Recognition Problems, Profinite Completions and Cube Complexes

Martin R Bridson

Mathematical Institute
University of Oxford

Park City, Utah, 2 July 2012.

Outline

What do the following questions have in common?

Question

How constrained are finitely presented subgroups of mapping class groups?

Question (Baumslag)

How diverse can the finitely generated groups within a given nilpotent genus be?

Question

Do there exist algorithms that, given a finitely presented group Γ, can determine whether

1. *there is a non-trivial linear representation $\Gamma \to \mathrm{GL}(d, \mathbb{C})$?*
2. *Γ is large (has a finite-index subgroup mapping onto F_2)?*
3. *Γ has a non-trivial finite quotient?*

. . . we'll see.

Subgroups of Mapping Class Groups

The mapping class group $\mathrm{Mod}(S)$ of a surface S consists of isotopy classes of homeomorphisms $S \to S$.

We will assume that S is orientable and of finite type: it is a closed surface, minus some points (punctures) and open discs. Homeomorphisms and isotopies restrict to the identity on the boundary.
Nielsen-Thurston theory describes the individual elements of $\mathrm{Mod}(S)$ in great detail, but less is known about the subgroups of $\mathrm{Mod}(S)$. One does know some things [Ivanov, Birman-Lubotzky-McCarthy, Kerchoff...]

1. solvable subgroups are finitely generated and virtually abelian;
2. every non-solvable subgroup contains a free group of rank 2 (so there are lots of subgroups of the form $F \times \cdots \times F$);
3. there are only finitely many conjugacy classes of finite subgroups; there are surface subgroups; ...

But until very recently, our knowledge was not much greater than this.

Five Problems Concerning Mapping Class Groups

This general paucity was widely commented on. Farb (2006) highlighted:

Question

Isomorphism problem for finitely presented sgps of $\mathrm{Mod}(S)$ *– solvable?*

Question

$\exists$? *finitely presented* $H < \mathrm{Mod}(S)$ *with unsolvable conjugacy problem?*

Question

$\exists$? *finitely presented* $H < \mathrm{Mod}(S)$ *with unsolvable membership problem?*

Question

$\exists$? *fin pres* $H < \mathrm{Mod}(S)$ *with* ∞ *conjugacy classes of finite subgroups?*

Question

Are finitely presented sgps automatic? Polynomial Dehn functions?

Über unendliche diskontinuierliche Gruppen (Dehn, 1912)

$$\Gamma \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$$

"The general discontinuous group is given [as above]. There are above all three fundamental problems. [Are there algorithms to solve...]

- The word problem: decide if words in the $a_i^{\pm 1}$ equal $1 \in \Gamma$.
- The conjugacy problem: decide if a pair of words are conjugate in Γ
- The isomorphism problem: decide which pair of finite presentations, define isomorphic groups

[...] One is already led to them by necessity with work in topology. Each knotted space curve, in order to be completely understood....

- The membership [Magnus] problem for a subgroup $H < G$ of a finitely generated group asks for an algorithm that, given a word w in the generators of G decides whether or not $w \in H$.

All undecidable without further hypotheses. But for nice groups,...
Do the finitely presented subgroups of $\mathrm{Mod}(S)$ form a *nice* class?

Five Problems Concerning Mapping Class Groups

Question

Isomorphism problem for finitely presented sgps of $\mathrm{Mod}(S)$ *– solvable?*

Question

$\exists$? *finitely presented* $H < \mathrm{Mod}(S)$ *with unsolvable conjugacy problem?*

Question

$\exists$? *finitely presented* $H < \mathrm{Mod}(S)$ *with unsolvable membership problem?*

Question

$\exists$? *fin pres* $H < \mathrm{Mod}(S)$ *with* ∞ *conjugacy classes of finite subgroups?*

Question

Are finitely presented sgps automatic? Polynomial Dehn functions?

3 Theorems and 2 Questions

If the genus of S is sufficiently large, then in $\mathrm{Mod}(S)\dots$

Theorem

... the isomorphism problem for finitely presented subgroups is unsolvable.

Theorem

$\exists$ *finitely presented* $H < \mathrm{Mod}(S)$ *with unsolvable conjugacy problem, and unsolvable membership problem.*

Question

$\exists$? *fin pres* $H < \mathrm{Mod}(S)$ *with* ∞ *conjugacy classes of finite subgroups?*

Question

Are finitely presented sgps automatic? Polynomial Dehn functions?

The Remaining 2 Problems

Question

Do there exist finitely presented subgroups of $\mathrm{Mod}(S)$ with infinitely many conjugacy classes of finite subgroups?

Farb asked if my construction of subgroups in $\mathrm{SL}(n, \mathbb{Z})$ with this property could be adapted to the mapping class group. Brady, Clay and Dani [2008] used Morse theory to produce the first examples. In fact, my original examples embed. And it follows readily from work of Leary, Nuncinkis and Hsu that..

Theorem

There exist groups of type VF that have infinitely many conjugacy classes of finite (cyclic) subgroups and which embed in mapping class groups.

NB: This time I did not say "all S of sufficiently high genus".

Isoperimetric functions

Question

Do all f.p. $H < \mathrm{Mod}(S)$ satisfy a polynomial isoperimetric inequality?

Theorem

NO! If S has sufficiently high genus, there are finitely presented subgroups of $\mathrm{Mod}(S)$ with exponential Dehn function.

Example: Let M be a hyperbolic 3-manifold that fibres over the circle. The fibre product of

$$1 \to \Sigma \to \pi_1 M \to \mathbb{Z} \to 1$$

is $(\Sigma \times \Sigma) \rtimes \mathbb{Z}$, the fundamental group of a closed aspherical 5-manifold. It's Dehn function is exponential.
"Subgroups of semihyperbolic groups", [B, 2001]

Right-Angled Artin Groups (RAAGs)

A finite graph with vertices V and edges $E \subset V \times V$ defines RAAG

$$\langle a_v \ (v \in V) \mid [a_u, a_v] = 1 \text{ if } \ \{u, v\} \in E \rangle.$$

Such a group is the fundamental group of a compact non-positively curved cube complex (sticking together standard tori along coordinate sub-tori).

Theorem

$\forall$ *RAAG* Γ, $\exists \Gamma \hookrightarrow \mathrm{Mod}(S)$ *whenver* $\mathrm{genus}(S)$ *is sufficiently large.*

Crisp and Wiest (?). More delicate results controlling geometry of embedding by Koberda, and Clay-Leininger-Mangahas,...

If a group Γ is special, then it embeds in a RAAG, and lots of groups are virtually special!!

The theorems for mapping class groups are consequences of the corresponding theorems for RAAGs and....

Virtual and Induced Embeddings into $\mathrm{Mod}(S)$

Propn

If S is a compact surface with non-empty boundary and G is a finite group, then there is a closed surface S_g and a monomorphism $\mathrm{Mod}(S) \wr G \to \mathrm{Mod}(S_g)$.

Corollary

If a group Γ has a subgroup of finite index that embeds in a RAAG, then Γ embeds in $\mathrm{Mod}(S_g)$ for infinitely many closed surfaces S_g.

Corollary (Agol)

M^3 hyperbolic, then $\pi_1 M^3$ embeds in $\mathrm{Mod}(S_g)$ for infinitely many g.

Considerable flexibility in the above construction, but one cannot get $\Gamma \hookrightarrow \mathrm{Mod}(S_g)$ for *all* $g >> 0$, even if Γ is finite.

The Isomorphism Problem

I'll sketch a proof of the following

Theorem

There exist RAAGs Γ and finitely presented subgroups $H_n = \langle S_n \mid R_n \rangle$, with explicit embeddings $H_n \hookrightarrow \Gamma$ such that there is no algorithm that can determine if $H_n \cong H_0$.

But first, what happened to Baumslag and nilpotent genus?!

Residually nilpotent groups

Γ is residually nilpotent (resp. *residually torsion-free nilpotent*) if

$$\forall \gamma \in \Gamma \smallsetminus \{1\} \ \ \exists \phi : \Gamma \to \text{nilpotent}, \quad \phi(\gamma) \neq 1$$

Equivalently, $\bigcap \Gamma_n = 1$, where Γ_n is the n-th term of the *lower central series* of Γ,

$$\Gamma_1 = \Gamma, \quad \Gamma_{n+1} = \langle [x, y] : x \in \Gamma_n, y \in \Gamma \rangle.$$

Residually nilpotent groups Γ and Λ have the same nilpotent genus if $\Gamma/\Gamma_c \cong \Lambda/\Lambda_c$ for all $c \geq 1$; equivalently, they have the same nilpotent quotients (same nilpotent completion).

Important examples:

Theorem (Droms)

RAAGs are residually torsion-free nilpotent.

Around parafree groups

Γ is parafree if it is residually nilpotent and same genus as some free group. Baumslag produces many examples of finitely presented parafree groups that are not free, but the nature of parafree groups remains a mystery.

$$G_{ij} = \langle a, b, c \mid a = [c^i, a].[c^j, b] \rangle.$$

cf: **Open problem:** Does there exist a finitely generated residually finite group Γ that has the same finite quotients as some free group, i.e. $\hat{\Gamma} \cong \hat{F}_r$

$$\hat{\Gamma} := \lim_{\leftarrow} \Gamma/N \qquad |\Gamma/N| < \infty$$

is the profinite completion of Γ.

Theorem (B,Conder, Reid)

If Γ is a Fuchsian group (e.g. a free group) and Λ is a lattice in a connected Lie group with $\hat{\Gamma} \cong \hat{\Lambda}$, then $\Gamma \cong \Lambda$.

Theorem (B,Conder, Reid)

If Γ is a Fuchsian group (e.g. a free group) and Λ is a lattice in a connected Lie group with $\hat{\Gamma} \cong \hat{\Lambda}$, then $\Gamma \cong \Lambda$.

Propn (B,Reid)

A non-free parafree group cannot be a lattice in a connected Lie group.

There **do** exist lattices in $\mathrm{PSL}(2, \mathbb{C})$ that are *virtually* residually nilpotent and have the same nilpotent quotients as a free group.

homology boundary links

$L \subset S^3$ of m components is called a *homology boundary link* if there exists an epimorphism $h : \pi_1(S^3 \setminus L) \to F$ where F is a free group of rank m. An old theorem of Stallings implies that this map induces an isomorphism of lower central series quotients.

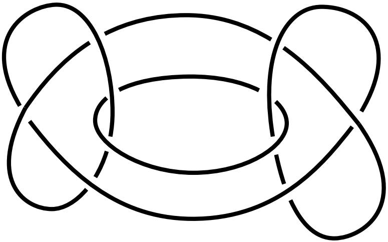

Figure: A boundary homoloy link

Baumslag's Problems

Problem

$\exists$? finitely generated, residually (t.f.) nilpotent groups, same nilpotent genus, one finitely presented and the other is not?

Problem

$\exists$? finitely presented, residually (t.f.) nilpotent groups, same nilpotent genus, s.t. one has a solvable conjugacy problem and the other does not?

Problem

$\exists$? Let G be a finitely generated parafree group and let $N < G$ be a finitely generated, non-trivial, normal subgroup. Must N be of finite index in G?

Problem

$\exists$? finitely presented, residually (t.f.) nilpotent groups, same nilpotent genus, one with finitely generated $H_2(-,\mathbb{Z})$ and the other not?

Answers [B, Reid]

Theorem

1. $\exists$ *finitely generated, residually t.f. nilpotent groups* $H \hookrightarrow D$ *of the same nilpotent genus s.t.* D *is finitely presented and* H *is not.*
2. $\exists$ *finitely presented, residually t.f. nilpotent groups* $P \hookrightarrow \Gamma$ *of the same nilpotent genus s.t.* Γ *has a solvable conjugacy problem and* P *does not.*
3. $\exists$ *finitely generated, residually t.f. nilpotent groups* $N \hookrightarrow \Gamma$ *of the same nilpotent genus s.t.* $H_2(\Gamma, \mathbb{Z})$ *is finitely generated but* $H_2(N, \mathbb{Z})$ *is not.*
4. *Let* G *be a finitely generated parafree group, and let* $N < G$ *be a non-trivial normal subgroup. If* N *is finitely generated,* G/N *is finite.*

(3) is connected to (but does not solve) the *parafree conjecture*, which asserts that the second homology of a parafree group should be trivial. I'll say a little about the idea of the proofs of (1) and (2), and more about (3). To prove (4) one considers ℓ_2-betti numbers and $\widehat{G}_{\mathrm{nil}}$.

We're going to have a quick look at the proofs of

Theorem

If the genus of S is sufficiently large, then the isomorphism problem for the finitely presented subgroups of mod *is unsolvable.*

and

Theorem

There exist pairs of finitely generated, residually torsion-free nilpotent groups $N \hookrightarrow \Gamma$ of the same nilpotent genus such that $H_2(\Gamma, \mathbb{Z})$ is finitely generated but $H_2(N, \mathbb{Z})$ is not.

with emphasis on ideas for ingredients, not details.

we're expecting (virtually) special groups to play a role.

but first an interlude. . .

The standard 2-complex

$$\Gamma \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle \equiv \mathcal{P}$$

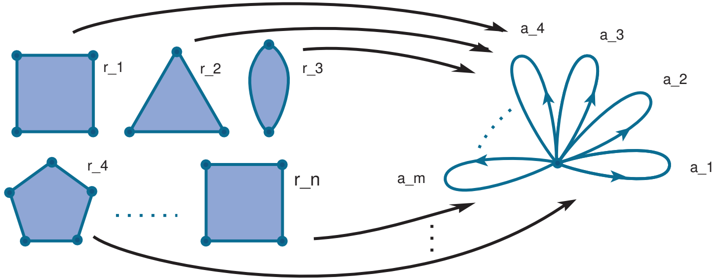

Figure: The standard 2-complex $K(\mathcal{P})$

The profinite triviality problem [B, Wilton]

One of our opening questions was:

Question

Does there exist an algorithm that, given a finitely presented group Γ, can determine whether or not Γ has a non-trivial finite quotient, i.e. if $\hat{\Gamma} = 1$.

I would to consider this in the following context: given

$$\mathcal{P} \equiv \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle \cong \Gamma$$

first take $K(\mathcal{P})$,
pass to $\pi_1 K(\mathcal{P})$ to get Γ.
Or regard $\mathcal{P}$ as a presentation of profinite $\hat{\Gamma}$ or pro-nilpotent $\hat{\Gamma}_{\rm nil}$.
Pasage $K(\mathcal{P})$ to Γ to $\hat{\Gamma}$ and $\hat{\Gamma}_{\rm nil}$ gives up information.
The triviality problem makes sense, as algorithmic problem, at each stage:
Is $K(\mathcal{P}) \simeq *$ (homotopically trivial)? Is $\Gamma \cong 1$? Is $\hat{\Gamma} = 1$? Is $\hat{\Gamma}_{\rm nil} = 1$?

The profinite triviality problem [B, Wilton]

Is $K(\mathcal{P}) \simeq *$ (homotopically trivial)? Is $\Gamma \cong 1$? Is $\hat{\Gamma} = 1$? Is $\hat{\Gamma}_{\text{nil}} = 1$?

It is unknown if the first problem is decidable (a problem of Magnus); the second is undecidable (Boone-Novikov; Adian-Rabin); the fourth is easy.

Theorem (B-Wilton '11)

$\nexists$ an algorithm that, given a finitely presented discrete group Γ can determine whether or not Γ has a non-trivial finite quotient or a non-trivial linear representation, or whether Γ is large.

This theorem does *not* involve RAAGs, but it does use many ideas that Ian Agol referred to the morning and that play a significant role in the study of virtually special groups: subgroup separability (LERF), the careful construction of finite-sheeted covers and virtual retractions (in the spirit of Stallings), variations on omnipotence for virtually-free groups,....

We're going to have a quick look at the proofs of

Theorem

If the genus of S is sufficiently large, then the isomorphism problem for the finitely presented subgroups of $\mathrm{Mod}(S)$ is unsolvable.

and

Theorem

There exist pairs of finitely generated, residually torsion-free nilpotent groups $N \hookrightarrow \Gamma$ of the same nilpotent genus such that $H_2(\Gamma, \mathbb{Z})$ is finitely generated but $H_2(N, \mathbb{Z})$ is not.

with emphasis on ideas for ingredients, not details.

we're expecting (virtually) special groups to play a role.

Warm up: Bridson-Grunewald [2006]

Question

If Γ is residually finite, what can one tell about it from it's set of finite homomorphic images, i.e. from its actions on all finite sets?

$$\hat{\Gamma} := \lim_{\leftarrow} \Gamma/N, \qquad |\Gamma/N| < \infty.$$

Question (Grothendieck 1970)

If Γ_i are fp, residually finite and $u : \Gamma_1 \hookrightarrow \Gamma_2$ induces an isomorphism $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$, must $u : \Gamma_1 \to \Gamma_2$ be an isomorphism?

Designer Groups, Rips and 1-2-3 Thm

$\exists$ algorithm with input a finite, aspherical presentation $\mathcal{Q}$ and output a FINITE presentation (by [BBMS]) for the fibre-product

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subset H \times H$$

associated to a s.e.s. (given by [Rips])

$$1 \to N \to H \xrightarrow{p} Q \to 1$$

with N fin gen, H 2-diml hyperbolic, $Q = |\mathcal{Q}|$ (to be cunninging invented).

"1-2-3 Thm" refers to fact that N, H and Q are of type $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ respectively. [Baumslag, B, Miller, Short]

Refinements (B-Haefliger, Wise, Haglund-Wise) place more stringent conditions on H, e.g. locally CAT(-1) or virtually special.

Solution of Grothendieck's Problem

Question

$u : \Gamma_1 \hookrightarrow \Gamma_2$ fp, rf, $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ iso, is $u : \Gamma_1 \to \Gamma_2$ iso ???

Grothendieck: yes in many cases, e.g. arithmetic groups. Platonov-Tavgen (later Bass–Lubotzky, Pyber): no for certain finitely generated groups.

Theorem (after B-Grunewald)

$\exists$ hyperbolic, virtually special H and finitely presented subgroup $P \hookrightarrow \Gamma := H \times H$ of infinite index, such that P is not abstractly isomorphic to Γ, but the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$.

PROOF: Build Q with finite aspherical presentation, no finite quotients and $H_1(Q) = H_2(Q) = 0$

Apply the virtually special Rips construction.

Use 1-2-3 Theorem to deduce that the fibre product is finitely presented.

Prove $\hat{P} \to \hat{H} \times \hat{H}$ is isomorphism

Bridson-Miller

Some of the ideas that go into the 1-2-3 Theorem establish the (easier):

Propn (B-Miller 2003)

If K_i is finitely generated for $i = 1, 2$, each L_i is free, and $K_i \rtimes L_i$ is finitely presented, and if the image of $\Phi : F \to L_1 \times L_2$ is a sub direct product *then $(K_1 \times K_2) \rtimes F$ is finitely presented.*

Subdirect products of free groups are nasty things! It is very hard to get at their properties algorithmically, so if one is in the above situation, then the structure of the group $(K_1 \times K_2) \rtimes F$ can be deceptively complicated. This is the first observation in the proof of....

B-Miller Undecidability Theorem, 2003

Theorem

Let $1 \to N \to \Gamma \to L \to 1$ *be an exact sequence of groups. Suppose that*

1. Γ *is torsion-free and hyperbolic,*
2. N *is infinite and finitely generated, and*
3. L *is a non-abelian free group.*

If F *is a non-abelian free group, then the isomorphism problem for finitely presented subgroups of* $\Gamma \times \Gamma \times F$ *is unsolvable.*

In more detail, there is a recursive sequence Δ_i $(i \in \mathbb{N})$ *of finite subsets of* $\Gamma \times \Gamma \times F$*, together with finite presentations* $\langle \Delta_i \mid \Theta_i \rangle$ *of the subgroups they generate, such that there is no algorithm that can determine whether or not* $\langle \Delta_i \mid \Theta_i \rangle \cong \langle \Delta_0 \mid \Theta_0 \rangle$.

Unsolvabilty of Iso for f.p. subgroups in a RAAG

Virtually special version of the Rips construction (Haglund-Wise), with Q free

$$1 \to N \to \Gamma \to Q \to 1$$

N finitely generated and Γ hyperbolic, virtually special.

Pass to finite-index $\Gamma_0 < G$ with $\Gamma_0 \hookrightarrow A$, a RAAG

$$1 \to N_0 \to \Gamma_0 \to Q_0 \to 1$$

$N_0 = N \cap G_0$. Let F be a finitely generated free group.

[B-Miller] tells us that the isomorphism problem is unsolvable among the finitely presented subgroups of $\Gamma_0 \times \Gamma_0 \times F$, which is a subgroup of the right angled Artin group $A \times A \times F$. $\square$

Baumslag's H_2 problem

Theorem (B-Reid, 2012)

There exists a pair of finitely generated residually torsion-free nilpotent groups $N \hookrightarrow \Gamma$ *that have the same nilpotent genus and the same profinite completion, but* Γ *is finitely presented while* $\dim H_2(N, \mathbb{Q}) = \infty$.

A Rips construction, spectral sequence calculations, and a designer group.

Proposition

$\exists$ *a torsion-free, finitely presented group* $\tilde{\Delta}$ *with no non-trivial finite quotients,* $H_1(\tilde{\Delta}, \mathbb{Z}) = H_2(\tilde{\Delta}, \mathbb{Z}) = 0$ *and* $\dim H_3(\tilde{\Delta}, \mathbb{Q}) = \infty$.

The group $\tilde{\Delta}$

$$B_p = \langle a_1, a_2, b_1, b_2 \mid$$

$$a_1^{-1} a_2^p a_1 a_2^{-p-1},\ b_1^{-1} b_2^p b_1 b_2^{-p-1},\ a_1^{-1}[b_2, b_1^{-1} b_2 b_1],\ b_1^{-1}[a_2, a_1^{-1} a_2 a_1]\rangle$$

The salient features of B_p are that it is finitely presented, acyclic over $\mathbb{Z}$, has no finite quotients, contains a 2-generator free group, F say, and is torsion-free.

$$\Delta := (A \times A) *_S (A \times A),$$

the double of $A \times A$ along $S < F \times F$, where $S = \ker(F \times F \to \mathbb{Z})$.

The universal central extension $\tilde{\Delta}$ is the group we seek.