PCMI LECTURE NOTES

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Introduction

Let M be a closed Riemannian manifold whose sectional curvatures are all negative, and denote by $\mathcal{C}$ the set of free homotopy classes of closed curves in M. Negative curvature implies that in each free homotopy class, there is a unique closed geodesic. This defines a *marked length spectrum* function $\ell\colon \mathcal{C} \to \mathbb{R}_{>0}$ which assigns to the class g the length $\ell(g)$ of this closed geodesic. Burns and Katok asked whether the function ℓ determines M, up to isometry [5]. This question remains open in general, but has been solved completely for surfaces by Otal [21] and slightly later but in more generality by Croke [6].

In these lectures, I'll explain in several steps a proof of this marked length spectrum rigidity for negatively curved surfaces:

Theorem 0.1 (Otal)**.** *Let S and S' be closed, negatively curved surfaces with the same marked length spectrum. Then S is isometric to S'.*

Remark: The (unmarked) length spectrum is defined to be the set of lengths $\{\ell(g) : g \in \mathcal{C}\}$, counted with multiplicity . The length spectrum does not determine the manifold up to isometry. Examples exist even for surfaces of constant negative curvature [25, 23].

Remark: There is a connection between length spectrum and spectrum of the Laplacian. On hyperbolic manifolds, the Selberg trace formula shows that the spectrum of the Laplacian determines the length spectrum. For generic Riemannian metrics, the spectrum of the Laplacian determines the length spectrum. The analogous spectral rigidity question for the spectrum of the Laplacian was posed by Kac. Such rigidity does not hold in general (one cannot "hear the shape of a drum") but does hold along deformations of negatively curved metrics [12, 7]. See [11, 24] for a discussion of these and related rigidity problems.

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1. Lecture 1

1.1. **Background on negatively curved surfaces.** Let S be a compact, negatively curved surface, and let $\tilde{S}$ be its universal cover. Since S is a surface, all notions of curvature coincide (sectional, Gaussian, Ricci...), and the curvature can thus be expressed as a function $k\colon S \to \mathbb{R}_{<0}$ which pulls back to a bounded function $k\colon \tilde{S} \to \mathbb{R}_{<0}$. The Riemann structure defines a Levi-Civita connection ∇, which in turn defines a notion of parallel translation. A vector field X along a curve $c(t)$ is *parallel* if $\nabla_{\dot{c}(t)} X(c(t)) = 0$ for all t.

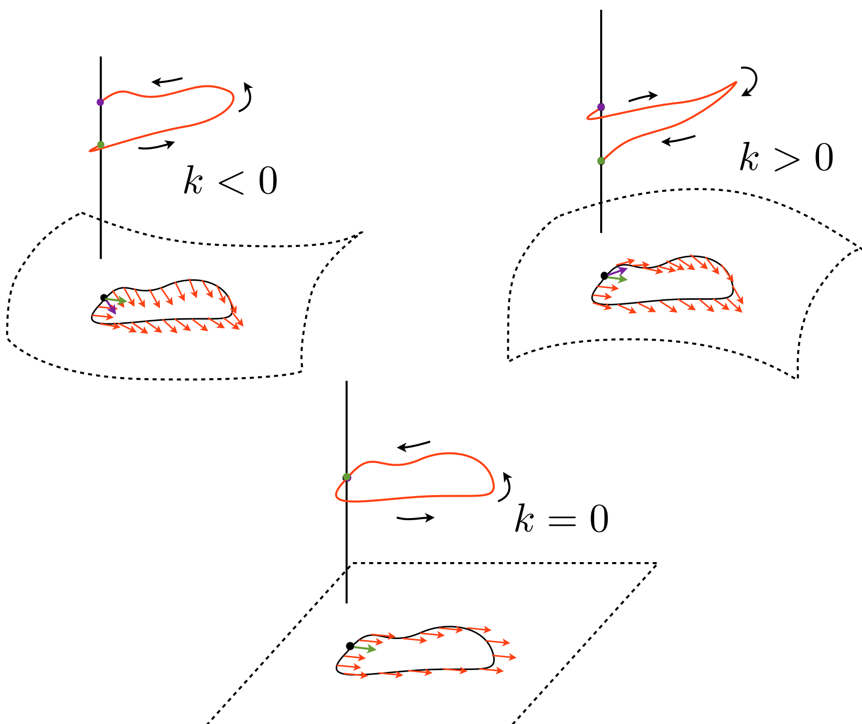

Figure 1. Parallel transport in negative, positive and zero curvature.

A curve γ is a *geodesic* if its velocity curve is parallel along itself:

$$\nabla_{\dot{\gamma}} \dot{\gamma} \equiv 0. \tag{1}$$

Regarded in local coordinates, equation (1) is a second-order linear ODE. A tangent vector $v \in T\tilde{S}$ supplies an initial value:

$$\dot{\gamma}(0) = v. \tag{2}$$

Since the connection is C^∞, the initial value problem given by (1) and (2) has a unique solution. Because the Riemann structure on $\tilde{S}$ is the pullback of a structure on a compact manifold, this solution is defined for all time. For a tangent vector $v \in T\tilde{S}$, we denote by $\gamma_v\colon (-\infty, \infty) \to \tilde{S}$ this unique geodesic with $\dot{\gamma}_v(0) = v$. (For background on the geodesic equation see, e.g. [16]). Solutions to ODEs depend smoothly on parameters, so the map

$$(t, v) \mapsto \gamma_v(t)$$

from $\mathbb{R} \times T\tilde{S} \to \tilde{S}$ is C^∞. As parallel transport preserves the Riemman structure, the speed $\|\dot{\gamma}_v(t)\|$ is constant, equal to $\|v\|$. One therefore obtains a $1-1$ correspondence between unit-speed geodesics and the unit tangent bundle $T^1\tilde{S}$ given by $v \leftrightarrow \gamma_v$.

The Cartan-Hadamard theorem states in this setting that for any $p \in \tilde{S}$, the exponential map

$$\exp_p\colon w \in T_p\tilde{S} \mapsto \gamma_w(1)$$

is a C^∞ diffeomorphism onto $\tilde{S}$. Consequently, $\tilde{S}$ is contractible, diffeomorphic to the plane $\mathbb{R}^2$.

1.2. **A key example.** A key example is the hyperbolic plane. The *Poincaré disk* (or hyperbolic disk) is the domain $\mathbb{D} = \{z : |z| < 1\}$ with the metric

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

The group of orientation-preserving isometries of $\mathbb{D}$ is

$$\{\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 \neq 0\},$$

which acts by Möbius transformations:

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

The hyperbolic disk is isometric via a Möbius transformation to the upper-half plane $\mathbb{H} = \mathrm{Im}(z) > 0$ with the metric

$$ds^2 = \frac{|dz|^2}{c(\mathrm{Im} z)^2}$$

(find c). The isometry group of $\mathbb{H}$ is

$$\mathrm{PSL}(2, \mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}/\{\pm I\},$$

also acting by Möbius transformations. The curvature of $\mathbb{H}$ is constant, equal to $-c$. We will refer to the $\mathbb{D}$ and $\mathbb{H}$ models interchangeably.

Hyperbolic geodesics in $\mathbb{D}$ are Euclidean circular arcs, perpendicular to $\partial\mathbb{D} = \{|z| = 1\}$. In $\mathbb{H}$, hyperbolic geodesics in $\mathbb{H}$ are Euclidean (semi) circular arcs, perpendicular to $\mathrm{Im}(z) = 0$ (where lines are Euclidean circles with infinite radius).

The stabilizer of a point under this left action is the compact subgroup $K = SO(2)/\{\pm I\}$, which gives an identification of $\mathbb{H}$ with the coset space of K:

$$\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})/K.$$

The derivative action of $PSL(2,\mathbb{R})$ on the unit tangent bundle $T^1\mathbb{H}$ is free and transitive, and gives an analytic identification between $T^1\mathbb{H}$ and $PSL(2,\mathbb{R})$. The action of $PSL(2,\mathbb{R})$ on $T^1\mathbb{H}$ by isometries corresponds to left multiplication in $\mathrm{PSL}(2,\mathbb{R})$.

If S is a closed orientable surface with $\tilde{S} = \mathbb{H}$, then $\pi_1(S)$ acts by isometries on $\mathbb{H}$ and hence embeds as a discrete subgroup $\Gamma <$ $\mathrm{PSL}(2,\mathbb{R})$. We thus obtain the following identifications:

$$S = \Gamma\backslash\mathbb{H} = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})/K,$$

and

$$T^1 S = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R}).$$

Endowing $\mathrm{PSL}(2,\mathbb{R})$ with a suitable left-invariant metric gives an isometry between $\mathrm{PSL}(2,\mathbb{R})/K$ and $\mathbb{H}$. This metric on $\mathrm{PSL}(2,\mathbb{R})$ also induces a metric on $T^1\mathbb{H}$, called the Sasaki metric (see the next section). In this metric, the lifts of geodesics in $\mathbb{H}$ via $\gamma \mapsto \dot{\gamma}$ gives Sasaki geodesics in $T^1\mathbb{H}$ (there are other Sasaki geodesics that do not project to geodesics in $\mathbb{H}$ but project to curves of constant geodesic curvature: for example, the orbits of the $SO(2)$ subgroup.)

Exercise 1.1. *If you have never done so before, verify these assertions about hyperbolic space. Useful fact: the curvature of a conformal metric $ds^2 = h(z)^2|dz|^2$ (where h is real-valued and positive) on a planar domain is given by the formula:*

$$k = -\frac{\Delta \log h}{h^2},$$

where Δ is the Euclidean Laplacian.

To verify the assertion about geodesics, it suffices to show that the curve $t \mapsto ie^t$ is a geodesic in $\mathbb{H}$ and then apply isometries. (note that this vertical ray in $\mathbb{H}$ is fixed pointwise by the (orientation-reversing) hyperbolic isometry $z \mapsto -\overline{z}$...). One can also find a formula for hyperbolic distance using this method.

To identify $T^1\mathbb{H}$ *with* $\mathrm{PSL}(2,\mathbb{R})$*, start by identifying the unit vertical tangent vector based at* i *with the identity matrix. It is helpful to understand the orbit of this vector under one-parameter subgroups that together generate* $\mathrm{PSL}(2,\mathbb{R})$*, for example, the groups in the Iwasawa (*KAN*) decomposition.*

Any closed orientable surface of genus 2 or higher admits a metric of constant negative curvature. There are a variety of methods to construct such a metric. One way is to find a discrete and faithful representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, and set

$$S = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})/K$$

as above, with $\Gamma = \rho(\pi_1(S))$. Using algebraic methods, one can find arithmetic subgroups (and so arithmetic surfaces) in this way.

A highly symmetric, hands-on way to construct a genus g surface is to take a regular hyperbolic $4g$-gon with sum of vertex angles equal to 2π and use hyperbolic isometries to glue opposite sides. Much more generally, hyperbolic structures are constructed by gluing together $2g$ hyperbolic "pairs of pants" of varying cuff lengths. There are $6g-6$ degrees of freedom in this construction (cuff lengths of pants and twist parameters in gluing). The space of all such structures is a $6g-6$-dimensional space diffeomorphic to a ball called *Teichmüller space*, and the space of of such structures modulo isometry is a $6g-6$ dimensional orbiford called *Moduli space*.

1.3. **Geodesics in negative curvature.** In the sequel, S is a closed, orientable negatively curved surface. Henceforth, all geodesics are unit speed, unless otherwise specified. To fix concepts, we endow the tangent bundle TS with a fixed Riemann structure called the Sasaki metric, which is compatible with the negatively curved structure on S. [1] (see, e.g. [16]). This pulls back to a Riemann structure on $T^1\tilde{S}$. In what follows, the distances d_{T^1S} and $d_{T^1\tilde{S}}$ on T^1S and $T^1\tilde{S}$ are measured in this Sasaki metric. This metric has the property that its restriction to any fiber of T^1S is just Lebesgue (angular) measure $d\theta$, and its restriction to any parallel vector field along a curve in S is just arclength along that curve. On the hyperbolic plane, the Sasaki metric is precisely the left-invariant metric on $\mathrm{PSL}(2,\mathbb{R})$.

[1] Briefly, the Sasaki structure in the tangent space T_vTS to $v \in TS$ is obtained using the identification $T_v(TS) \simeq T_{\pi(v)}S \times T_{\pi(v)}S$ given by the connection. The two factors are endowed with the original Riemann inner product and declared to be orthogonal.

We say $\gamma_1 \sim \gamma_2$ if one is an orientation-preserving reparamentrization of the other: $\gamma_2(t) = \gamma_1(t + t_0)$, for some $t_0 \in \mathbb{R}$. Denote by $[\gamma]$ the equivalence class of the parametrized unit speed geodesic γ.

Proposition 1.2. *The universal cover $\tilde{S}$ has the following properties:*

(1) **Strict Convexity:** *If γ_1, γ_2 are distinct unit speed geodesics, then*

$$t \mapsto d(\gamma_1(t), \gamma_2(\mathbb{R})) \quad \text{and} \quad t \mapsto d_{T^1\tilde{S}}(\dot{\gamma}_1(t), \dot{\gamma}_2(\mathbb{R}))$$

are C^∞, strictly convex functions acheiving their minimum at the same time t_0. Thus the distance between two unit speed geodesics is realized at a unique point, where the (Sasaki) distance between velocity curves is also minimized.

(2) **Geodesic rays are asymptotic or diverge:** *If $\gamma_1, \gamma_2 \colon [0, \infty) \to \tilde{S}$ are geodesic rays with*

$$\limsup_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty,$$

then there exists $t_0 \in \mathbb{R}$ such that

$$\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t + t_0)) = 0.$$

(in fact, this convergence to 0 is uniformly exponentially fast, with rate determined by the curvature k).

(3) **Distinct geodesics diverge:** *For every $C, \epsilon > 0$, there exists $T > 0$ such that, for any two unit speed geodesics: $\gamma_1, \gamma_2 \colon (-\infty, \infty) \to \tilde{S}$, if*

$$\max\{d(\gamma_1(-T), d(\gamma_2(-T)), d(\gamma_1(T), \gamma_2(T))\} < C,$$

then

$$d_{T^1\tilde{S}}(\dot{\gamma}_1(0), \dot{\gamma}_2([-T, T])) < \epsilon.$$

In particular, if

$$d(\gamma_1(t), \gamma_2(t)) < C$$

for all t, then $\gamma_1 \sim \gamma_2$.

1.4. **The geodesic flow.** The *geodesic flow* $\varphi \colon T\tilde{S} \times \mathbb{R} \to T\tilde{S}$ sends (v, t) to $\varphi_t(v) := \dot{\gamma}_v(t)$. This projects to a geodesic flow on TS, since the geodesic flow commutes with isometries. As remarked above, the dependence of $\dot{\gamma}_v(t)$ on v and t is C^∞, and uniqueness of solutions to the initial value problem (1) and (2) implies that φ_t is a flow on $T^1\tilde{S}$, i.e. a 1-parameter group of diffeomorphisms under composition:

$$\varphi_0 = Id, \quad \text{and } \varphi_{s+t} = \varphi_s \circ \varphi_t, \quad \forall s, t.$$

Since geodesics have constant speed, the geodesic flow restricts to a geodesic flow on the unit tangent bundles T^1S, $T^1\tilde{S}$.

Exercise 1.3. *An additional symmetry of the geodesic flow is* flip invariance*:*

$$\varphi_{-t}(-v) = -\varphi_t(v).$$

Another way to state this is that φ_t is conjugate to the reverse time flow φ_{-t} via the involution on $I\colon T^1\tilde{S} \to T^1S$ defined by:

$$I(v) = -v.$$

Verify this.

Another useful way to describe the geodesic flow is as the flow of a Hamiltonian vector field on TS (and similarly on $T\tilde{S}$). To do this, one first recalls that the cotangent bundle T^*S carries a canonical symplectic structure (it is $\omega = d\theta$, where θ is the canonical 1-form on T^*S). This pulls back to a (noncanonical) symplectic form ω on the tangent bundle via the Riemann structure. Let $E\colon TS \to \mathbb{R}$ be the Hamiltonian (energy) function given by the square of the Riemannian metric:

$$E(v) = \|v\|^2$$

Then the symplectic gradient X_E of this Hamiltonian is defined by:

$$dE = i_{X_E}\omega.$$

The vector field X_E on TS then generates the geodesic flow; i.e.,

$$\frac{d}{dt}\varphi_t(v)|_{t=t_0} = X_E(\varphi_{t_0}(v)),$$

for all v, t_0 (this is just another formulation of the geodesic equation given by (1) and (2)).

From standard properties of Hamiltonian flows, one reads off immediately properties of the geodesic flow:

(1) $E \circ \varphi_t = E$, for all t. That is, the geodesic flow preserves length.
(2) $\varphi_t^*\omega = \omega$, for all t. That is, $\{\varphi_t\}$ is a 1-parameter group of symplectomorphisms. In particular, φ_t preserves the volume form $\omega \wedge \omega$.
(3) The restriction of φ_t to the unit tangent bundle $T^1\tilde{S} = E^{-1}(1)$ preserves the contact 1-form α given by

$$\alpha = i_{\nabla E}\omega.$$

In particular, φ_t preserves the volume form $d\lambda = \alpha \wedge d\alpha$ on T^1S (Liouville's Theorem). This volume λ is called the *Liouville measure*. It is the product of Riemannian measure on S with arclength on the fibers of T^1S.

(4) Since T^1S is compact, its total volume is finite:

$$\lambda(T^1S) = \int_{T^1S} |\alpha \wedge d\alpha| < \infty.$$

Poincaré Recurrence implies that for almost every $v \in T^1S$ (with respect to volume):

$$\liminf_{t \to \pm\infty} d_{T^1S}(\varphi_t(v), v) = 0.$$

More generally, the machinery of smooth ergodic theory can be applied to geodesic flows to prove things like ergodicity, mixing etc.

(See [15] for more details. Most of what is written here is completely general and applies to any Riemannian manifold.)

Exercise 1.4. *Show that on $T^1\mathbb{H} = PSL(2, \mathbb{R})$, the geodesic flow is given by right multiplication by the* 1*-parameter subgroup:*

$$A = \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

To be continued...

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