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cf. Mod$(S)$!
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\[
\begin{array}{c|cccccc}
   n & 2 & 3 & 4 & 5 & 6 \\
   \chi_{\text{orb}}(\text{Out}(F_n)) & -\frac{1}{24} & -\frac{1}{48} & -\frac{161}{5760} & -\frac{367}{5760} & -\frac{120257}{580608} \\
\end{array}
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2. A generating function for these orbifold Euler characteristics is known. (Kontsevich, Smillie-Vogtmann)
$$H_k(\text{Aut}(F_n); \mathbb{Q})$$

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Jim Conant Univ. of Tennessee joint w/ Marl

Hairy Graphs and the Homology of $\text{Out}(F_n)$

July 11, 2012 5 / 29
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2. $\epsilon_k \in H_{4k+3}(\text{Aut}(F_{2k+3}); \mathbb{Q})$ (CKV). It is unknown if $\epsilon_k \neq 0$ unless $k = 1, 2$. We could call these *Eisenstein classes*. 
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Let $S_{2k}$ be the space of cusp forms for $\text{SL}(2, \mathbb{Z})$ of weight $2k$. (This can be defined as follows. Spaces of all modular forms of weight $2k$, $M_{2k}$, are defined by $\mathbb{Q}[x_4, x_6] \cong \bigoplus_k M_k$. Then $\dim(S_{2k}) = \dim(M_{2k}) - 1$ for $k \geq 2$.) There is an embedding

$$\bigwedge^2 S^*_{2k} \hookrightarrow Z_{4k-2}(\text{Out}(F_{2k+1}); \mathbb{Q}).$$

The first potential class lies in $Z_{46}(\text{Out}(F_{25}))$!
The main goal of this talk is to give an idea how these classes are produced.
The Lie operad

\[ \text{Lie}((5)) = \mathbb{Q} \left\{ \begin{array}{c} 0 \quad 1 \quad 4 \\ 3 \quad 2 \end{array} \right\} / \text{IHX} + \text{AS} \]

1. IHX:

\[ J_3 \quad J_2 \]

2. AS:

\[ J_1 \quad ( -1)^{|\sigma|} \]

\[ J_\sigma(1) \quad J_\sigma(2) \]
The free Lie algebra over the vector space $V$ is

$$L(V) = \bigoplus_n L_n(V).$$
1. \( L_n(V) := \text{Lie}((n+1)) \otimes \Sigma_n V \otimes^n. \)

The free Lie algebra over the vector space \( V \) is

\[
L(V) = \bigoplus_n L_n(V).
\]

2. \( h_V[d] := \text{Lie}((d + 2)) \otimes \Sigma_{d+2} V \otimes^{(d+2)}. \)

\[
h_V := \bigoplus_{d \geq 1} h_V[d].
\]
If $V = (V, \omega)$ has a nondegenerate antisymmetric form (symplectic), then $\mathfrak{h}_V$ is itself a Lie algebra.
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$\mathfrak{h}_V$ is remarkably ubiquitous in low-dimensional topology.

1. If $V = H_1(S_{g,1}; \mathbb{Q})$, then $\mathfrak{h}_V$ is the target of the (associated graded) Johnson homomorphism on $\text{Mod}(S_{g,1})$ and on 3-dimensional homology cylinders. (Johnson, S. Morita, J. Levine)
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   2. If $V = \mathbb{Q}^m$ then $\mathfrak{h}_V$ parameterizes Milnor invariants of $m$-component links in $S^3$. (Over $\mathbb{Z}$, $\mathfrak{h}_V$ measures the failure of the Whitney move in 4 dimensions. (C.-Schneiderman-Teichner))
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   - If $V$ is a direct limit of finite-dimensional symplectic vector spaces, then

     $$PH^*(h_V)^{Sp} = \bigoplus_{r \geq 2} H_*(\text{Out}(F_r); \mathbb{Q}). (\text{Kontsevich})$$

     $$PH_*(h_V; L(V))^{Sp} = \bigoplus_{r \geq 2} H^*(\text{Aut}(F_r); \mathbb{Q}). (\text{Gray})$$
Informally, one obtains a \textit{Lie graph} by gluing elements of the Lie operad into the vertices of a template graph:
Informally, one obtains a *Lie graph* by gluing elements of the Lie operad into the vertices of a template graph:

More formally, for every template graph $G$, define

$$G_G = \left[ \bigotimes_{v \in V(G)} \text{Lie}(\text{val}(v)) \right]_{\text{Aut}(G)}$$
$g_k^{(n)} := \bigoplus_G G$, where $G \simeq \bigvee_{i=1}^n S^1$, has $k$ vertices, and all vertices have valence $\geq 3$. 
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Boundary operator: $\partial : G_k^{(n)} \to G_{k-1}^{(n)}$
Theorem

\[ H_k(G^{(n)}) \cong H^{2n-3-k}(\text{Out}(F_n); \mathbb{Q}) \]

This was observed by Kontsevich. It is proven using double complex spectral sequence applied to the spine of Outer space. That the signs match up is very surprising. (C.-Vogtmann has a complete proof.)
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1. This was observed by Kontsevich. It is proven using double complex spectral sequence applied to the spine of Outer space. That the signs match up is very surprising. (C.-Vogtmann has a complete proof.)
2. This is how the isomorphism to the homology of \( h_V \) is proven.
The Lie$_V$ operad and Lie$_V$-graphs

$\text{Lie}_V((3)) = \mathbb{Q} \left\{ \begin{array}{c} 0 \\ v_1 \\ v_2 \\ 1 \\ 2 \end{array} \right\} / \text{IHX + AS}$
The $\text{Lie}_V$ operad and $\text{Lie}_V$-graphs

1. \[ \text{Lie}_V((3)) = \mathbb{Q} \left\{ \begin{array}{c} 0 \quad v_1 \quad v_2 \\ 1 \quad 2 \end{array} \right\} / \text{IHX} + \text{AS} \]

2. A hairy Lie graph is a $\text{Lie}_V$-graph.
The hairy graph complex

1. The *hairy graph complex* $\mathcal{H}_V$ is spanned by all (not necessarily connected) hairy Lie graphs. $P\mathcal{H}_V$ is the subspace spanned by connected graphs.
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The hairy graph complex $\mathcal{H}_V$ is spanned by all (not necessarily connected) hairy Lie graphs. $P\mathcal{H}_V$ is the subspace spanned by connected graphs.

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3. \( \partial : \mathcal{H}_V \to \mathcal{H}_V \) is defined similarly to before.

4. \( G \subseteq \mathcal{H}_V \) is a subcomplex.
Let \((V, \omega)\) be a finite-dimensional symplectic vector space with symplectic basis \(B\).
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Define a chain map $S: \mathcal{H}_V \to \mathcal{H}_V$

$$\sum_{x \in B} x^* x + \sum_{x \in B} x x^*$$
Let \((V, \omega)\) be a finite-dimensional symplectic vector space with symplectic basis \(B\).

Define a chain map \(S : \mathcal{H}_V \to \mathcal{H}_V\)

\[
\sum_{x \in B} v_2 v_1 v_3 v_4 x^* x + \sum_{x \in B} x^* x
\]

The chain map

\[
\exp(S) = \sum_{i=0}^{n} \frac{S^i}{i!} : \mathcal{H}_V \to \mathcal{H}_V \cong \mathbb{Q}[\mathcal{P} \mathcal{H}_V]
\]

is a sum over snipping some subset of the black edges of a hairy graph and labeling the new ends by paired vectors \(x, x^*\).
Dualizing the restriction to $\mathcal{G} \subset \mathcal{H}_V$, we get a degree-preserving assembly map

$$\exp(S)^*: S(PH^*(\mathcal{H}_V)) \to H^*(\mathcal{G})$$

which glues together “small” classes to make bigger ones.
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From now on we restrict to the simplest piece $H^1(\mathcal{H}_V)$ or $H_1(\mathcal{H}_V)$. (Note $PH_1(\mathcal{H}_V) = H_1(\mathcal{H}_V)$.)
The cohomological assembly map

1. Dualizing the restriction to $G \subset H_V$, we get a degree-preserving assembly map

$$\exp(S)^* : S(PH^*(H_V)) \to H^*(G)$$

which glues together “small” classes to make bigger ones.

2. From now on we restrict to the simplest piece $H^1(H_V)$ or $H_1(H_V)$. (Note $PH_1(H_V) = H_1(H_V)$.)

3. $H_1(H_V)$ is graded by first Betti number:

$$H_1(H_V) \cong \bigoplus_{k=0}^{\infty} H_1(H_V)^{(k)}$$
Theorem (CKV)

$$H_1(\mathcal{H}_V)^{(0)} \cong \bigwedge^3 V$$
Theorem (CKV)

1. \( H_1(\mathcal{H}_V)^{(0)} \cong \bigwedge^3 V \)

2. \( H_1(\mathcal{H}_V)^{(1)} \cong \bigoplus_{k=0}^{\infty} S^{2k+1} V \)

Here, \( \text{Out}(F_r) \) acts on \( \mathbb{Q}_r \otimes V \) via the standard \( \text{GL}(r, \mathbb{Z}) \) action twisted by the determinant.
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2. \( H_1(\mathcal{H}_V)^{(1)} \cong \bigoplus_{k=0}^{\infty} S^{2k+1} V \)

3. \( H_1(\mathcal{H}_V)^{(r)} \cong H^{2r-3}(\text{Out}(F_r); S(\mathbb{Q}^r \otimes V)) \) for \( r \geq 2 \).

Here \( \text{Out}(F_r) \) acts on \( \mathbb{Q}^r \) via the standard \( \text{GL}(r, \mathbb{Z}) \) action twisted by the determinant.
So we get an assembly map that takes formal products of homology classes in lower rank groups (with twisted coefficients) and produces homology classes with rational coefficients.

\[
S \left( \bigwedge^3 V \bigoplus \bigoplus_{k=0}^{\infty} S^{2k+1} V \bigoplus \bigoplus_{r=2}^{\infty} H_{2r-3}(\text{Out}(F_r); S(\mathbb{Q}^r \otimes V)) \right)_{\text{Sp}}
\]

\[
\downarrow
\]

\[
\bigoplus_{r=2}^{\infty} H_*(\text{Out}(F_r); \mathbb{Q})
\]
Theorem (CKV)

\[ H_1(H_V)^{(2)} \cong H^1(\text{Out}(F_2); S(\mathbb{Q}^2 \otimes V)) \cong \bigoplus_{k > \ell \geq 0} S_{(k,\ell)} V \otimes W_{(k,\ell)} \]

where \( W_{(k,\ell)} = \begin{cases} 
S_{k-\ell+2} & \text{if } k, \ell \text{ are even.} \\
M_{k-\ell+2} & \text{if } k, \ell \text{ are odd.} \\
0 & \text{if } k + \ell \text{ is odd.}
\end{cases} \)

- Recall that \( S_r \) are cusp forms of weight \( r \) and \( M_r \) is one higher dimension, including the Eisenstein series.
Theorem (CKV)

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- \( S_\lambda V := P_\lambda \otimes_{\Sigma_n} V \otimes^n \), where \( \lambda \) is a partition of \( n \) and \( P_\lambda \) is the irreducible \( \Sigma_n \)-representation corresponding to \( \lambda \).
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- \( S_\lambda V := P_\lambda \otimes \Sigma_n V \otimes^n \), where \( \lambda \) is a partition of \( n \) and \( P_\lambda \) is the irreducible \( \Sigma_n \)-representation corresponding to \( \lambda \).
- The proof uses the Eichler-Shimura computation
  \[ H^1(\text{SL}(2, \mathbb{Z}); S^{2k}(\mathbb{Q}^2)) \cong M_{2k+2} \oplus S_{2k+2}. \]
Morita classes

\[(\bigwedge^2 S^{2k+1}V)^{Sp} \cong \mathbb{Q}\{\mu_k\}\]

\[v_1 v_2 v_3 \in S^3 V\]

\[\mu_1 \in \left(\bigwedge^2 S^{2k+1}V\right)^{Sp}\]
Eisenstein classes

\[ V \otimes V^{2k} \hookrightarrow S_{(2k+1,1)} V \]

\[ \left( S^{(2k+1)} V \otimes (V \otimes V^{2k}) \right)^{Sp} \cong \mathbb{Q}\{\epsilon_k\} \]

\[ \epsilon_1 \in \left( S^{(2k+1)} V \otimes (V \otimes V^{2k}) \right)^{Sp} \]
Doubled cusp form classes

\[
\left[ \bigwedge^2 \left( S_{(2m,0)} \oplus S_{2m+2} \right) \right]^{\text{Sp}} \cong \bigwedge^2 S_{2m+2}
\]
Doubled cusp form classes

1. \[ \bigwedge^2 (S_{(2m,0)} \vee \otimes S_{2m+2})^{Sp} \cong \bigwedge^2 S_{2m+2} \]

2. \( \dim S_{2m+2} \approx m/6. \) The first time the dimension is \( \geq 2 \) is \( m = 11. \)
Doubled cusp form classes

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\left[ \bigwedge^2 \left( S_{(2m,0)} \vee \otimes S_{2m+2} \right) \right]^{Sp} \cong \bigwedge^2 S_{2m+2}
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2 \( \dim S_{2m+2} \approx m/6. \) The first time the dimension is \( \geq 2 \) is \( m = 11. \)

This will give a class in \( H_{46}(\text{Out}(F_{25} \mathbb{Q})). \) Too large to test by computer. :(
Doubled cusp form classes

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   1. This will give a class in \( H_{46}(\text{Out}(F_{25}\mathbb{Q})) \). Too large to test by computer. :( 
   2. This gives a growing family of cycles in dimension \( vcd - 1 \). If these survive in homology, it would contradict a conjecture of Church-Farb-Putman that the homology stabilizes in fixed codimension.
Let $A$ be an algebra with involution, $a \mapsto \bar{a}$.
Let $\mathcal{A}$ be an algebra with involution, $a \mapsto \bar{a}$.

There is a chain complex $CD_k(A) = [A \otimes_k]_{D_{2k}}$, where $D_{2k}$ acts with certain signs, imagining a copy of $A$ at each corner of a $k$-gon.

$\partial: CD_k(A) \to CD_{k-1}(A)$. This is induced by multiplication of algebra elements along edges of the polygon.

(Twisted) Dihedral homology is defined as $HD_k(A) = H_k(CD_* (A), \partial)$.

(Loday)
Let $\mathcal{A}$ be an algebra with involution, $a \mapsto \bar{a}$.

There is a chain complex $CD_k(\mathcal{A}) = [\mathcal{A} \otimes k]_{D_{2k}}$, where $D_{2k}$ acts with certain signs, imagining a copy of $\mathcal{A}$ at each corner of a $k$-gon.

\[
a_1 \otimes \cdots \otimes a_n \mapsto (-1)^{n-1} a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}
\]

\[
a_1 \otimes \cdots \otimes a_n \mapsto (-1)^{n+\binom{n}{2}} \bar{a}_n \otimes \cdots \otimes \bar{a}_1
\]
Dihedral homology

1. Let $A$ be an algebra with involution, $a \mapsto \bar{a}$.
2. There is a chain complex $CD_k(A) = [A \otimes^k]_{D_{2k}}$, where $D_{2k}$ acts with certain signs, imagining a copy of $A$ at each corner of a $k$-gon.

![Diagram of a hexagon with vertices labeled $a_1$ to $a_6$ connected by edges, and the boundary operator mapping $a_1 \otimes \cdots \otimes a_6 \in V \otimes^6$.]

3. Boundary operator: $\partial : CD_k(A) \to CD_{k-1}(A)$. This is induced by multiplication of algebra elements along edges of the polygon.
4. (twisted) Dihedral homology is defined as

$$HD_k(A) = H_k(CD_{\cdot}(A), \partial). \text{ (Loday)}$$
Consider the algebra \( \mathcal{A} = S(V) \), the free commutative algebra on \( V \), with involution defined on generators \( v \mapsto -v \).
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\textbf{Theorem}

\[ H_k(\mathcal{H}_V)^{(1)} = \text{HD}_k(S(V)) \]
Consider the algebra $\mathcal{A} = S(V)$, the free commutative algebra on $V$, with involution defined on generators $v \mapsto -v$.

**Theorem**

$$H_k(\mathcal{H}_V)^{(1)} = HD_k(S(V))$$

So dihedral homology class could combine with other hairy graph homology classes to produce classes in $\text{Out}(F_n)$. 
Morita, who was studying $h_V$ in connection with the Johnson homomorphisms of mapping class groups, constructed a trace homomorphism:

$$\text{Tr}_M : h_V \to \bigoplus_{k=1}^{\infty} S^{2k+1} V.$$

$$\begin{align*}
\mapsto & \omega(v_2, v_6) \\
& + \cdots \mapsto \omega(v_2, v_6)v_1[v_3, v_4]v_5 + \cdots
\end{align*}$$

$$h_V \to T(V)_{\mathbb{Z}_2} \to S(V)_{\mathbb{Z}_2}$$
Morita conjectured that the trace homomorphism (the range being abelian) induces an isomorphism on the abelianization:

$$h^\text{ab}_V \cong \bigwedge^3 V \oplus S(V)\mathbb{Z}_2$$
Morita conjectured that the trace homomorphism (the range being abelian) induces an isomorphism on the abelianization:

\[ \mathfrak{h}_V^{ab} \cong \bigwedge^3 V \oplus S(V)\mathbb{Z}_2 \]

We recognize that the middle term in the above trace map definition is an element of a hairy graph complex, and lift \( \text{Tr}^M \) to a map \( T : \mathfrak{h}_V \to \mathcal{H}_V \) defined by summing over contractions:

\[ T \mapsto \omega(v_2, v_6) + \cdots \]
On a Conjecture of Morita

Theorem (CKV)

\[ \text{Tr} = \exp(T) \] induces an monomorphism \( \mathfrak{h}_V^{ab} \hookrightarrow H_1(\mathcal{H}_V). \)

\((\dim V = \infty)\)
On a Conjecture of Morita

Theorem (CKV)

1. \( \text{Tr} = \exp(T) \) induces an monomorphism \( \mathfrak{h}_V^{ab} \hookrightarrow H_1(\mathcal{H}_V) \).
   \( (\dim V = \infty) \)

2. Let \( V^+ < V \) be a Lagrangian subspace. There is a natural projection \( \pi : H_1(\mathcal{H}_V) \rightarrow H_1(\mathcal{H}_{V^+}) \). Then \( \pi \circ \text{Tr} \) is an epimorphism. (Note that \( H_1(\mathcal{H}_V) \cong H_1(\mathcal{H}_{V^+}) \) as \( \text{GL} \)-modules!)

Corollary

\( h^{ab}_V \sim = \bigoplus_{k=0}^{\infty} h^{ab}_V[k] \), where \( h^{ab}_V[0] \sim = \bigwedge^3 V \), \( h^{ab}_V[1] = S(\bigotimes^2 V) Z_2 \), and \( h^{ab}_V[k] \) is a "large" subspace of \( H_2 \mathcal{H}_{k-3}(\text{Out}(F_k); S(\bigotimes^2 V)) \). In particular, it contains infinitely many irreducible \( \text{Sp} \)-modules when \( k = 2 \), contradicting Morita’s conjecture.
On a Conjecture of Morita

**Theorem (CKV)**

1. \( \text{Tr} = \exp(T) \) induces an monomorphism \( \mathfrak{h}^{ab}_V \hookrightarrow H_1(\mathcal{H}_V) \).
   \((\dim V = \infty)\)

2. Let \( V^+ < V \) be a Lagrangian subspace. There is a natural projection \( \pi: H_1(\mathcal{H}_V) \rightarrow H_1(\mathcal{H}_{V^+}) \). Then \( \pi \circ \text{Tr} \) is an epimorphism. \((\text{Note that } H_1(\mathcal{H}_V) \cong H_1(\mathcal{H}_{V^+}) \text{ as } \text{GL}-\text{modules!})\)

**Corollary**

\( \mathfrak{h}^{ab}_V \cong \bigoplus_{k=0}^{\infty} \mathfrak{h}^{ab}_V[k] \), where \( \mathfrak{h}^{ab}_V[0] \cong \wedge^3 V \), \( \mathfrak{h}^{ab}_V[1] = S(V) \mathbb{Z}_2 \), and \( \mathfrak{h}^{ab}_V[k] \) is a “large” subspace of \( H^{2k-3}(\text{Out}(F_k); S(\mathbb{Q}^2 \otimes V)) \). In particular, it contains infinitely many irreducible \( \text{Sp} \)-modules when \( k = 2 \), contradicting Morita’s conjecture.
Questions

1. Are all $\mu_k, \epsilon_k, \cdots$ nontrivial?
2. How (non)-trivial is the cohomological assembly map?
3. Does the assembly map have a more direct definition/interpretation?
4. What is $H_1(\mathcal{H}_V)^{(k)}$ for $k \geq 3$? We know it is highly nontrivial for $k = 3$, and can describe it fairly explicitly. We don’t know for $k \geq 4$. Likely these are related to modular and automorphic forms.
5. Construct new elements of the cokernel of the Johnson filtration?