{AS}$

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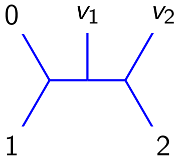
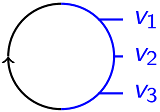


2. A *hairy Lie graph* is a Lie_V-graph.



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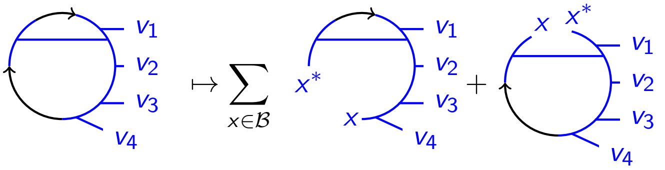
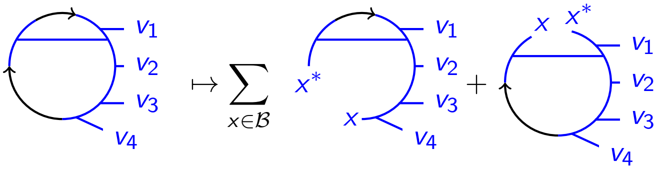


1. Let (V, ω) be a finite-dimensional symplectic vector space with symplectic basis $\mathcal{B}$.
2. Define a chain map $S\colon \mathcal{H}_V \to \mathcal{H}_V$



3. The chain map

$$\exp(S) = \sum_{i=0}^{n} \frac{S^i}{i!}\colon \mathcal{H}_V \to \mathcal{H}_V \cong \mathbb{Q}[P\mathcal{H}_V]$$

is a sum over snipping some subset of the black edges of a hairy graph and labeling the new ends by paired vectors x, x^*.

The cohomological assembly map

1. Dualizing the restriction to $\mathcal{G} \subset \mathcal{H}_V$, we get a degree-preserving *assembly* map

$$\exp(S)^*\colon S(PH^*(\mathcal{H}_V)) \to H^*(\mathcal{G})$$

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3. $H_1(\mathcal{H}_V)$ is graded by first Betti number:
$$H_1(\mathcal{H}_V) \cong \bigoplus_{k=0}^{\infty} H_1(\mathcal{H}_V)^{(k)}$$

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3. $$H_1(\mathcal{H}_V)^{(r)} \cong H^{2r-3}(\mathrm{Out}(F_r); S(\mathbb{Q}^r \otimes V)) \textit{ for } r \geq 2.$$

Here $\mathrm{Out}(F_r)$ *acts on* $\mathbb{Q}^r$ *via the standard* $\mathrm{GL}(r, \mathbb{Z})$ *action twisted by the determinant.*

So we get an assembly map that takes formal products of homology classes in lower rank groups (with twisted coefficients) and produces homology classes with rational coefficients.

$$S\left(\bigwedge^3 V \oplus \bigoplus_{k=0}^{\infty} S^{2k+1}V \oplus \bigoplus_{r=2}^{\infty} H_{2r-3}(\mathrm{Out}(F_r); S(\mathbb{Q}^r \otimes V))\right)_{\mathrm{Sp}}$$

$$\downarrow$$

$$\bigoplus_{r=2}^{\infty} H_*(\mathrm{Out}(F_r); \mathbb{Q})$$

Theorem

(CKV)

$$H_1(\mathcal{H}_V)^{(2)} \cong H^1(\mathrm{Out}(F_2); S(\mathbb{Q}^2 \otimes V)) \quad \cong \bigoplus_{k>\ell\geq 0} \mathbb{S}_{(k,\ell)} V \otimes W_{(k,\ell)}$$

$$\textit{where } W_{(k,\ell)} = \begin{cases} \mathcal{S}_{k-\ell+2} & \textit{if } k,\ell \textit{ are even.} \\ \mathcal{M}_{k-\ell+2} & \textit{if } k,\ell \textit{ are odd.} \\ 0 & \textit{if } k+\ell \textit{ is odd.} \end{cases}$$

- Recall that $\mathcal{S}_r$ are cusp forms of weight r and $\mathcal{M}_r$ is one higher dimension, including the Eisenstein series.

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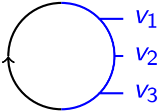
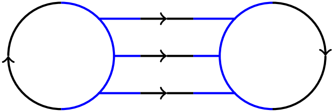
- $\mathbb{S}_\lambda V := P_\lambda \otimes_{\Sigma_n} V^{\otimes n}$, where λ is a partition of n and P_λ is the irreducible Σ_n-representation corresponding to λ.
- The proof uses the Eichler-Shimura computation $H^1(\mathrm{SL}(2,\mathbb{Z}); S^{2k}(\mathbb{Q}^2)) \cong \mathcal{M}_{2k+2} \oplus \mathcal{S}_{2k+2}$.

Morita classes

$$\left(\bigwedge^2 S^{2k+1}V\right)^{\mathrm{Sp}} \cong \mathbb{Q}\{\mu_k\}$$

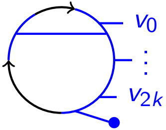


$$\longleftrightarrow \quad v_1 v_2 v_3 \in S^3 V$$



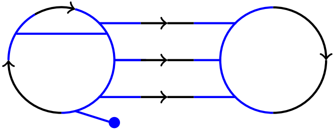
$$\longleftrightarrow \quad \mu_1 \in \left(\bigwedge^2 S^{2k+1}V\right)^{\mathrm{Sp}}$$

Eisenstein classes



$$\longleftrightarrow V \otimes V^{2k} \hookrightarrow \mathbb{S}_{(2k+1,1)} V$$

$$\left(S^{(2k+1)} V \otimes (V \otimes V^{2k})\right)^{\mathrm{Sp}} \cong \mathbb{Q}\{\epsilon_k\}$$



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Doubled cusp form classes

1. $$\left[\bigwedge^2(\mathbb{S}_{(2m,0)}V \otimes \mathcal{S}_{2m+2})\right]^{\mathrm{Sp}} \cong \bigwedge^2 \mathcal{S}_{2m+2}$$

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    2. This gives a growing family of cycles in dimension $vcd-1$. If these survive in homology, it would contradict a conjecture of Church-Farb-Putman that the homology stabilizes in fixed codimension.

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1. Let $\mathcal{A}$ be an algebra with involution, $a \mapsto \bar{a}$.

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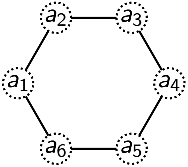

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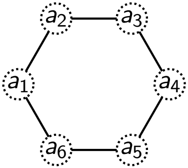

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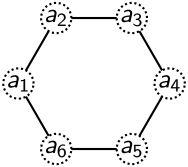

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3. Boundary operator: $\partial\colon CD_k(\mathcal{A}) \to CD_{k-1}(\mathcal{A})$. This is induced by multiplication of algebra elements along edges of the polygon.
4. *(twisted) Dihedral homology* is defined as

$$HD_k(\mathcal{A}) = H_k(CD_\bullet(\mathcal{A}), \partial). \text{ (Loday)}$$

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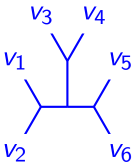
- So dihedral homology class could combine with other hairy graph homology classes to produce classes in $\mathrm{Out}(F_n)$.

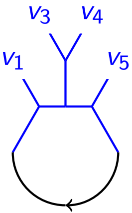
On a Conjecture of Morita

Morita, who was studying $\mathfrak{h}_V$ in connection with the Johnson homomorphisms of mapping class groups, constructed a trace homomorphism:

$$\mathrm{Tr}_M \colon \mathfrak{h}_V \to \bigoplus_{k=1}^{\infty} S^{2k+1} V.$$



$\mapsto \omega(v_2, v_6)$



$+ \cdots \mapsto \omega(v_2, v_6) v_1 [v_3, v_4] v_5 + \cdots$

$$\mathfrak{h}_V \to T(V)_{\mathbb{Z}_2} \to S(V)_{\mathbb{Z}_2}$$

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Morita conjectured that the trace homomorphism (the range being abelian) induces an isomorphism on the abelianization:

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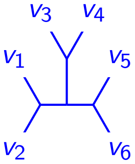
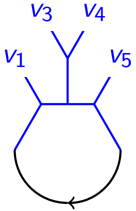
- We recognize that the middle term in the above trace map definition is an element of a hairy graph complex, and lift Tr^M to a map $T\colon \mathfrak{h}_V \to \mathcal{H}_V$ defined by summing over contractions:



$$\overset{T}{\mapsto} \omega(v_2, v_6)$$



$$+\cdots$$

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Corollary

$\mathfrak{h}_V^{\mathrm{ab}} \cong \bigoplus_{k=0}^{\infty} \mathfrak{h}_V^{\mathrm{ab}}[k]$, *where* $\mathfrak{h}_V^{\mathrm{ab}}[0] \cong \bigwedge^3 V$, $\mathfrak{h}_V^{\mathrm{ab}}[1] = S(V)_{\mathbb{Z}_2}$, *and* $\mathfrak{h}_V^{\mathrm{ab}}[k]$ *is a "large" subspace of* $H^{2k-3}(\mathrm{Out}(F_k); S(\mathbb{Q}^2 \otimes V))$. *In particular, it contains infinitely many irreducible* Sp*-modules when* $k = 2$, *contradicting Morita's conjecture.*

Questions

1. Are all $\mu_k, \epsilon_k, \cdots$ nontrivial?
2. How (non)-trivial is the cohomological assembly map?
3. Does the assembly map have a more direct definition/interpretation?
4. What is $H_1(\mathcal{H}_V)^{(k)}$ for $k \geq 3$? We know it is highly nontrivial for $k = 3$, and can describe it fairly explicitly. We don't know for $k \geq 4$. Likely these are related to modular and automorphic forms.
5. Construct new elements of the cokernel of the Johnson filtration?